TJMM
16 (2024), No. 1-2, 51-57

AN APPLICATION OF BANACH'S FIXED POINT THEOREM IN A PERTURBED METRIC SPACE TO AN INTEGRAL EQUATION

MARIA DOBRIŢOIU

Abstract. Using the Banach's fixed point theorem in a perturbed metric space, in this paper we present a new existence and uniqueness result for the solution of a nonlinear Fredholm integral equation. Finally, we give an example to illustrate the application of the obtained abstract result.

1. Introduction

Most studies on the existence and uniqueness of integral equations solutions in a metric space have used as a basic tool Banach's fixed point theorem (also known as the Contracting Principle) or its generalizations. These generalizations have been made in two ways: the first one, by introducing generalized contraction functions ([3]), and the second one, by using the contraction function in a metric space endowed with a generalized metric, i.e. in a generalized metric space ([5]). By applying some generalizations of the Contraction Principle ([4], [11]), the study of the solutions of integral equations was supplemented with new existence and uniqueness results. In order to obtain another result of existence and uniqueness for the solution of an nonlinear Fredholm integral equation, in this paper we will use the Banach 's fixed point theorem in a perturbed metric space, i.e. in a nonempty set endowed with a perturbed metric.

In this paper we will consider the following nonlinear Fredholm integral equation:

$$x(t) = \int_a^b K(t, s, x(s), x(a), x(b))ds + f(t), \quad t \in [a, b], \tag{1}$$

where $a, b \in \mathbb{R},\ a < b$; $K : [a, b] \times [a, b] \times \mathbb{R}^3 \to \mathbb{R}$, $f : [a, b] \to \mathbb{R}$ and $x \in C[a, b]$.

Some properties of the solution of this integral equation were studied using known classical theorems and the obtained results can be found in several papers, of which we mention [1] and [6]–[9]. To obtain these properties, some results from [2], [4] and [13] were used.

Now, we will add to the study of the nonlinear Fredholm integral equation (1) a new result for the existence and uniqueness of its solution in a perturbed metric space (notion was introduced in [12]) and obtained by applying a generalization of Banach's fixed point theorem ([11], [12]).

For some results related to integral equations, papers [2] and [10] have been consulted. Also, for certain results from the fixed point theory, papers [3], [5], [9] and [13] have been consulted.

This paper is organized into four sections.

The section 1 contains a brief introduction to the topic of this article and the nonlinear Fredholm integral equation (1), the solution of which will be studied in section 3, is presented.

2010 *Mathematics Subject Classification.* MSC2020 subject classifications: 47H10, 45G10.
Key words and phrases. perturbed metric, perturbed metric space, integral equation, fixed point.

In section 2, Preliminaries, we recall the definitions concerning some new notions ([12]), that will be used in order to obtain another existence and uniqueness theorem for the solution of the nonlinear Fredholm integral equation (1). Also, this section present an extension of Banach's fixed point theorem to a perturbed metric space ([12]).

The main result of this paper is presented in section 3, as an existence and uniqueness theorem for the solution of the nonlinear Fredholm integral equation (1) in a perturbed metric space.

The last section contains some brief conclusions on the result presented in this paper.

2. Preliminaries

Following [11] and [12], we present the notions and the theorems that will be used in order to obtain the main result of this paper, presented in the next section.

First of all, the following notions will be recalled: perturbed metric on a nonempty set X, perturbed convergent sequence, perturbed metric space, perturbed Cauchy sequence, complete perturbed metric space and perturbed continuous function.

Let X be a nonempty set.

Definition 1 ([12]). *Let $D, P : X \times X \to [0, +\infty)$ be two given functions. The function D is called a perturbed metric on X with respect to P, if the function*

$$D - P : X \times X \to \mathbb{R},$$

defined by the relation:

$$(D - P)(x, y) = D(x, y) - P(x, y), \text{ for all } x, y \in X$$

is a metric on X, i.e. for all $x, y, z \in X$, it satisfies the following conditions:

(i) $(D - P)(x, y) \geq 0$;
(ii) $(D - P)(x, y) = 0$ *if and only if* $x = y$;
(iii) $(D - P)(x, y) = (D - P)(y, x)$;
(iv) $(D - P)(x, y) \leq (D - P)(x, z) + (D - P)(z, y)]$.

P is called a perturbing function and respectively, $D - P$ is an exact metric.

The set X endowed with D, a perturbed metric with respect to P, respectively, (X, D, P) is called a perturbed metric space.

Remark 1. *A perturbed metric on X is not always also a metric on X.*

In [12] an interesting example of a perturbed metric defined on the space $C[0, 1]$ is presented.

In what follows we present another example of a perturbed metric and respectively a perturbed metric space.

Example 1. *In the continuous functions set defined on the interval $[0, 1]$, i.e. $C[0, 1]$, we consider the function $D : C[0, 1] \times C[0, 1] \to [0, +\infty)$ defined by the relation:*

$$D(f, g) := \int_0^1 |f(s) - g(s)| ds + \big(f(1) - g(1)\big)^2, \quad f, g \in C[0, 1].$$

In this example D is a perturbed metric on $C[0, 1]$ with respect to the perturbing function $P : C[0, 1] \times C[0, 1] \to [0, +\infty)$, defined by the relation:

$$P(f, g) := \big(f(1) - g(1)\big)^2, \quad f, g \in C[0, 1],$$

and the exact metric is the function $d : C[0, 1] \times C[0, 1] \to [0, +\infty)$ defined by the relation:

$$d(f, g) := \int_0^1 |f(s) - g(s)| ds, \quad f, g \in C[0, 1].$$

Therefore, $(C[0,1], D, P)$ *is the perturbed metric space and* $(C[0,1], d)$ *is the standard metric space.*

Next we recall the definition of the following notions: perturbed convergent sequence, perturbed Cauchy sequence and perturbed continuous function, all these in a perturbed metric space (X, D, P). Also, we recall the definition of a complete perturbed metric space (X, D, P).

Definition 2 ([12])**.** *Let* (X, D, P) *be a perturbed metric space,* $\{z_n\}$ *a sequence in* X*, and* $T : X \to X$*.*

(i) We say that $\{z_n\}$ *is a perturbed convergent sequence in* (X, D, P)*, if* $\{z_n\}$ *is a convergent sequence in the metric space* (X, d)*, where* $d = D - P$ *is the exact metric.*

(ii) We say that $\{z_n\}$ *is a perturbed Cauchy sequence in* (X, D, P)*, if* $\{z_n\}$ *is a Cauchy sequence in the metric space* (X, d)*.*

(iii) We say that (X, D, P) *is a complete perturbed metric space, if* (X, d) *is a complete metric space, or, equivalently, if every perturbed Cauchy sequence in* (X, D, P) *is a perturbed convergent sequence in* (X, D, P)*.*

(iv) We say that T *is a perturbed continuous function, if* T *is continuous with respect to the exact metric* d*.*

We recall below, the Banach's fixed point theorem (Contraction Principle) in a standard metric space.

Theorem 1. *(Contraction Principle) (*[13]*) Let* (X, d) *be a complete metric space and* $A : X \to X$ *an* α*-contraction,* $(\alpha < 1)$*. Then* A *has a unique fixed point* $x^* \in C[a, b]$*.*

Banach's fixed point theorem above has been extended to a complete perturbed metric space ([12]). We present this theorem below.

Theorem 2. *(*[12]*) Let* (X, D, P) *be a complete perturbed metric space and* $T : X \to X$ *be a given function. Assume that the following conditions hold:*

(i) T *is a perturbed continuous function;*

(ii) There exists $\alpha \in (0, 1)$ *such that*

$$D(Tu, Tv) \leq \alpha D(u, v), \text{ for all } u, v \in X.$$

Then, T *has a unique fixed point.*

Remark 2. *The fixed point theorem above includes the Banach's fixed point theorem in an usually metric space (Theorem 1).*

In the next section we will use a complete perturbed metric space to obtain the main result of this paper.

3. Main result

Now, we will apply the Banach's fixed point theorem in a perturbed metric space (Theorem 2) in order to study the existence and uniqueness for the solution of the nonlinear Fredholm integral equation (1):

$$x(t) = \int_a^b K(t, s, x(s), x(a), x(b))ds + f(t), \quad t \in [a, b],$$

where $a, b \in \mathbb{R},\ a < b$; $K : [a, b] \times [a, b] \times \mathbb{R}^3 \to \mathbb{R},\ f : [a, b] \to \mathbb{R}$ and $x \in C[a, b]$.

We consider the continuous functions set defined on $[a, b]$, namely

$$C[a, b] = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuous on } [a, b]\}$$

and the function $D : C[a,b] \times C[a,b] \to [0,+\infty)$, defined by the relation:

$$D(x,y) := \max_{t\in[a,b]} \big|x(t) - y(t)\big| + \tau\big(x(a) - y(a)\big), \quad \tau \geq -1, \ x,y \in C[a,b]. \tag{2}$$

The function D is a perturbed metric on $C[a,b]$ with respect to the perturbing function $P : C[a,b] \times C[a,b] \to [0,+\infty)$, defined by the relation:

$$P(x,y) := \tau\big(x(a) - y(a)\big), \quad \tau \geq -1, \ x,y \in C[a,b]. \tag{3}$$

The function $d : C[a,b] \times C[a,b] \to [0,+\infty)$ defined by the relation:

$$d(x,y) := \max_{t\in[a,b]} \big|x(t) - y(t)\big|, \quad x,y \in C[a,b] \tag{4}$$

is an exact metric on $C[a,b]$.

Consequently, $(C[a,b], D, P)$ is a perturbed metric space and $(C[a,b], d)$ is a standard metric space.

Now, using the Banach's fixed point theorem in a complete standard metric space, we will recall below an existence and uniqueness theorem for the solution of the integral equation (1) in the complete metric space $(C[a,b], d)$, where the metric is the function d defined by the relation (4). This result was obtained and published in 1978 ([1]).

Theorem 3. *([1]) Let $(C[a,b], d)$ be a complete metric space and consider the nonlinear Fredholm integral equation (1). If the following conditions are satisfied:*

(i) $K \in C([a,b] \times [a,b] \times \mathbb{R}^3)$*;*
(ii) $f \in C[a,b]$*;*
(iii) there exists $L > 0$ *such that*

$$\big|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)\big| \leq L\big(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|\big),$$

for all $t,s \in [a,b], \ u_i, v_i \in \mathbb{R}, i = 1,2,3;$
(iv) $3L(b-a) < 1$,

then the integral equation (1) has a unique solution $x^* \in C[a,b]$.

In addition to the metric d defined by the relation (4), in the proof of this theorem, Chebashev norm:

$$||x||_C = \max_{t\in[a,b]} \big|x(t)\big|, \quad x \in C[a,b]$$

was also used.

Using the Banach's fixed point theorem in a complete perturbed metric space (Theorem 2) we will present below another existence and uniqueness result for the solution of the integral equation (1).

Theorem 4. *Let $(C[a,b], D, P)$ be a complete perturbed metric space, where D is a perturbed metric defined by (2) and P is a perturbing function defined by (3).*

In this complete perturbed metric space we consider the nonlinear Fredholm integral equation (1) and suppose that the following conditions hold:

(i) $K \in C([a,b] \times [a,b] \times \mathbb{R}^3)$*;*
(ii) $f \in C[a,b]$*;*
(iii) there exists $L > 0$ *such that*

$$\big|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)\big| \leq L\big(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|\big),$$

for all $t,s \in [a,b], \ u_i, v_i \in \mathbb{R}, i = 1,2,3;$
(iv) $3L(b-a) < 1$*, for all* $\tau \geq -1$.

Then the integral equation (1) has a unique solution $x^* \in C[a,b]$.

Proof. It is known that the set $C[a,b]$ endowed with the metric d defined by the relation (4), i.e. $(C[a,b],d)$ is a complete metric space. Consequently, by (iii) of Definition 2, it results that $(C[a,b],D,P)$ is a complete perturbed metric space.

We attach to the integral equation (1), the operator $A : C[a,b] \to C[a,b]$, defined by

$$A(x)(t) := \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t), \tag{5}$$

for all $x \in C[a,b]$ and $t \in [a,b]$.

The set of the solutions of the integral equation (1) coincides with the set of the fixed points of the operator A.

Using the properties of an integral operator from the definition of the operator A, defined by the relation (5) and the hypotheses (i), (ii), it results that the operator A is continuous with respect to the exact metric d, defined by the relation (4). Then by (iv) of Definition 2, it results that A is a perturbed continuous operator in $(C[a,b],D,P)$.

Next, to satisfy hypothesis (ii) of Theorem 2, we estimate the difference:

$$\Big|A(x)(t) - A(y)(t)\Big| = \Big|\int_a^b K(t,s,x(s),x(a),x(b))ds - \int_a^b K(t,s,x(s),x(a),x(b))ds\Big|$$

$$= \Big|\int_a^b \big[K(t,s,x(s),x(a),x(b)) - K(t,s,x(s),x(a),x(b))\big]ds\Big|$$

$$\le \int_a^b \Big|K(t,s,x(s),x(a),x(b)) - K(t,s,x(s),x(a),x(b))\Big|ds$$

and using the hypothesis (iii) it results

$$\Big|A(x)(t) - A(y)(t)\Big| \le L\int_a^b \Big(\big|x(s)-y(s)\big| + \big|x(a)-y(a)\big| + \big|x(b)-y(b)\big|\Big)ds. \tag{6}$$

In what follows, we will establish the conditions under which the hypothesis (ii) of Theorem 2 is fulfilled. For this, using (6), we will determine a constant $\alpha \in (0,1)$ such that

$$D(A(x),A(y)) \le \alpha D(x,y), \ \textit{for all} \ x,y \in C[a,b].$$

We have

$$D(A(x),A(y)) = \max_{t\in[a,b]} \big|A(x)(t) - A(y)(t)\big| + \tau\big(A(x)(a) - A(y)(a)\big)$$

and

$$D(x,y) = \max_{t\in[a,b]} \big|x(t)-y(t)\big| + \tau\big(x(a)-y(a)\big).$$

Now, using (6) and the Chebyshev norm we obtain

$$3L(b-a)(1+\tau)\|x-y\|_C \le (1+\tau)\|x-y\|_C, \quad x,y \in C[a,b],$$

that is true for $3L(b-a) < 1$ and for all $\tau \ge -1$. The proof is complete. □

Finally, we present an application of the Theorem 4 in the following example.

Example 2. *Consider the following nonlinear Fredholm integral equation:*

$$x(t) = \int_0^1 \Big[\frac{x(s)}{t+s+4} + \frac{tx(0)+sx(1)}{4}\Big]ds + sint, \quad t \in [0,1], \tag{7}$$

where $K : [0,1]\times[0,1]\times\mathbb{R}^3 \to \mathbb{R}$, $K(t,s,u_1,u_2,u_3) = \frac{u_1}{t+s+4} + \frac{tu_2+su_3}{4}$, $f : [0,1] \to \mathbb{R}$, $f(t) = \sin t$ *and* $x \in C[0,1]$.

In this case, we will use the space $C[0,1]$ *endowed with the metric* $D : C[0,1]\times C[0,1] \to [0,+\infty)$ *defined by the relation:*

$$D(x,y) = \max_{t\in[0,1]} \big|x(t) - y(t)\big| + \tau\big(x(0) - y(0)\big), \quad \tau \geq -1,\ x,y \in C[0,1].$$

D *is a perturbed metric on* $C[0,1]$ *with respect to the perturbing function* $P : C[0,1] \times C[0,1] \to [0,+\infty)$*, defined by the relation:*

$$P(f,g) = \tau\big(x(0) - y(0)\big), \quad \tau \geq -1,\ x,y \in C[0,1].$$

The exact metric is the function $d : C[0,1] \times C[0,1] \to [0,+\infty)$ *defined by the relation:*

$$d(x,y) = \max_{t\in[0,1]} \big|x(t) - y(t)\big|, \quad x,y \in C[0,1].$$

$(C[0,1],D,P)$ *is a complete perturbed metric space and* $(C[0,1],d)$ *is a complete standard metric space.*

To this integral equation we attach the operator $A : C[0,1] \to C[0,1]$ *defined by the relation:*

$$A(x)(t) := \int_0^1 \Big[\frac{x(s)}{t+s+4} + \frac{tx(0)+sx(1)}{4}\Big]ds + sint, \quad t \in [0,1],\ x \in C[0,1].$$

We verify the conditions of Theorem 4 and obtain

$$L = \frac{3}{4} < 1, \quad and \quad \tau \geq -\frac{1}{2}, \quad [-\frac{1}{2},+\infty) \subset [-1,+\infty).$$

Therefore, the integral equation (7) has a unique solution in the perturbed metric space $(C[0,1],D,P)$*.*

4. Conclusions

The Fredholm integral equation is one of the most well known integral equations. For the nonlinear Fredholm integral equation (1) studied in this paper, new conditions of the existence and uniqueness of the solution were established (Theorem 4), using the notions of perturbed metric, perturbed metric space, complete perturbed metric space introduced in [xx] and applying the Banach's theorem in a perturbed metric space.

Conflict of interest

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] Ambro, M., The approximation of the solutions of integral equations with delay argument, *Studia Univ. Babeş-Bolyai, Mathematica* **1978**, *XXIII*, 2, 26–32 (in Romanian).

[2] Banach, S., Sur les opérations dans les ensembles abstraits et leur application aux équations integrales, *Fund. Math.*, 1922, 3, 133–181.

[3] Berinde, V., *Generalized contractions and applications*, Cub Press 22, Baia Mare, 1997.

[4] Ćirić, L.B., A generalization of Banach's contraction principle, Proc. Amer. Math. Soc., 1974, 45(2), 267–273.

[5] Czerwik, S., Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav., 1993, 1(1), 5–11.

[6] Dobriţoiu, M., Analysis of an integral equation with modified argument, *Studia Univ. Babeş-Bolyai, Mathematica* **51** (2006), 1, 81–94.

[7] Dobriţoiu, M., *Integral Equations with Modifed Argument*, Cluj University Press, Cluj-Napoca, 2009 (in Romanian).

[8] Dobriţoiu, M., A Nonlinear Fredholm Integral Equation, *Transylvanian Journal of Mathematics and Mechanics*, **1** (2009), 1-2, 25–32.

[9] Dobriţoiu, M., *Some Integral Equations with Modified Argument*, WSEAS Press, 2016.

[10] Guo, D., Lakshmikantham, V., Liu, X., *Nonlinear integral equations in abstract spaces*, Kluwer, London, 1996.

[11] Jleli, M., Samet, B., A new generalization of the Banach contraction principle, *J. Inequal. Appl.*, **38** (2014), 1–8.

[12] Jleli, M., Samet, B., On Banach's fixed point theorem in perturbed metric spaces, *J. of Applied Analysis and Computation*, **15** (2025), 2, 993–1001.

[13] Rus, I.A., *Principii şi aplicaţii ale teoriei punctului fix*, Cluj-Napoca: Dacia Publishing House, 1979. (in Romanian)

University of Petroşani
Department of Mathematics and Computer Science
Str. Universităţii, 20, 332006, Petroşani, Romania
E-mail address: mariadobritoiu@yahoo.com