TJMM
16 (2024), No. 1-2, 71-82

HARMONIC NONCONVEX VARIATIONAL INEQUALITIES

MUHAMMAD ASLAM NOOR AND KHALIDA INAYAT NOOR

Abstract. In this paper, we introduce and study the harmonic nonconvex variational inequalities. The auxiliary principle technique is applied to suggest and analyze some inertial iterative schemes for solving harmonic nonconvex variational inequalities. The convergence criteria of the proposed methods is discussed. Results obtained in this paper continue to hold for various new and known classes of harmonic variational inequalities and related optimization problems. The ideas and techniques of this paper may inspire further research in various branches of pure and applied sciences.

1. Introduction

The variational principles have been one of the major branches of mathematical and engineering sciences, the origin of which can be traced back to Euler, Newton, Lagrange and the Bernoulli's brothers. The ideas and techniques are being applied in a variety of diverse areas of sciences and prove to be productive and innovative. The variational principles can be applied to interpret the basic principles of mathematical and physical sciences in the form of simplicity and elegance. During this period, the variational principles have played an important and significant part as a unifying influence in the development of the general theory of relativity, gauge field theory in modern particle physics and soliton theory. Stampacchia [51] proved that the minimum of the energy (potential) functional associated with the obstacle problems arising in the potential on the convex set can be be characterised by the inequality, called the variational inequality. We would like to point out that the variational inequality theory describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics, regional and engineering sciences. For the applications, formulations, extensions, generalizations, dynamical systems, sensitivity analysis, numerical results, climate change, solar penal designing, and other aspects of variational inequalities, see [1, 2, 3, 5, 7, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 37, 40, 42, 43, 44, 45, 46, 47, 51] and the references therein.

Convexity theory contains a wealth of novel ideas and techniques, which have played the significant role in the development of almost all the branches of pure

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2010 *Mathematics Subject Classification.* MSC2020 subject classifications: 49J40, 26D15, 26D10, 90C23.

Key words and phrases. Harmonic convex functions, monotone operators, harmonic variational inequalities, Iterative methods.

and applied sciences. Several new generalizations and extensions of the convex functions and convex sets have been introduced and studied to tackle unrelated complicated and complex problems in a unified manner. Harmonic functions and harmonic convex sets are important generalizations of the convex functions and convex sets. Anderson et al. [4] have investigated several aspects of the harmonic convex functions. The harmonic means have novel applications in electrical circuits theory, stock exchange [3] and played in developing parallel algorithms for solving complicated problems. Noor et al[32] have shown that the minimum of the differentiable harmonic convex function on the harmonic convex set can be characterized by a class of variational inequalities, known as harmonic variational inequalities. For the formulation, motivation, numerical methods, generalizations and other aspects of harmonic convex functions and harmonic variational inequalities, see [1, 2, 3, 11, 12, 32, 33, 34, 35, 36, 38, 39, 40].
It is worth mentioning that almost all the results regarding the existence and iterative schemes for variational inequalities, which have been investigated and considered, if the underlying set is a convex set. This is because all the techniques are based on the properties of the projection operator over convex sets, which may not hold in general, when the sets are nonconvex. Poliquin et al.[49] and Clarke et al. [6] have introduced and studied a new class of nonconvex sets, which is called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. The uniformly prox-regular sets include the convex sets as a special case.

It is natural to study these different problems in a unified framework. This motivated us to introduce and consider some new classes of harmonic nonconvex variational inequalities. It is well known that projection method, resolvent method and descent methods are not applicable to propose numerical methods for solving harmonic variational inequalities. We apply the auxiliary principle technique, which is mainly due to Lions et al [13] and Glowinski et al[9]. Noor [15, 20, 21] and Noor et al[32, 33, 34, 41, 42, 43, 45] have used this technique to develop some iterative schemes for solving various classes of variational inequalities and equilibrium problems. We point out that this technique does not involve any projection and resolvent of the operator and is flexible. In this paper, we show that the auxiliary principle technique can be applied to suggest and analyze some new classes of inertial iterative methods for solving harmonic nonconvex variational inequalities. We also prove that the convergence of these new methods requires pseudomonotonicity, which is weaker conation than monotonicity. As special cases, one obtain several known and new results for harmonic variational inequalities, variational inequalities and related optimization problems. Results obtained in this paper, represent an improvement and refinement of the known results for harmonic variational inequalities and their variant forms.

2. Basic Results and Formulation

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$, respectively.

First of all, we recall the following concepts and results from convex analysis and nonsmooth analysis, see [4, 8, 14, 49]. For the sake of completeness and to convey the main ideas, we include the relevant details.

Definition 1. [4] *A set C_h is said to be a harmonic convex set, if*

$$\frac{uv}{v + \lambda(u - v)} \in \mathcal{C}_h, \quad \forall u, v \in C_h, \quad \lambda \in [0, 1].$$

Definition 2. [4] *A function ϕ on the harmonic convex set C_h is said to be harmonic convex, if*

$$\phi(\frac{uv}{v + \lambda(u - v)}) \leq (1 - \lambda)\phi(u) + \lambda\phi(v), \quad \forall u, v \in C_h \quad \lambda \in [0, 1].$$

A function ϕ is said to be a harmonic concave function, if $-\phi$ is harmonic convex function.

We recall that the minimum of a differentiable harmonic convex function on the harmonic convex set C_h can be characterized by the variational inequality. This is result is due to Noor and Noor [32].

Theorem 1. [32] *Let ϕ be a differentiable harmonic convex function on the harmonic convex set C_h. Then $u \in C_h$ is a minimum of ϕ, if and only if, $u \in C_h$ satisfies the inequality*

$$\langle \phi'(u), \frac{uv}{u - v} \rangle \geq 0, \quad \forall v \in C_h. \tag{1}$$

The inequality of type (1) is called the harmonic variational inequality.

We would like to mention that Theorem 1 implies that harmonic optimization programming problem can be studied via the harmonic variational inequality (2). Using the ideas and techniques of Theorem 1, we can derive the following result.

Theorem 2. *Let ϕ be a differentiable harmonic convex functions on the harmonic convex set C_h. Then*

$$(i). \qquad \phi(v) - \phi(u) \geq \langle \phi'(u), \frac{uv}{u - v} \rangle, \quad \forall u, v \in C_h.$$

$$(ii). \qquad \langle \phi'(u) - \phi'(v), \frac{uv}{v - u} \rangle \geq 0, \quad \forall u, v \in C_h.$$

Motivated by Theorem 1 and Theorem 2, we introduce some new concepts.

Definition 3. *An operator T is said to be a harmonic monotone operator, if and only if,*

$$\langle Tu - Tv, \frac{uv}{u - v} \rangle \geq 0, \quad \forall u, v \in H.$$

Definition 4. *An operator T is said to a harmonic pseudomonotone operator, if*

$$\langle Tu, \frac{uv}{u-v}\rangle \geq 0 \quad \Rightarrow \quad -\langle Tv, \frac{uv}{u-v}\rangle \geq 0, \quad \forall u, v \in H.$$

An harmonic monotone operator is a harmonic pseudomonotone operator, but the converse is not true.

Definition 5. [3, 44] *The proximal normal cone of C at $u \in H$ is given by*

$$N_C^P(u) := \{\xi \in H : u \in P_C[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_C[u] = \{u^* \in C : d_C(u) = \|u - u^*\|\}.$$

Here $d_C(.)$ is the usual distance function to the subset C, that is

$$d_K(u) = \inf_{v \in C} \|v - u\|.$$

The proximal normal cone $N_C^P(u)$ has the following characterization.

Lemma 1. [6, 49] *Let C be a nonempty, closed and convex subset in H. Then $\zeta \in N_C^P(u)$,*
if and only if, there exists a constant $\alpha > 0$ such that

$$\langle \zeta, v - u\rangle \leq \alpha\|v - u\|^2, \quad \forall v \in C.$$

Definition 6. [6, 49] *The Clarke normal cone, denoted by $N_C^C(u)$, is defined as*

$$N_C^C(u) = \overline{co}[N_C^P(u)],$$

where $\overline{co}$ means the closure of the convex hull.

Clearly $N_C^P(u) \subset N_C^C(u)$, but the converse is not true. Note that $N_C^P(u)$ is always closed and convex, whereas $N_C^C(u)$ is convex, but may not be closed [6, 49].

Poliquin et al.[49] and Clarke et al. [6] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions.

Definition 7. [6, 49] *For a given $r \in (0, \infty]$, a subset C_r is said to be normalized uniformly r-prox-regular, if and only if, every nonzero proximal normal to C_r can be realized by an r-ball, that is, $\forall u \in C_r$ and $0 \neq \xi \in N_{C_r}^P(u)$, one has*

$$\langle (\xi)/\|\xi\|, v - u\rangle \leq (1/2r)\|v - u\|^2, \quad \forall v \in C_r.$$

Remark 1. *It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see* [6, 49]*. It is clear that if $r = \infty$, then uniformly prox-regularity of C_r is equivalent to the convexity of C. It is known that if C_r is a uniformly prox-regular set, then the proximal normal cone $N_{C_r}^P(u)$ is closed as a set-valued mapping. Thus, we have $N_{C_r}^P(u) = N_{C_r}^C(u)$. Using the* *idea of Definition 7, we define the concept of normalized uniformly pros-regular harmonic convex subset C_{rh}, which is called harmonic nonconvex set. In recent years, quantum calculus and fuzzy function techniques are being applied in the convex analysis, which is another area of future research.*

In many applications, the inequalities of the type (1) may not arise as the minimum of the differentiable harmonic convex functions. These facts motivated us to consider more general harmonic variational inequality, which contains the inequalities (1) as a special case.

For a given nonlinear continuous operator $T : H \longrightarrow H$, we consider the problem of finding $u \in C_{rh}$ such that

$$\langle Tu, \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in C_{rh}, \tag{2}$$

which is called the *harmonic nonconvex variational inequality.*
We now discuss some important special cases of the harmonic nonconvex variational inequalities (2).
(i). If $C_{rh} = C_h$, harmonic convex set in H, then problem (2) is equivalent to fining $u \in C_h$, such that

$$\langle Tu, \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in C_h, \tag{3}$$

which is called the harmonic variational inequality, introduced and studied by Noor [32].

(ii). If $(C_{rh})^{\star} = \{u \in \mathcal{H} : \langle u, \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in C_{rh}\}$ is a polar harmonic convex cone of the harmonic convex C_{rh}, then problem (2) is equivalent to fining $u \in H$, such that

$$\frac{uv}{u-v} \in C_{rh}, \quad Tu \in C^{\star}_{rh}, \quad \langle Tu, \frac{uv}{u-v}\rangle = 0, \tag{4}$$

is called the harmonic complementarity problem. For the applications, numerical methods and other aspects of complementarity problems, see [7, 16, 17, 20, 41, 42, 43, 46] and the references therein.
(iii). If $C_{rh} = H$, then problem (2 is equivalent to fining $u \in H$, such that

$$\langle Tu, \frac{uv}{u-v}\rangle = 0, \quad \forall v \in H, \tag{5}$$

which is called the weak formulation of the harmonic boundary value problem.

(iv). For $C_{rh} = C$, convex set in H, then the problem (2) reduces to finding $u \in C$ such that

$$\langle Tu, v-u\rangle \geq 0, \quad \forall v \in C, \tag{6}$$

is called the variational inequality. For the recent applications, motivation, numerical methods, sensitivity analysis and local uniqueness of solutions of harmonic variational inequalities and related optimization problems, see [1, 2, 3, 11, 12, 32, 33, 34, 35, 36, 38, 39, 40] and the references therein.

This show that the problem (2) is quite and unified one. Due to the structure and nonlinearity involved, one has to consider its own. It is an open problem to develop unified implementation numerical methods for solving the harmonic variational inequalities and related optimization problems.

3. Main Results

In this section, we apply the auxiliary principle technique, which is mainly due to Lions et. al.[13] and Glowinski et al [10] as developed in [1, 2, 12, 15, 18, 20, 21, 22, 23, 24, 25, 29, 30, 30, 35, 36, 40, 41, 42, 48], to suggest and analyze some inertial iterative methods for solving harmonic nonconvex variational inequalities (2).

For a given $u \in C_{rh}$ satisfying (2), consider the problem of finding $w \in C_{rh}$ such that

$$\langle \rho T(w + \eta(u - w)), \frac{uw}{u - w}\rangle + \langle M(w) - M(u), v - w\rangle \geq 0, \quad \forall v \in C_{rh}, \tag{7}$$

where $\rho > 0, \eta \in [0, 1]$ are constants and $M : H \longrightarrow H$, is an arbitrary operator. Inequality of type (7) is called the modified auxiliary harmonic nonconvex variational inequality involving an arbitrary operator, which is mainly due to Noor [18].

If $w = u$, then w is a solution of (2). This simple observation enables us to suggest the following iterative method for solving (2).

Algorithm 1. *For a given $u_0 \in C_{rh}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(u_{n+1} + \eta(u_n - u_{n+1})), \frac{u_n u_{n+1}}{u_n - u_{n+1}}\rangle + \langle M(u_{n+1}) - M(u_n), v - u_{n+1}\rangle \geq 0, \quad \forall v \in C_{rh}.$$

Algorithm 1 is called the hybrid proximal point algorithm for solving harmonic nonconvex variational inequalities(2).

Special Cases

We now consider some cases of Algorithm 1.

(I). For $\eta = 0$, Algorithm 1 reduces to:

Algorithm 2. *For a given $u_0 \in C_{rh}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T u_{n+1}, \frac{u_n u_{n+1}}{u_n - u_{n+1}}\rangle + \langle M(u_{n+1}) - M(u_n), v - u_{n+1}\rangle \geq 0, \quad \forall v \in C_{rh}. \tag{8}$$

(II). If $\eta = 1$, then Algorithm 1 reduces to:

Algorithm 3. *For a given $u_0 \in C_{rh}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(u_n, \frac{u_n u_{n+1}}{u_n - u_{n+1}}\rangle + \langle M(u_{n+1}) - M(u_n), v - u_{n+1}\rangle \geq 0, \quad \forall v \in C_{rh}.$$

(III). If $\eta = \frac{1}{2}$, then Algorithm 1 collapses to:

Algorithm 4. *For a given $u_0 \in C_{rh}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(\frac{u_{n+1}+u_n}{2}), \frac{u_n u_{n+1}}{u_n - u_{n+1}}\rangle + \langle M(u_{n+1}) - M(u_n), v - u_{n+1}\rangle \geq 0, \quad \forall v \in C_{rh}.$$

which is called the mid-point proximal method for solving the problem (2).

For the convergence analysis of Algorithm 2, we recall the following concepts and results.

Definition 8. *An operator $T : H \to H$ is said to be:*
(i). harmonic monotone, if and only if,

$$\langle Tu - Tv, \frac{uv}{u-v}\rangle \geq 0, \quad \forall u, v \in H.$$

(ii) harmonic pseudomonotone if and only if,

$$\langle Tu, \frac{uv}{u-v}\rangle \geq 0 \quad \Longrightarrow \quad \langle Tv, \frac{uv}{u-v}\rangle \geq 0, \quad \forall u, v \in H.$$

It is known that harmonic monotonicity implies harmonic pseudomonotonicity; but the converse is not true. Consequently, the class of harmonic pseudomonotone operators is bigger than the one of harmonic monotone operators.

We now consider the convergence criteria of Algorithm 2 using the idea and technique developed in [1, 2, 35, 36]. We include the proof for the sake of completeness and to convey an idea of the technique involved.

Theorem 3. *Let $u \in C_{rh}$ be a solution of (2) and let u_{n+1} be the approximate solution obtained from Algorithm 2. Let the operators T and A are harmonic. If the operator M is a strongly monotone with constant $\xi > 0$ and Lipschitz continuous with consatnt $\zeta > 0$, then*

$$\xi \|u_{n+1} - u_n\| \leq \zeta \|u_n - u\|. \tag{9}$$

Proof. Let $u \in C_{rh}$ be a solution of (2). Then

$$\langle Tv, \frac{uv}{v-u}\rangle \geq 0, \quad \forall v \in C_{rh}. \tag{10}$$

since T is a harmonic pseudomonotone operator.
Now taking $v = u_{n+1}$ in (10), we have

$$\langle Tu, \frac{uu_{n+1}}{u_{n+1} - u}\rangle \geq 0. \tag{11}$$

Taking $v = u$ in (8), we get

$$\langle \rho T(u_{n+1}, \frac{u_n u_{n+1}}{u_n - u_{n+1}}\rangle + \langle M(u_{n+1}) - M(u_n), u - u_{n+1}\rangle \geq 0,$$

which can be written as

$$\langle M(u_{n+1}) - M(u_n), u - u_{n+1}\rangle \geq \langle \rho T u_{n+1}, \frac{uu_{n+1}}{u_{n+1} - u}\rangle \geq 0, \tag{12}$$

where we have used (11).
From (12), we have

$$0 \leq \langle M(u_{n+1}) - M(u_n), u - u_{n+1}\rangle$$
$$= -\langle M(u_{n+1} - M(u_n), u_{n+1} - u_n\rangle + \langle M(u_{n+1}) - M(u_n), u - u_n\rangle,$$

which implies that

$$\langle M(u_{n+1} - M(u_n), u_{n+1} - u_n\rangle \leq \langle M(u_{n+1}) - M(u_n), u - u_n\rangle.$$

Since the operator M is strongly monotone with constant $\xi > 0$ and Lipschitz continuous with constant $\zeta > 0$, we obtain

$$\xi\|u_{n+1} - u_n\| \leq \zeta\|u - u_n\|,$$

the required result (9). □

Theorem 4. *Let H be a finite dimensional space and all the assumptions of Theorem 3 hold. Then the sequence $\{u_n\}_1^\infty$ given by Algorithm 2 converges to a solution u of (2).*

Proof. Let $u \in K$ be a solution of (2). From (9), it follows that the sequence $\{\|u - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{13}$$

Let $\hat{u}$ be the limit point of $\{u_n\}_0^\infty$; whose subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_0^\infty$ converges to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (8), taking the limit $n_j \longrightarrow \infty$ and using (13), we have

$$\langle T\hat{u}, \frac{\hat{u}v}{v - \hat{u}}\rangle \geq 0, \quad \forall v \in \mathcal{C}_h,$$

which implies that $\hat{u}$ solves the harmonic hemivariational inequality (2) and

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2.$$

Thus, it follows from the above inequality that $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} (u_n) = \hat{u}.$$

the required result. □

Recently, inertial methods for solving the variational inequalities are being considered for finding the speeding the convergence criteria, the origin of which can be traced back to Polyak [50]. We again consider the auxiliary principle technique to suggest some hybrid inertial proximal point methods for solving the problem (2).

For a given $u \in \mathcal{C}_h$ satisfying (2), consider the problem of finding $w \in C_{rh}$ such that

$$\langle \rho T(w+\eta(u-w)), \frac{uw}{u-w}\rangle + \langle M(w)-M(u)+\alpha(u-u), v-w\rangle \geq 0, \quad \forall v \in C_{rh}, \tag{14}$$

where $\rho > 0, \alpha, \xi, \eta, \in [0,1]$ are constants and M is an arbitrary operator.

Clearly, for $w = u, \quad w$ is a solution of (2). This fact motivated us to to suggest the following inertial iterative method for solving (2).

Algorithm 5. *For given $u_0, u_1 \in C_{rh}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho T(u_{n+1}+\eta(u_n-u_{n+1})), \frac{u_n u_{n+1}}{u_n-u_{n+1}}\rangle \\ &\quad + \langle M(u_{n+1})-M(u_n)+\alpha(u_n-u_{n-1}, v-u_{n+1}\rangle \geq \quad \forall v \in C_{rh}. \end{aligned}$$

which is known as the inertial iterative method.

Note that for $\alpha = 0, \quad \alpha = 0$, Algorithm 5 is exactly the Algorithm 1. Using essentially the technique of Theorem 3, Alshejari et al. [1, 2] and Noor et al. [35, 36], one can study the convergence analysis of Algorithm 5.

For different and appropriate values of the parameters, η, α, the operators T, M and spaces, one can obtain a wide class of inertial type iterative methods for solving the harmonic variational inequalities and related optimization problems.

Conclusion: Some new classes of harmonic nonconvex variational inequalities are introduced in this paper. It is shown that several important problems such as harmonic complementarity problems, variational inequalities and related problems can be obtained as special cases. The auxiliary principle technique involving an arbitrary operator is applied to suggest several inertial type methods for solving harmonic variational inequalities with suitable modifications. We note that this technique is independent of the projection and the resolvent of the operator. Moreover, we have studied the convergence analysis of these new methods under weaker conditions. We have only considered the theoretical aspects of the hybrid inertial iterative methods. It is an interesting problem to develop some numerically implemntable methods and compare with other iterative schemes.

Conflict interest: The authors declare that they have no competing interests.

Authors contributations: All authors contributed equally, design the work, review, edit and agreed to the publish the manuscript.

Acknowledgments: The authors wish to express their deepest gratitude to their respected teachers, students, colleagues, collaborators, referees and friends, who have directly or indirectly contributed in the preparation of this paper.

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Department of Mathematics
COMSATS University Islamabad, Islamabad
Pakistan
E-mail address: noormaslam@gmail.com

Department of Mathematics
COMSATS University Islamabad, Islamabad
Pakistan
E-mail address: khalidan@gmail.com