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THE USE OF THE PARTIAL CONVOLUTION PRODUCT IN THE STUDY OF VISCOELASTIC ROD VIBRATIONS

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Abstract. The study of viscoelastic rod vibrations leads to a class of partial differential equations, defined with the help of a linear operator having the particularity that its coefficients are related to the operator by the partial convolution product. The considered operator can describe the longitudinal, transverse and torsional vibrations of viscoelastic rods.

1. Introduction

The study and solution of boundary problems in the dynamics of viscoelastic solids leads to a class of partial differential equations with the particularity that its coefficients are distributions that are related to the unknown distribution by means of the partial convolution product. This is due to the fact that the constitutive law of viscoelastic solids in the linear theory of viscoelasticity is expressed in the most general form by means of the partial convolution product with respect to the time variable.

We shall denote by $\mathcal{D}(\mathbb{R}^n)$ Schwartz' space of indefinitely differentiable functions with compact support, and by $\mathcal{D}'(\mathbb{R}^n)$ the set of linear continuous functionals defined on $\mathcal{D}(\mathbb{R}^n)$.

The notion of partial convolution product was introduced in [1] and represents a law of composition of two distributions from different spaces. Its properties were studied in [2], [16], [3], [13].

We shall remember the following:

Definition 1. *Let be the distributions $f(x,t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ and $g(x) \in \mathcal{D}'(\mathbb{R}^n)$. We call partial convolution product of the distribution f with g, the distribution denoted $f(x,t) \underset{x}{\otimes} g(x) \in \mathcal{D}'(\mathbb{R}^n)$, defined by the formula*

$$f(x,t) \underset{x}{\otimes} g(x) = f(x,t) * (g(x) \times \delta(t)), \tag{1}$$

where $\delta(t) \in \mathcal{D}'(\mathbb{R}^m)$ is Dirac's delta distribution.

The symbol $\underset{x}{\otimes}$ for the convolution product denotes that the convolution is performed only with respect to the variable $x \in \mathbb{R}^n$, common to the distributions $f(x,t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ and $g(x) \in \mathcal{D}'(\mathbb{R}^n)$, considered in different spaces.

In the right member of the formula (1), the convolution product denoted by the symbol $*$ obviously refers to the variables $(x,t) \in \mathbb{R}^n \times \mathbb{R}^m$.

In the case of existence of the partial convolution product, the latter is a distribution from $\mathcal{D}'(\mathbb{R}^{n+m})$, hence $f(x,t) \underset{x}{\otimes} g(x) \in \mathcal{D}'(\mathbb{R}^{n+m})$.

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Taking into account the definition of the commutativity of the partial convolution product, we will not distinguish between the distributions $f(x,t)\underset{x}{\otimes} g(x)$ and $g(x)\underset{x}{\otimes} f(x,t)$.

Proposition 1. *Let be $f(x,t)\in\mathcal{D}'(\mathbb{R}^{n+m})$ and $g(x)\in\mathcal{D}'(\mathbb{R}^{n})$. The partial convolution product $f(x,t)\underset{x}{\otimes} g(x)\in\mathcal{D}'(\mathbb{R}^{n+m})$ exists if one of the distributions f, g has compact support.*

From the above considerations, it follows that the partial convolution product denoted by the symbol $\underset{x}{\otimes}$ is a new law of composition for the distributions $f(x,t)\in\mathcal{D}'(\mathbb{R}^{n+m})$ and $g(x)\in\mathcal{D}'(\mathbb{R}^{n})$ with respect to the common variable $x\in\mathbb{R}^n$.

This convolution product has wide applications in the deformable solid mechanics and in particular in visco-elasticity [4], [5], [6], [7], [8], [9], [10], [11] .

The structure relation of the partial convolution product is showed as follows:

Proposition 2. ***(Representation formula)*** *Let be the distributions $f(x,t)\in\mathcal{D}'(\mathbb{R}^{n+m})$ and $g(x)\in\mathcal{D}'(\mathbb{R}^{n})$. If the partial convolution product $f(x,t)\underset{x}{\otimes} g(x)\in\mathcal{D}'(\mathbb{R}^{n+m})$, then the relation*

$$\left(f(x,t)\underset{x}{\otimes} g(x),\ \varphi\right)=\left(f(x,t),\ g^{\nu}\underset{x}{\otimes}\varphi(x,t)\right),\ \forall\varphi(x,t)\in\mathcal{D}(\mathbb{R}^{n+m}),\tag{2}$$

takes place, where g^{ν} is the symmetric with respect to the origin of the distribution $g(x)\in\mathcal{D}'(\mathbb{R}^{n})$.

The partial convolution product is a generalization of the ordinary convolution product.

Definition 2. *Let be $f\in\mathcal{D}'(\mathbb{R}^{n})$ and $\mathrm{D}^{\alpha}=\mathrm{D}^{|\alpha|}\Big/\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}$ the derivative operator of order $|\alpha|=\sum\limits_{i=1}^{n}\alpha_i$. We call the derivative of order $|\alpha|$ of the distribution f, the distribution denoted $\mathrm{D}^{\alpha}f$ and given by the relation*

$$(\mathrm{D}^{\alpha}f,\ \varphi)=(-1)^{|\alpha|}(f,\ \mathrm{D}^{\alpha}\varphi),\ \forall\varphi\in\mathcal{D}(\mathbb{R}^{n}).\tag{3}$$

We emphasize that the distributions' derivative does not depend on the order of derivation, so that there is the relation

$$\mathrm{D}^{\alpha+\beta}f=\mathrm{D}^{\alpha}\left(\mathrm{D}^{\beta}f\right)=\mathrm{D}^{\beta}\left(\mathrm{D}^{\alpha}f\right),\ f\in\mathcal{D}'.\tag{4}$$

Below, we give some properties of the partial convolution product.

Proposition 3. *Let be the distributions $f(x,t)\in\mathcal{D}'(\mathbb{R}^{n+m})$, $g(x)\in\mathcal{D}'(\mathbb{R}^{n})$. If the product $f(x,t)\underset{x}{\otimes} g(x)\in\mathcal{D}'(\mathbb{R}^{n+m})$ exists and D_x^{α}, D_t^{β} are derivation operators with respect to the variables $x\in\mathbb{R}^n$, $t\in\mathbb{R}^m$, respectively, then the following formulae take place*

$$\mathrm{D}_x^{\alpha}\left[f(x,t)\underset{x}{\otimes} g(x)\right]=\mathrm{D}_x^{\alpha}f(x,t)\underset{x}{\otimes} g(x)=f(x,t)\underset{x}{\otimes}\mathrm{D}_x^{\alpha}g(x),\tag{5}$$

$$\mathrm{D}_t^{\beta}\left[f(x,t)\underset{x}{\otimes} g(x)\right]=\mathrm{D}_t^{\beta}f(x,t)\underset{x}{\otimes} g(x),\tag{6}$$

$$\left(\mathrm{D}_t^{\beta}f(x,t)\underset{x}{\otimes} g(x),\ \varphi(x)\right)=\mathrm{D}_t^{\beta}\left(f(x,t)\underset{x}{\otimes} g(x),\ \varphi(x)\right),\ \forall\varphi\in\mathcal{D}(\mathbb{R}^{n}).\tag{7}$$

Remark 1. *A similar relation takes place for the distributions depending on the parameter $t\in\mathbb{R}^m$. Thus, if $f_t(x)$, $g(x)\in\mathcal{D}'(\mathbb{R}^{n})$ and $f_t(x)*g(x)\in\mathcal{D}'(\mathbb{R}^{n})$ exists, then $\forall\varphi(x)\in\mathcal{D}(\mathbb{R}^{n})$ and we have*

$$\left(\mathrm{D}_t^{\beta}f_t(x)*g(x),\ \varphi(x)\right)=\mathrm{D}_t^{\beta}\left(f_t(x)*g(x),\ \varphi(x)\right).\tag{8}$$

Proposition 4. *Let be the distributions* $f \in \mathcal{D}'(\mathbb{R}^n),\ h \in \mathcal{E}'(\mathbb{R}^n),\ g \in \mathcal{D}'(\mathbb{R}^m)$. *Then we have*

$$(f(x) \times g(t)) \underset{x}{\otimes} h(x) = (f * h)(x) \times g(t). \tag{9}$$

The partial convolution product has the property of continuity as the usual convolution product.

We have seen that the partial convolution product exists if one of the factors is a distribution with compact support. Another case of existence of the partial convolution product which has particular importance in mechanics is given by [16]:

Proposition 5. *If* $f(x,t) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^m)$, $g(x) \in \mathcal{D}'(\mathbb{R})$ *and* $\operatorname{supp}(f) = (a,\infty) \times T$, $\operatorname{supp}(g) \subset (b,\infty)$, $T \subset \mathbb{R}^m$, *then* $f(x,t) \underset{x}{\otimes} g(x) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^m)$ *exists.*

Proposition 6. *Let be* $f(x,t) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^m)$, $g(x) \in \mathcal{D}'(\mathbb{R})$. *If* $\operatorname{supp}(f) \subset \Omega \times T$, Ω*-compact,* $T \subset \mathbb{R}^m$ *and* $\operatorname{supp}(g) = \Omega'$ *arbitrary, then the partial convolution product* $f(x,t) \underset{x}{\otimes} g(x) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^m)$ *exists.*

A property that expresses a certain relation between the partial convolution product and the usual one is given by

Proposition 7. *Let be the distributions* $f(x,t) \in \mathcal{D}'(\mathbb{R}^{n+m})$ *and* $g_1(x),\ g_2(x) \in \mathcal{E}'(\mathbb{R}^n)$. *We have*

$$f \underset{x}{\otimes} (g_1 * g_2) = \left(f \underset{x}{\otimes} g_1\right) \underset{x}{\otimes} g_2 = \left(f \underset{x}{\otimes} g_2\right) \underset{x}{\otimes} g_1. \tag{10}$$

2. Properties of the operator

The rod theory constitutes an unidimensional theory of the solids which describes the behavior of thin three-dimensional solid bodies, by a system of equations having only two independent variables, namely, a curve parameter and the time.

In the vibrations study of homogeneous and straight isotropic one-dimensional viscoelastic rods with constant cross-section, the operator $\mathrm{L} : (\mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2)$ defined by

$$\begin{aligned} \mathrm{L}(\partial_t, \partial_x) = \partial_t \partial_x^4 \underset{t}{\otimes} a_1(t) + \partial_t \partial_x^2 \underset{t}{\otimes} a_2(t) + \partial_t^2 \partial_x^2 \underset{t}{\otimes} a_3(t) + \\ \partial_t \underset{t}{\otimes} a_4(t) + \partial_t^2 \underset{t}{\otimes} a_5(t) + \partial_x^2 \underset{t}{\otimes} a_6(t), \end{aligned} \tag{11}$$

where $a_i(t) \in \mathcal{D}'_+, i = \overline{1,6}$, so there are distributions with supports in $[0,\infty]$, has wide applications.

Let be the distributions $u(x,t), v(x,t) \in \mathcal{D}'(\mathbb{R}^2)$, null for $t < 0$, that is, $\operatorname{supp}(u,v) \subset \mathbb{R} \times [0,\infty)$. If the convolution product $u * v \in \mathcal{D}'(\mathbb{R}^2)$ exists, then the following relations hold

$$\mathrm{L}(\partial_t, \partial_x)(\alpha u + \beta v) = \alpha\, \mathrm{L}(\partial_t, \partial_x)(u) + \beta\, \mathrm{L}(\partial_t, \partial_x)(v), \tag{12}$$

$$\mathrm{L}(\partial_t, \partial_x)(u * v) = \mathrm{L}(\partial_t, \partial_x)(u) * v = u * \mathrm{L}(\partial_t, \partial_x)(v). \tag{13}$$

We note that the $\mathrm{L}(\partial_t, \partial_x)$ operator, although it is a linear operator, its coefficients are related to the operator using the partial convolution product. Properties (12) and (13) highlight the structural characteristics of the operator.

It is said that the constitutive law of the viscoelastic solid is of differential type if between the stress $\sigma(x,t) \in \mathcal{D}'(\mathbb{R}^2)$ and the specific deformation $\varepsilon(x,t) \in \mathcal{D}'(\mathbb{R}^2)$ take place:

$$\mathrm{P}(\partial_t)\sigma = \mathrm{Q}(\partial_t)\varepsilon, \tag{14}$$

where P, Q are linear differential operators with constant coefficients in the distribution space $\mathcal{D}'(\mathbb{R}^2)$.

We have the following result:

If the constitutive law of the viscoelastic solid is of differential type (14), then the relaxation distribution $\psi(t) \in \mathcal{D}'_+$ satisfies the equation:

$$\mathrm{P}(\partial_t)\psi'(t) = \mathrm{Q}(\partial_t)\delta(t). \tag{15}$$

This result suggests the assumption that the coefficients $a_i(t) \in \mathcal{D}'_+, i = \overline{1,6}$ of the operator (11) satisfy the differential type relations:

$$\begin{gathered} \mathrm{P}_i(\partial_t)a_i'(t) = \mathrm{Q}_i(\partial_t)\delta(t), i = \overline{1,5}, \\ \mathrm{P}_6(\partial_t)a_6(t) = \mathrm{Q}_6(\partial_t)\delta(t), \end{gathered} \tag{16}$$

where P_i and Q_i, $i = \overline{1,6}$, are linear differential operators with constant coefficients.

We say that the distribution $E(x,t) \in \mathcal{D}'(\mathbb{R}^2)$ is the fundamental solution for the operator $\mathrm{L}(\partial_t, \partial_x)$ defined by (11) if it satisfies the following relation:

$$\mathrm{L}(\partial_t, \partial_x)E(x,t) = \delta(x,t), \tag{17}$$

where $\delta(x,t) = \delta(x) \times \delta(t)$ represents the Dirac distribution.

3. Applications of the operator in the study of viscoelastic rod vibrations

The considered operator can describe the longitudinal, transverse and torsional vibrations of viscoelastic rods. We will exemplify this below.

I. The generalized equation of longitudinal vibrations of viscoelastic rods that also takes into account the influence of tangential stresses [15], [5], [3] is:

$$\mathrm{L}_1(\partial_t, \partial_x)u(x,t) = F_1(x,t), F_1 \in \mathcal{D}'(\mathbb{R}^2) \tag{18}$$

where the operator $\mathrm{L}_1 : \mathcal{D}'(\mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2)$ has the expression

$$\mathrm{L}_1(\partial_t, \partial_x) = \partial_t\partial_x^4 \underset{t}{\otimes} \alpha_0\psi(t) - \partial_t\partial_x^2 \underset{t}{\otimes} \psi(t) - \beta_0\partial_t^2\partial_x^2 + \rho\partial_t^2, \alpha_0 = \frac{\nu^2 r_0^2}{2(1+\nu)}, \beta_0 = \rho\nu^2 r_0^2 \tag{19}$$

where $\psi \in \mathcal{D}'_+$, represents the relaxation distribution, $\rho \in \mathbb{R}$ the density and r_0 the radius of gyration.

The operator L_1 is obtained from (11) considering $a_1(t) = \alpha_0\psi(t), a_2(t) = -\psi(t), a_3(t) = -\beta_0\delta(t), a_4 = 0, a_5 = \rho\delta(t)$ and a_6=0.

II. The equation of torsional vibrations of viscoelastic rods [3], [14] is

$$\mathrm{L}_2(\partial_t, \partial_x)\theta(x,t) = F_2(x,t), F_2 \in \mathcal{D}'(\mathbb{R}^2) \tag{20}$$

in which the operator $\mathrm{L}_2 : \mathcal{D}'(\mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2)$ has the expression

$$\mathrm{L}_2(\partial_t, \partial_x) = -\partial_t\partial_x^2 \underset{t}{\otimes} \frac{\psi(t)}{2(1+\nu)} + \rho\partial_t^2. \tag{21}$$

The operator L_2 is obtained from (11) considering $a_1(t) = 0, a_2(t) = -\frac{\psi(t)}{2(1+\nu)}, a_3(t) = a_4 = a_6 = 0, a_5 = \rho\delta(t)$.

III. The equation of transversal vibrations of viscoelastic rods on viscoelastic foundation (the Winkler hypothesis is adopted) [17], [3] is

$$\mathrm{L}_3(\partial_t, \partial_x)v(x,t) = F_3(x,t), F_3 \in \mathcal{D}'(\mathbb{R}^2) \tag{22}$$

where the operator $\mathrm{L}_3 : \mathcal{D}'(\mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2)$ has the expression

$$\mathrm{L}_3(\partial_t, \partial_x) = \partial_t\partial_x^4 \underset{t}{\otimes} I\psi(t) + \partial_t \underset{t}{\otimes} k_0\psi_f(t) + \rho\partial_t^2, \tag{23}$$

where $I \in \mathbb{R}_+$ - the axial moment of inertia of the rod, $k_0 \in \mathbb{R}$ - the stiffness coefficient of the foundation and $\psi(t), \psi_f(t) \in \mathcal{D}'_+$ represent the relaxation distributions corresponding to the rod and the viscoelastic foundation, respectively.

The operator L_3 is obtained from (11) considering $a_1(t) = I\psi(t), a_2 = a_3 = a_6 = 0, a_4(t) = k_0\psi_f(t), a_5 = \rho\delta(t)$.

4. Conclusions

The study of viscoelastic rod vibrations leads to a class of partial differential equations, defined with the help of the operator (11). As we saw in the last chapter, many mathematical models for the viscoelastic rod vibrations can described using the operator (11).

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