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A QUASI-RELAXATION TRANSFORMS PAIR

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Abstract. The functions obtained though the quasi-relaxation transforms are the extensions to a meta-stable state of the material to their creep fuction $\Psi(t)$, and relaxation function $\phi(t)$, respectively, being one or other (the transform) in each case its spectral density of meta-stability. Likewise, $QX\{\phi(t)\}$, is $\Psi_c(\tau)$, the quasi-relaxation transform of $\phi(t)$ and $QX^{-1}\{\Psi_c(\tau)\}$ is the function $\phi_c(t)$,the anti-quasi-relaxation transform or inverse quasi-relaxation transform. Likewise are obtained these quasi-relaxation transforms pair, where in the extension of load, beyond of the stability limit of a material, can appear poles or singularities after of certain limit during the quasi-relaxation process. A short table of load functions and their corresponding quasi-relaxation transforms pair is obtained.

Likewise we have the integral transforms pair

$$\Psi_c(\tau) = QX\{\phi(t)\} = \int_0^\infty \phi(t)e^{-\frac{t}{\tau}}\,dt$$

$$\phi_c(t) = Q^{-1}X\{\Psi(\tau)\} = \int_{-\infty}^\infty \Psi(\tau)e^{\frac{t}{\tau}}\,d\tau.$$

1. The Physical Problem of the Quasi-relaxation

The experimental technique in the last 40 years, to the characterization of materials with the use of testing machines has stablished an interesting phenomena derived from controlled stress through deformation information on metallic specimens submitted in the conventional machines through several essays, where the specimen is previously loaded up to an initial level of the stress, for that after the motorized system of the machine is disconnected, which is observed a spontaneous fall of stress, though the length of specimen is conserved. This defines a quasi-relaxation state in the proof specimen.

Likewise, the kinetic of the fall of the stress is registered during all the process of the essays [1]. Then a similar experiment must be executed in a programmed specially machine, in which during the essay of automatic manage stays constant the longitude of the specimen, is to say, the condition of the essay in regime of quasi-relaxation can be expressed in the following form

$$l = cte, \tag{1}$$

or well, in terms of constant deformation

$$\frac{d\epsilon}{dt} = 0. \tag{2}$$

The condition (2) defines the meta-stability as a state of constant deformation only in their plastic characteristics in the initial process of dislocations [2,3], where the energy of the nano-crystals accumulates the enough energy to maintain the specimen in a stable range of recovering to the original state, in a very short time interval [2,4].

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Definition 1. *The limit before of the dislocations in the system described by quasi-relaxation (in a plastic work) on* $0 \leq s \leq t$*, comes given by the linear hereditary integrals:*

$$\varepsilon(t) = \int_{-\infty}^{t} \frac{d\sigma(s)}{ds} \psi(t-s)ds, \text{[1]}$$

$$\sigma(t) = \int_{-\infty}^{t} \frac{d\varepsilon(s)}{ds} \phi(t-s)ds,$$

where the functions $\phi(t)$*, and* $\psi(t)$*, are the relaxation and creep functions respectively, and here* $\phi(-\infty) = \psi(-\infty) = 0$*.*

Likewise, is necesary realice a deep study of the trace of deformation tensor in function of the stress tensor corresponding to the plastic deformation and use a functional of energy [5], consigning to our problem of quasi-relaxation phenomena in the following theorem.

Theorem 1. ***(F. Bulnes-Y. Yermishkin)*** *The quantity of accumulated energy* ΔG*, during the quasi-relaxation is determined by the work of plastic deformation during the application of the system machine-specimen and* $\Delta G = \Gamma t$*.*

Proof. Using the Lagrangian action, the conserved energy satisfies

$$G = \Im - L, \tag{3}$$

which is the energy of the mono-crystals under the plastic deformation tensor ε, to along of Γ, establishing the action (on the signals $\varepsilon(t)$ in the material M):

$$\int_M L\left(\varepsilon(t)\right) d\varepsilon(t) = \int_0^{\infty} \left\{ \int_M d\varepsilon_{pT}(t) \right\} \Psi(\tau) e^{t/\tau} d\tau, \tag{4}$$

From lemma 2.2 of [6], we have that the variation of ε_{pT}, is:

$$\dot{\varepsilon}_{pT} = \int_0^{\infty} \varepsilon_p d\varepsilon_p = \int_0^{\infty} d\varepsilon_{pT} = \varepsilon(t) - \varepsilon_0, \tag{5}$$

but its plastic deformation is a reflex of the work that realizes the machine to obtain the meta-stability conditions that defines the quasi-relaxation in the circuit machine-specimen [7-9]. Thus, the form to measure and determine the quasi-relaxation is through the work of plastic deformation during the application of the machine-specimen system. But is got imposing the condition (2) that defines the meta-stability:

$$\dot{\varepsilon}_{pT} = \varepsilon(t) - \varepsilon_0 = 0. \tag{6}$$

Then (4) can be written as:

$$\int_0^{\infty} \{\varepsilon(t) - \varepsilon_0\} \Psi(\tau) e^{t/\tau} d\tau = 0, \tag{7}$$

[1] We take the initial condition to the differential equation to the Maxwell model $\sigma(-\infty) = 0$. Also the corresponding to the differential equation to Kevin model, $\varepsilon(-\infty) = 0$.

or equivalently

$$\varepsilon_0\phi(t)=\varepsilon_0\int_0^\infty \Psi(\tau)e^{t/\tau}d\tau=\int_0^\infty \varepsilon(t)\Psi(\tau)e^{t/\tau}d\tau=\sigma(t), \tag{8}$$

but this is the functional of stress-strain along the time t,

$$\int_0^\infty \{\dot{\varepsilon}_{pT}\}\Psi(\tau)e^{t/\tau}d\tau=\int_0^\infty\left\{\int_0^t \sigma(t)d\varepsilon(t)\right\}\Psi(\tau)e^{t/\tau}d\tau=\Gamma\left(\sigma-\varepsilon,t\right). \tag{9}$$

For other way,

$$\int_0^t \sigma(t)d\varepsilon(t)\approx\widehat{\Delta G},$$

where the stress and deformation have done the same work along the time. Then

$$\Gamma\left(\sigma-\epsilon,t\right)=\widehat{\Delta G}\int_0^\infty \Psi(\tau)e^{t/\tau}d\tau=\widehat{\Delta G}\phi_c(t).$$

However $\widehat{\Delta G}$, represents the work due the quasi-relaxation in a time t, only. We want the net work due to the plastic deformation during the application of the system machine-specimen. Then we have considering (9) that:

$$\Gamma\left(\sigma-\epsilon,t\right)=\widehat{\Delta G}\phi_c(t)=\frac{\Delta G\phi_c(t)}{\phi_c(t)t},$$

where is trivial that $\Delta G=\Gamma\left(\sigma-\epsilon,t\right)t$. □

The integral in the right member from Equation (9), is an integral transform, if the expression between the brackets is a function with analytic properties that is joined with $e^{t/\tau}$, [10,11], which is the kernel of the integral transform and characterize it like a quasi-relaxation transform.

Our functional of energy (9) represents the evaluations of the field of plastic deformation considering the quantity of energy of liberated plastic deformation by the specimen for unit of time, in the generated dislocations in the specimen in the regimens of quasi-relaxation. If we consider the average energy of the longitude unit for line of dislocation, the integral form equation (9), takes the form of the Burgers vector b, and the initial reserve of elastic energy in the specimen [12][2]

$$\Gamma\left(\sigma-\varepsilon,t\right)=\int_0^\infty G_0-\frac{Gb^2\rho(t)}{2}\Psi(\tau)e^{t/\tau}d\tau. \tag{10}$$

But by the meta-stability condition we have that:

$$\varepsilon(t)-\varepsilon_0=G_0-\frac{Gb^2\rho(t)}{2}=0. \tag{11}$$

But $G_0=\frac{\sigma^2}{2E}$, and not consider loss of mass, $\rho(t)=cte$, then (11) takes the form:

$$\frac{\sigma^2}{2E_r}=\frac{Gb^2\rho}{2}, \tag{12}$$

[2]Functional of plastic energy that promote the dislocations.

where E_r, is the value of modulus of normal elastic relaxation, that is to say, the valued energy considering the elasticity of the essay machine, ρ_{dym}, the sensity of the dynamical dislocations, expressed by the right part of the Equation (12). Then we rewrite the formula (12) as:

$$\sigma = b\sqrt{GE_\gamma \rho_{dym}}. \tag{13}$$

For other side, the energy functional $\Gamma(\sigma - \varepsilon, t)$, satisfies in a meta-stability interval that:

$$\Gamma(\sigma - \varepsilon, t) = b\sqrt{GE_\gamma \rho_{dym}} \int\limits_{-\infty}^{\infty} \Phi(\tau) e^{t/\tau} d\tau = b\sqrt{GE_\gamma \rho_{dym}} \phi_c(t).$$

However the dislocation work inside of the energy functional satisfies

$$\widehat{\Delta G}\phi_0(t) = \alpha \Gamma_0,$$

where Γ_0, is the value of the energy functional before of the appearing of the dislocations (here the work realized $\widehat{\Delta G}$, is of dislocations), being $\alpha\Gamma_0 = \Gamma(\sigma - \varepsilon, t)$. Then finally:

$$b\sqrt{GE_\gamma \rho_{dym}} \phi_c(t) = \alpha \int\limits_{-\infty}^{\infty} \Psi(\tau) e^{t/\tau} d\tau.$$

We can consider withouth loss generality that E_r, and G, are equal to 1. Then we have:

$$\phi_c(t) = \frac{\alpha}{b\sqrt{\rho_{dym}}} \int\limits_{-\infty}^{\infty} \Psi(\tau) e^{t/\tau} d\tau,$$

that in the successive will be the inverse quasi-relaxation transform wich will be detailed in the next section.
Likewise the quasi-relaxation transform obtain the spectra of meta-stability states at the time t, of a material submitted to a stress regime in an interval very short of time. However must consider all strain/stress history (hereditary integral) during the experiment.
To what quasi-relaxation transform serves?
This serves to refer the viscous-elasticity state that a material has in the time of relaxation field proper of the material.
Likewise the kernel of the quasi-relaxation is $K(t, \tau) = e^{-t/\tau}$. That is to say, all quasi-relaxation phenomena consider functions of type $e^{\gamma t}$, even experimentally. This we can demonstrate it in the following lemma.

Lemma 1. *Whole quasi-relaxation experiment has function of physical system dynamic of Laplace type.*

Proof. We consider the quasi-relaxation integral

$$\phi_c(t) = \int_{-\infty}^{\infty} \sigma(\tau) e^{-\gamma |t/\tau|} d\tau, \tag{14}$$

which is propitious due to that the energy or work of plastic deformation during the application of the system machine-specimen satisfies $\forall t$, given by the theorem 1.1. We observe that inside the integral we can to define a special Laplace function[3]:

$$\sigma(\tau) = \frac{\gamma}{2} e^{-\gamma|t/\tau|}, \qquad \gamma > 0, \tag{15}$$

where along of the real the line $-\infty \leq t \leq \infty$, is described whole quasi-relaxation experiment. Of fact, we can describe it. In the first step of the experiment is applied a load which goes increasing accords to $e^{t/\tau}$, in the interval $(-\infty, 0]$. After stop the loading in a very short interval around of $(-\xi, \xi)$, to very small ξ $(\xi \to 0)$. Finally the quasi-relaxation phenomena is determined from $t = 0$, untill $t = \infty$, accords to $e^{-t/\tau}$.

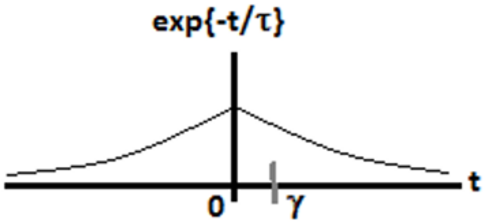

Figure 1. Curve of whole development of quasi-relaxation experiment.

For other side, the experimental quasi-relaxation curves comes given for (see the figure 2).

$$\sigma(t) = \frac{\sigma_0 - \sigma_1}{1 + K(\sigma_0 - \sigma_1)} + \sigma_1 e^{-\gamma(t/\tau)}, \tag{16}$$

where the hyperbolic term can be justified through the Laplace transform when $\tau = 1$, (functional of stress-deformation to along the time t),

$$\mathscr{L}\{e^{-t}\} = \int_0^\infty e^{-K(\sigma_0 - \sigma_1)t/\tau} e^{-t} dt = \frac{1}{1 + K(\sigma_0 - \sigma_1)}, \tag{17}$$

and by other side, using $e^{-\tau t}$, $\tau > 1$ (that is to say, for example $\tau = \sigma_1$), we have:

$$\mathscr{L}\{e^{-\tau}\} = \int_0^\infty e^{-\tau} e^{-t/\sigma_1} dt = \sigma_1 e^{-\tau}, \tag{18}$$

[3]This special function is defined as:

$$e^{-|t/\tau|} = \begin{cases} e^{t/\tau}, & t < 0, \\ e^{-t/\tau}, & t \geq 0, \end{cases}, \ \tau > 1,$$

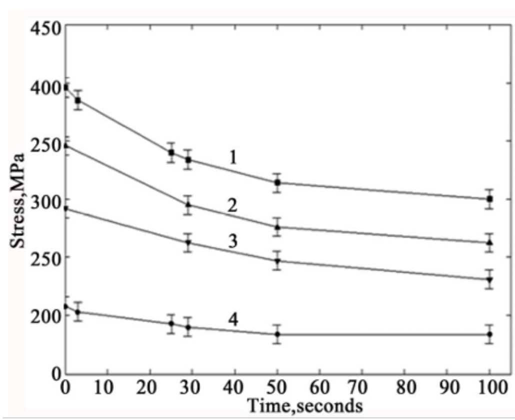

Figure 2. Experimental quasi-relaxation curves [12]

Then whole quasi-relaxation experiment has function of physical system dynamics of Laplace type. □

Lemma 2. *The nucleus $K(t,\tau)$, defined to the operator $I_{t\tau}$, whose integral is*

$$I_{t\tau} = \int_{specimen} \xi(\tau)K(t,\tau)d\tau, \tag{19}$$

verifies

$$I_{t\tau} = \int_{specimen} |K(t,\tau)|d\sigma(\tau) \leq C_q\|\Omega\| \leq 1. \tag{20}$$

Proof. We consider the all history of the quasi-relaxation process, and using inequalities properties, by (5) and considering that exist all quasi-relaxation for all $\tau > 1$, then we have:

$$\begin{aligned}\int_{\text{specimen}} |K(t,\tau)|d\sigma(\tau) &= \int_{\text{specimen}} |e^{-\gamma t}e^{1/\tau}|d\sigma = \int_{\text{specimen}} |e^{-\gamma t}||e^{1/\tau}|d\sigma \\ &\leq |e^{-\gamma t}||e^{1/\tau}| \int_{\text{specimen}} V d\epsilon_{pT} \leq C_q\|\Omega\| \leq 1,\end{aligned}$$

where $\Omega = \int_{\text{specimen}} V d\epsilon_{pT}$, $|e^{-\gamma t}||e^{1/\tau_q}| \leq C_q$, $\gamma > 0$. □

2. Quasi-Relaxation Transforms

We consider the following definitions on the base of the quasi-relaxation experiments.

Postulate 1 The quasi-relaxation transform realizes a pause on the creep function when realizes a load on the specimen, and the inverse quasi-relaxation transform interrupts[4] the relaxation when realizes a load on the specimen.
In all metals, when realizes a load on the specimen the inverse quasi-relaxation transform interrupts the relaxation on the metals [13]. Then the relaxation will have one or more poles or singularities.

[4]Deforms the specimen untill the grade of provokes some rips or pre-dislocations.

The duality defines the direct and inverse transformations of the creep and relaxation functions.

Lemma 3. ***(Duality of the Relaxation and Creep functions)*** *Let $\phi(t)$, and $\Psi(\tau)$, be the relaxation and creep functions defined on the interval $(-\infty, s]$. The functions are dual.*

Proof. If these functions are dual then must comply the following relation:

$$\phi_c(t) = \frac{1}{\sqrt{\Lambda}} T\left(\Psi(\tau)\right), \tag{21}$$

that is to say, the functional transformation of the creep function is the relaxation function. The functional transformation of the creep function on domain τ, is the relaxation function of domain t. Indeed, we consider as functional transformation T, the given for the quasi-relaxation process:

$$T : \hat{M}\left(., \Psi\right) \otimes D\left(., \dot{\sigma}\right) \to D\left(., \dot{\varepsilon}\right)^{5} \tag{22}$$

with the correspondence rule (considering the convolution product $\circ$ between spaces), where T, is the usual convolution

$$T : \frac{d\sigma(s)}{ds} \Psi(t-s) \mapsto \int_{-\infty}^{t} \dot{\sigma}(s)\Psi(t-s)ds. \tag{23}$$

Then in particular by (8) we have:

$$\varepsilon_0 \phi(t) = \varepsilon_0 \int_{0}^{\infty} \Psi(\tau) e^{-t/\tau} d\tau, \tag{24}$$

where clearly, we have $\phi_c = T\left(\Psi(\tau)\right)$. Now in the inverse way, we have that exists T^{-1}, such that

$$\Psi_c(\tau) = \sqrt{\Lambda} T^{-1}\left(\phi(t)\right). \tag{25}$$

Then the composition of (21) with (25) is the identity. □

Lemma 4. ***(F. Bulnes)***
We can consider in an indistinct order the relaxation and creep functions and their transforms (also the respective transforms in the theorem 4.1 [6] are fulfilled).

Proof. We consider the self-dual relation between functions $\phi(t)$, and $\Psi(\tau)$, and their inverse Laplace transforms on all real straight line. Then

$$\begin{aligned} \mathcal{L}^{-1}\left[\int_{-\infty}^{\infty} \phi(t)\Psi(\tau) - \Phi(\tau)\psi(t)dt\right] &= \int_{-\infty}^{\infty} \{\phi(t)\mathcal{L}^{-1}\{\Psi(\tau)\} - \mathcal{L}^{-1}\{\Phi(\tau)\}\psi(t)\}d\tau \\ &= \int_{.\infty}^{\infty} \{\phi(t)\psi(t) - \phi(t)\psi(t)\}dt = 0. \end{aligned} \tag{26}$$

Then

$$\int_{-\infty}^{\infty} \phi(t)\Psi(\tau) - \Phi(\tau)\psi(t)dt = 0, \tag{27}$$

If only yes

$$\{\phi(t)\Psi(\tau) - \Phi(\tau)\psi(t)\} = 0, \tag{28}$$

[5]The tensor product in their topological vector spaces is restricted to a convolution product.

which happens. Then we consider in an indistinct order the quasi-relaxation transforms pair. □

Who is $\sqrt{\Lambda}$? This is a constant discussed in the introduction I. 1.
We consider real constants $M > 0$, and γ, such that $\forall\ t > N$, we have

$$|e^{-t/\gamma}\phi(t)| < M, \tag{29}$$

or equivalently,

$$|\phi(t)| < Me^{t/\gamma}. \tag{30}$$

In this case the relaxation function $\phi(t)$, is a function of exponential order $1/\gamma$, when $t \to \infty$.

Theorem 2. *Let $\phi(t)$, piecewise continuos in each finite interval $[0, N]$, of exponential order $1/\gamma$, for all $t > N$. Then exists the quasi-relaxation transform $\Psi_c(\tau)$, for all $\tau > 1/\gamma$, $\gamma \neq 0$.*

Proof. For any positive integer N, we have

$$\int_0^{-\infty} \phi(t)e^{-t/\tau}dt = \int_0^N \phi(t)e^{-t/\tau}dt + \int_N^{\infty} \phi(t)e^{-t/\tau}dt. \tag{31}$$

Due to that the relaxation function $\phi(t)$, is piecewise constinuos in each finite interval $[0, N]$, the first integral of the right side of (31) exists, since that $\phi(t)$, is the exponential order $1/\gamma$, for $t > N$. Indeed we consider the length:

$$\left|\int_N^{\infty} \phi(t)e^{-t/\tau}dt\right| \leq \int_N^{\infty} |\phi(t)e^{-t/\tau}|dt \leq \int_0^{\infty} e^{-t/\tau}|\phi(t)|dt \leq \int_0^{\infty} e^{-t/\tau}Me^{t/\gamma}dt \leq \frac{M}{\left(\gamma - \frac{1}{\tau}\right)}. \tag{32}$$

Likewise, the quasi-relaxation transform exists for $\tau > 1/\gamma$, $\gamma \neq 0$. □

Definition 2. *Let $\phi(t)$, be an integrable function in $t \geq 0$, with $\tau \neq 0$. Then the integral transform $\Psi_c(\tau)$, of $\phi(t)$, is defined by*

$$\Psi_c(\tau) = QX\{\phi(t)\} = \int_0^{\infty} \phi(t)e^{-\frac{t}{\tau}}dt, \tag{33}$$

which exists in the space $D_{\Psi_c(\tau)} = \{\tau | \tau > 1/\gamma,\ \gamma \neq 0\}$. Without of this space the integral transform $\Psi_c(\tau)$, don´t exist.

In (33) we have that the quasi-relaxation transform is the transformation of the hereditary integral of its stress history $\varepsilon(t)$, due to the relaxation $\phi(t)$, along $(0, \infty)$. In resume we can to say that the quasi-relaxation transform is the relaxation in the meta-stable state.

Example 1. *Determine the quasi-relaxation transform considering the load $U(t - t')$. We consider the definition of inverse transform and the corresponding hereditary integral* *to obtain*

$$\Psi_c(\tau)=\int_{-\infty}^{\infty}\phi(t)e^{-t/\tau}dt=\int_{-\infty}^{\infty}\varepsilon(t)e^{-t/\tau}dt=\int_{-\infty}^{\infty}\int_{-\infty}^{t}\frac{d\sigma(t')}{dt'}\Psi(t-t')dt'e^{-t/\tau}d\tau=$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{t}\sigma_0\delta(t')\Psi(t-t')dt'e^{t/\tau}d\tau=\sigma_0\int_{-\infty}^{\infty}\Psi(t)U(t-t')e^{-t/\tau}d\tau, \tag{34}$$

where has been considered from the functional analysis, the property

$$\int_{-\infty}^{t}\Psi(s)\delta(s-t)ds=\Psi(t)U(t-s) \tag{35}$$

and also considering the property,

$$\int_{-\infty}^{t}\Psi(t')\delta(t'-t_0)dt'=U(t-t_0)\int_{t_0}^{t}\Psi(t')dt'. \tag{36}$$

Then the extreme right of (34) takes the form

$$QX\{\phi(t)\}=\sigma_0U(t-t_0)\int_{-\infty}^{\infty}\Psi(t)e^{-t/\tau}d\tau=\sigma_0U(t-t')\phi(\tau). \tag{37}$$

If we consider $t'=0$, the direct quasi-relaxation transform reduces to the load case $U(t)$.
By the lemma 2.1, and the unicity of the quasi-relaxation transform we can enunciate the inverse quasi-relaxation transform as the recovering of the quasi-relaxation function through the creep function:

$$\phi_c(t)=\frac{\alpha}{b\sqrt{\rho_{dym}}}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT}\Psi(\tau)e^{t/\tau}d\tau. \tag{38}$$

The before integral relation can prove it that is a Bromwich equality. The criteria of complex contour are satisfied and we can observe from the material science the following fact:

$$\alpha\approx\frac{1}{2\pi}=0.1587101,$$

and we consider a hypothetical value (only as functional coefficient of contour) of the density $\rho_{dym}=-1$.
and realize the contour construct considering in this little neighborhood that (without loss generality) the Burger's coefficient $b=1$.[6]
The evaluation of the inverse transform is not easy, and involves non-elemental functions[7].

[6]The Burger´s coefficient is given by the empirical formula $b=\frac{R_D}{L}$, where L, is the horizontal length scale and R_D, is the Rossby deformation radius.
[7]$Ei(t)$, is the exponential integral

$$Ei(t)=-\int_{t}^{\infty}\frac{e^{-t}}{t}dt,$$

In some cases could be evaluated only by numerical methods. Likewise the integral of its inverse transform involves the term $e^{t/\tau}$, which is its nucleus whose integration is:

$$\int_0^{\infty} e^{t/\tau} d\tau = \left\{ e^{-t/\tau}\tau \Big|_0^{\infty} - Ei\left(\frac{t}{\tau}\right) \right\}. \tag{39}$$

Example 2. *Determine the inverse quasi-relaxation transform of the creep function considered in the example 2.1.*
We consider the definition of inverse transform and the conrresponding hereditary integral to obtain

$$\begin{aligned} \phi_c(t) &= \int_{-\infty}^{\infty} \Psi(\tau)e^{t/\tau}d\tau = \int_{-\infty}^{\infty} \sigma(\tau)e^{t/\tau}d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{t} \frac{d\varepsilon(t')}{dt'}\phi(t-t')dt'e^{t/\tau}d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{t} \varepsilon_0\delta(t)\phi(t-t')dt'e^{t/\tau}d\tau = \varepsilon_0 \int_{-\infty}^{\infty} \phi(t')U(t-t')e^{t/\tau}d\tau, \end{aligned} \tag{40}$$

where has been considered the functional analysis property

$$\int_{-\infty}^{t} \phi(\tau)\delta(\tau - t)d\tau = \phi(t)U(t-\tau). \tag{41}$$

Then considering the following property to

$$\int_{-\infty}^{t} \phi(t')\delta(t'-t_0)dt' = U(t-t_0)\int_{t_0}^{t} \phi(t')dt'. \tag{42}$$

Then the extreme right of (40) takes the form

$$\begin{aligned} \varepsilon_0 \int_{-\infty}^{\infty} \phi(t')U(t-t')e^{t/\tau}d\tau &= \varepsilon_0 U(t-t')\int_{-\infty}^{\infty} \phi(t')e^{t/\tau}d\tau \\ &= \varepsilon_0 U(t-t')\phi(t')\int_{-t_0}^{t} e^{t/\tau}d\tau = \varepsilon_0\phi(t)\int_{-t_0}^{t} e^{t/\tau}d\tau, \end{aligned} \tag{43}$$

but

$$\int_{t_0}^{t} e^{t/\tau}d\tau = e^{t/\tau}\tau - Ei\left(\frac{t}{\tau}\right)\Big|_{t_0}^{t} = et - Ei(1) - \left[e^{t/t_0}t_0 - Ei\left(\frac{t}{t_0}\right)\right]. \tag{44}$$

Finally, the inverse quasi-relaxation transform is[8]:

$$\phi_c(t) = \varepsilon_0\phi(t)\left\{ et - Ei(1) - e^{t/t_0}t_0 - Ei\left(\frac{t}{t_0}\right) \right\}.^9 \tag{45}$$

[8]We can verify

$$\Psi_c(\tau) = \sqrt{\Lambda}T^{-1}\left(\frac{1}{\sqrt{\Lambda}}T(\Psi_c(\tau))\right) = \frac{1}{\sqrt{\Lambda}}\sqrt{\Lambda}QX^{-1}\left\{\varepsilon_0\phi(t)\left\{et - Ei(1) - e^{t/t_0}t_0 - Ei\left(\frac{t}{t_0}\right)\right\}\right\}$$

Example 3. *Determine the quasi-relaxation transform considering the load $\delta(t)$. We consider our integral transform*

$$\Psi_c(\tau) = \int_{-\infty}^{\infty} \phi(t) e^{-t/\tau} dt = \int_{-\infty}^{\infty} \varepsilon(t) e^{-t/\tau} dt = \int_{-\infty}^{\infty} \int_{-\infty}^{t} \frac{d\sigma(t')}{dt'} \Psi(t-t') dt' e^{-t/\tau} d\tau = \\ = \sigma_0 \int_{-\infty}^{\infty} \int_{-\infty}^{t} \frac{d\delta(t')}{dt'} \Psi(t-t') dt' e^{-t/\tau} d\tau. \tag{46}$$

Considering the integral by parts $\int_{-\infty}^{t} \frac{d\delta(t')}{dt'} \Psi(t-t') dt'$, *choosing the substitutions:*

$$dv = \delta'(t')dt', \qquad u = \Psi(t-t'), \qquad v = \delta(t'), \qquad du = \Psi'(t-t')dt', \tag{47}$$

we have

$$\int_{-\infty}^{t} \frac{d\delta(t')}{dt'} \Psi(t-t') dt' = \delta(t')\Psi(t-t') \Big|_{-\infty}^{t} - \int_{-\infty}^{t} \delta(t')\Psi'(t-t')dt'. \tag{48}$$

In the last integral of the right side from (48) we consider the convolution property to Dirac's impulse function:

$$\int_{-\infty}^{t} \delta(t')\Psi(t-t')dt' = \int_{-\infty}^{t} \delta(t-t')\Psi(t')dt' = \Psi(t). \tag{49}$$

Further the function is symmetric, to know; $\delta(t-t') = \delta(t'-t)$. *Then*

$$\int_{-\infty}^{t} \frac{d\delta(t')}{dt'} \Psi(t-t') dt' = \delta(t)\Psi(0) - \Psi(t), \tag{50}$$

where finally and due to the duality between the creep and relaxation functions, we have:

$$\Psi_c(\tau) = \sigma_0 \int_{-\infty}^{\infty} \{\delta(t)\Psi(0) - \Psi(t)\} e^{-t/\tau} d\tau = \sigma_0 \left(e^{-1}\Psi(0) - \phi(\tau) \right). \tag{51}$$

□

Now its inverse quasi-relaxation transformation will be

$$\phi_c(t) = \int_{-\infty}^{\infty} \Psi(\tau) e^{t/\tau} d\tau = \int_{-\infty}^{\infty} \sigma(\tau) e^{t/\tau} d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{t} \frac{d\varepsilon(t')}{dt'} \phi(t-t') dt' e^{t/\tau} d\tau = \\ = \int_{-\infty}^{\infty} \int_{-\infty}^{t} \varepsilon_0 \frac{d\delta(t')}{dt'} \phi(t-t') dt' e^{t/\tau} d\tau = \varepsilon_0 \int_{-\infty}^{\infty} \phi(t') U(t-t') e^{t/\tau} d\tau, \tag{52}$$

9

$$Ei(1) = 1.89511781\ldots$$

where by (48) we have

$$\int_{-\infty}^{t} \frac{d\delta(t')}{dt'}\phi(t-t')dt' = \delta(t)\phi(0) - \delta(-\infty)\phi(t+\infty) - \int_{-\infty}^{t} \delta(t')\phi(t-t')dt'. \tag{53}$$

In the last integral we consider the convolution property to Dirac's impulse function:

$$\int_{-\infty}^{t} \delta(t')\phi(t-t')dt' = \int_{-\infty}^{t} \delta(t-t')\phi'(t')dt' = \phi'(t). \tag{54}$$

From (52) we have:

$$\begin{aligned} \phi_c(t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{t} \varepsilon_0 \frac{d\delta(t')}{dt'}\phi(t-t')dt' e^{t/\tau} d\tau \\ &= \varepsilon_0 \int_{-\infty}^{\infty} \left\{ \delta(t)\phi(0) - \delta(-\infty)\phi(t+\infty) - \int_{-\infty}^{t} \phi'(t)dt \right\} e^{t/\tau} d\tau, \end{aligned} \tag{55}$$

where

$$\phi_c(t) = \varepsilon_0\phi(0)\delta(t)\left\{ \int_{-\infty}^{t} e^{t/\tau}d\tau - \int_{-\infty}^{t} \phi(t)e^{t/\tau}d\tau \right\} = \phi(0)\delta(t)\left\{ e\tau - Ei\left(\frac{t}{\tau}\right)\Bigg|_{-\infty}^{t} - \phi(t) \right\}. \tag{56}$$

Finally, the inverse quasi-relaxation transform is:

$$\phi_c(t) = \varepsilon_0\phi(0)\left\{ et - Ei\left(\frac{t}{t_0}\right) - \phi(t) \right\}. \tag{57}$$

□

3. Some properties with derivatives and integrals and integral transforms.

The quasi-relaxation transform has the following derivatives properties:

i) $QX\{\phi'(t)\} = \tau\Psi(\tau) - \phi(0)$
ii) $QX\{\phi''(t)\} = \tau^2\Psi(\tau) - \tau\phi(0) - \phi'(0)$
iii) $QX\{\phi^{(n)}(t)\} = \tau^n\Psi(\tau) - \sum\limits_{k=0}^{n-1} \tau^{1-n+k}\phi^{(k)}(0)$.

We demonstrate the properties.

Proof. i) We have

$$QX\{\phi'(t)\} = \int_{0}^{\infty} \phi'(t)e^{-t/\tau}dt.$$

Integrating by parts we have:

$$\int_{0}^{\infty} \phi'(t)e^{-t/\tau}dt = e^{-t/\tau}\phi(t)\Big|_{0}^{\infty} + \tau\Psi(\tau), \tag{58}$$

where finally

$$QX\{\phi'(t)\} = \tau\Psi(\tau) - \phi(0).$$

ii) Let $\lambda(t) = \phi''(t)$, then

$$QX\{\lambda'(t)\} = \tau QX\{\lambda(t)\} - \lambda(0). \tag{59}$$

Then

$$QX\{\phi''(t)\} = \tau\left[\zeta\Psi(\tau) - \phi(0)\right] - \lambda(0) = \tau^2\Psi(\tau) - \tau\phi(0) - \lambda(0).$$

iii) The proof of (iii) can be proved by induction on n.

iv) $QX\left\{\int\limits_0^t \phi(u)du\right\} = \tau\Psi(\tau)$.

Indeed, we have integrating by parts that

$$QX\left\{\int\limits_0^t \phi(u)du\right\} = \int\limits_0^\infty e^{-t/\tau}\left[\int\limits_0^t \phi(u)du\right]dt = \int\limits_0^t \phi(u)du\left(-\tau e^{-t/\tau}\right)\Big|_0^\infty + \tau\int\limits_0^\infty \phi(t)e^{-t/\tau}dt = \tau\Psi(\tau). \tag{60}$$

□

Some relations with other integral transforms are the following:

(a). $\tau QX\{\phi(t)\} = E\{\phi(t)\}$, where $E\{\phi(t)\}$ is the Elzaki transform.

(b). If $f(t) = \phi(t)$, (i. e, if the function is a relaxation function) and $p = -1/\tau$, then the Laplace transform is a quasi-relaxation transform.

(c). Let be

$$C\{\phi(t)\} = p\int\limits_0^\infty \phi(t)e^{-pt}dt, \tag{61}$$

the Carson transform of the relaxation function $\phi(t)$. We consider that the Carson transform has the following relation with the Laplace transform:

$$C\{\phi(t)\} = p\Phi(p) = p\mathcal{L}\{\phi(t)\}. \tag{62}$$

If we elect $p = \frac{1}{\tau}$, then

$$C\{\phi(t)\} = \frac{1}{\tau}\int\limits_0^\infty \phi(t)e^{-t/\tau}dt = \frac{1}{\tau}\Psi(\tau) = \frac{1}{\tau}\mathcal{L}\{\phi(t)\}. \tag{63}$$

Now we consider a differential equations problem . We consider the differential equation in the Maxwell model in the viscoelasticity problem to a relaxation function on the interval $[0, \tilde{t}]$. Then we consider

$$\frac{d\sigma}{dt} + \frac{E}{\eta}\sigma = E\frac{d\varepsilon}{dt}. \tag{64}$$

Firstly, we observe that the equation is a first order differential equation which can be solved using the standard integrating factor. The initial condition from the hereditary integrals to $\sigma(t)$, can be taken as $\sigma(-\infty) = \sigma(0) = 0$, where we consider $\sigma(0) = 0$, to apply the direct quasi-relaxation transform. Solving by the integrating factor $e^{Et/\tau}$, we have:

$$\frac{d}{dt}\left(e^{Et/\tau}\phi\right) = Ee^{Et/\eta}\frac{d\varepsilon(t)}{dt}, \tag{65}$$

where integrating both sides on $[0,\tau]$, we have

$$\left(e^{Et/\tau}\sigma\right)\Bigg|_{\tilde{t}} - \left(e^{Et/\tau}\sigma\right)\Bigg|_{-\infty} = Ee^{Et/\eta}\frac{d\varepsilon(t)}{dt}. \tag{66}$$

Finally

$$\sigma(t) = \int_{-\infty}^{\tau} Ee^{Et/\tau}\frac{d\varepsilon(t)}{dt}dt = \int_{-\infty}^{\tau} E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau, \tag{67}$$

where $E(t)$, the relaxation modulus for the Maxwell model, is

$$E(t) = Ee^{-Et/\eta}, \tag{68}$$

which is the solution to differential equation on said interval. Now we use our quasi-relaxations transforms considering $\tau = \eta/E$, and remembering by hereditary integrals, with $\sigma(-\infty) = 0$, that

$$\varepsilon(\tau) = \phi_c(\tau) = \int_0^{\infty} \Psi(t)e^{-Et/\eta}dt = \int_0^{\infty} \sigma(t)e^{-Et/\eta}dt. \tag{69}$$

Then by the derivatives properties

$$QX\Big\{\frac{d\sigma}{dt}\Big\} + \frac{E}{\eta}QX\{\sigma(t)\} = QX\Big\{E\frac{d\varepsilon}{dt}\Big\}, \tag{70}$$

or equivalently

$$\varepsilon(\tau) = \frac{\eta}{\eta\tau - \eta + E}\int_0^{\infty} E\frac{d\varepsilon}{dt}e^{Et/\tau}dt. \tag{71}$$

Now we apply the respective inverse transform

$$\begin{aligned}\phi_c(t) = QX^{-1}\{\varepsilon(\tau)\} &= \int_{-\infty}^{\tau}\left\{\frac{\eta}{\tau\eta - \eta + E}\int_0^{\infty} E\frac{d\varepsilon}{dt}e^{-Et/\eta}dt\right\}e^{Et/\eta}d\tau^{10} \\ &= \int_0^{\tau} \varepsilon(\tau)e^{Et/\eta}d\tau. \end{aligned}\tag{72}$$

Then the last integral (72) takes the form finally:

$$\phi_c(t) = \int_{-\infty}^{\tau} E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau, \tag{73}$$

[10] We demonstrate the following identity of the terms in the integrating of the integrals (72) $E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau} = \varepsilon(\tau)e^{Et/\eta}$. We start with the implication $\Rightarrow$)

$$\frac{d}{d\tau}\{\varepsilon(\tau)e^{Et/\eta}\} = e^{Et/\eta}\frac{d\varepsilon(\tau)}{d\tau} = e^{E(\tilde{t}-\tau)/\eta}\frac{d\varepsilon(\tau)}{d\tau} = e^{-E(\tau-\tilde{t})/\eta}\frac{d\varepsilon(\tau)}{d\tau},$$
$$t = \tilde{t} - \tau \qquad\qquad e^{E(\tilde{t}-\tau)/\eta} = e^{-E(\tau-\tilde{t})/\eta}$$

If we consider $E_0 = e^{\tilde{t}}$, then $e^{-E(\tau-\tilde{t})/\eta} = e^{\tilde{t}}e^{-E(t-\tau)/\eta}$. Then

$$e^{-E(\tau-\tilde{t})/\eta}\frac{d\varepsilon(\tau)}{d\tau} = E_0e^{-E(t-\tau)/\eta} = E(t-\tau).$$

The solution es the same solution obtained by classic methods to a first order differential equation.

4. Short Table of Quasi-relaxation transforms

Function of Load or Strain	Direct Quasi-relaxation Transform	Inverse Quasi-relaxation Transform
$U(t)$	$\Psi_c(\tau) = \sigma_0\phi(\tau)$	$\phi_c(t) = \phi(t')\tau e^{t/\tau}$
$U(t-t')$	$\Psi_c(\tau) = \sigma_0 U(t-t')\phi(\tau)$	$\phi_c(t) = \varepsilon_0\phi(t)\left\{ et - Ei\,(1) - e^{t/t_0}t_0 - Ei\left(\frac{t}{t_0}\right)\right\}$
$\delta(t)$	$\Psi_c(\tau) = \sigma_0\left(e^{-1}\Psi(0) - \phi(\tau)\right)$	$\phi_c(t) = \varepsilon_0\phi(0)\left\{ et - Ei\left(\frac{t}{t_0}\right) - \phi(t)\right\}$
$\phi(\tau)$	$\Psi_c(t)$	$\phi_c(\tau)$
$\Psi(t)$	$\phi_c(\tau)$	$\Psi_c(t)$

NOTES. Has been used the properties of convolution with delta function inside hereditary integrals:

a. $\int\limits_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = \int\limits_{-\infty}^{\infty} f(t-\tau)\delta(t-\tau)d\tau = f(t)$.

b. $\varepsilon(t) = \sigma(0)J(t) + \hat{\sigma}J(t-T)$.

c. We use $\sigma(t) = \hat{\sigma}U(t-a)$ in $\varepsilon(t) = \sigma(0)J(t) + \int\limits_0^t J(t-\tau)\dfrac{d\sigma(\tau)}{d\tau}d\tau$.

5. Applications and Experimental Proof

We consider the following creep function extracted from the Table 1:

$$\Psi_c(\tau) = \sigma_0 U(t-t_0)\phi(\tau),$$

where we consider

$$\phi(\tau) = Ee^{-E\tau/\eta},$$

being E, the Young's modulus and η, is a viscocity of the material. We consider the following values, $E = 1$, and $\eta = 2$, in the material submitted under the load of $\sigma_0 = 5\ kg/mm^2$. Then we have in particular the quasi-relaxation transform:

$$\Psi_c(\tau) = 5U(t-5)e^{-\tau/2},$$

whose curve is given in the figure 3.
In a tension von Mises Modulus model, the spectra given in the figure 3, can viewed as the figure 4.

Now, we consider the following creep spectra or inverse quasi-relaxation transform:

$$\phi_c(t) = 6e^{-3\tau/2}\left(e\tau - 1.8951178 - e^{\tau/5}\right),$$

where we have considered previously the function extracted from the Table 1, $\phi_c(t) = \varepsilon_0\phi(t)\left\{ et - Ei(1) - e^{t/t_0}t_0 - Ei\left(\frac{t}{t_0}\right)\right\}$. We have took the value $Ei(1)$, expressed in the footnote 8, and the logarithmic expression approximation of the exponential integral given in the footnote 6 to $Ei\left(\frac{t}{t_0}\right)$. Then we have the following spectra curve $\phi_c(t)$ in the figure 5, which we can observer with the experimental serious studies in material sciences realized in [14, 15], to spectroscopy of polymers, where the goal was obtain the

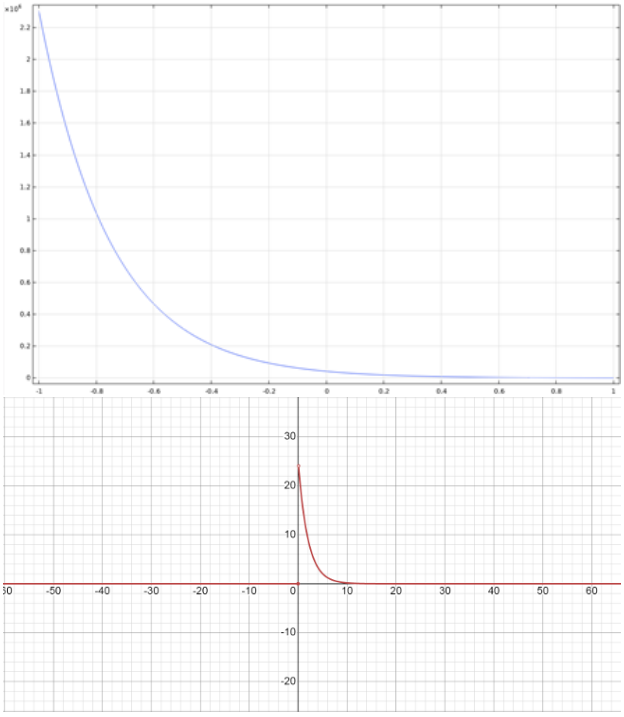

Figure 3. Quasi relaxation transform with the load function $U(t-5)$.

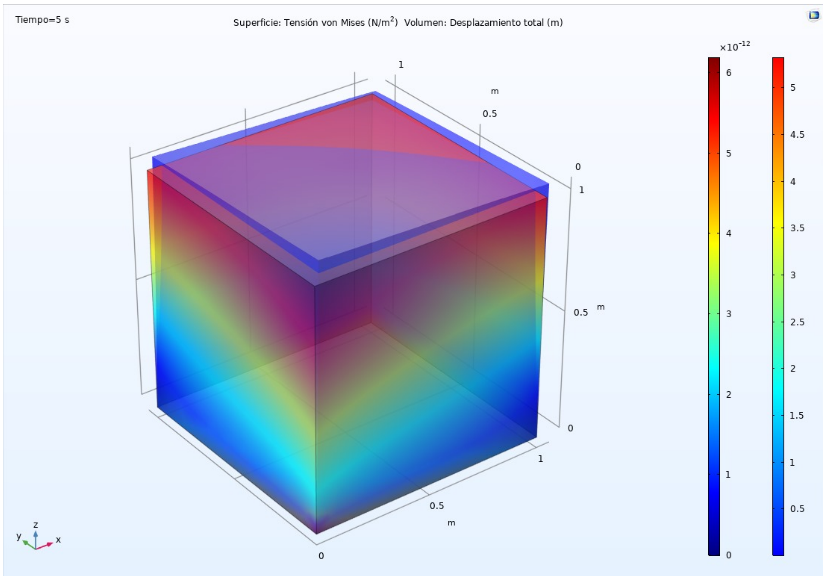

Figure 4. Tension Von Mises modulus model to the spectra given in the figure 3. The displacement is given in millimeters during 5 seconds.

molecular stress distribution function determining the stress relaxation modulus or creep compliances.

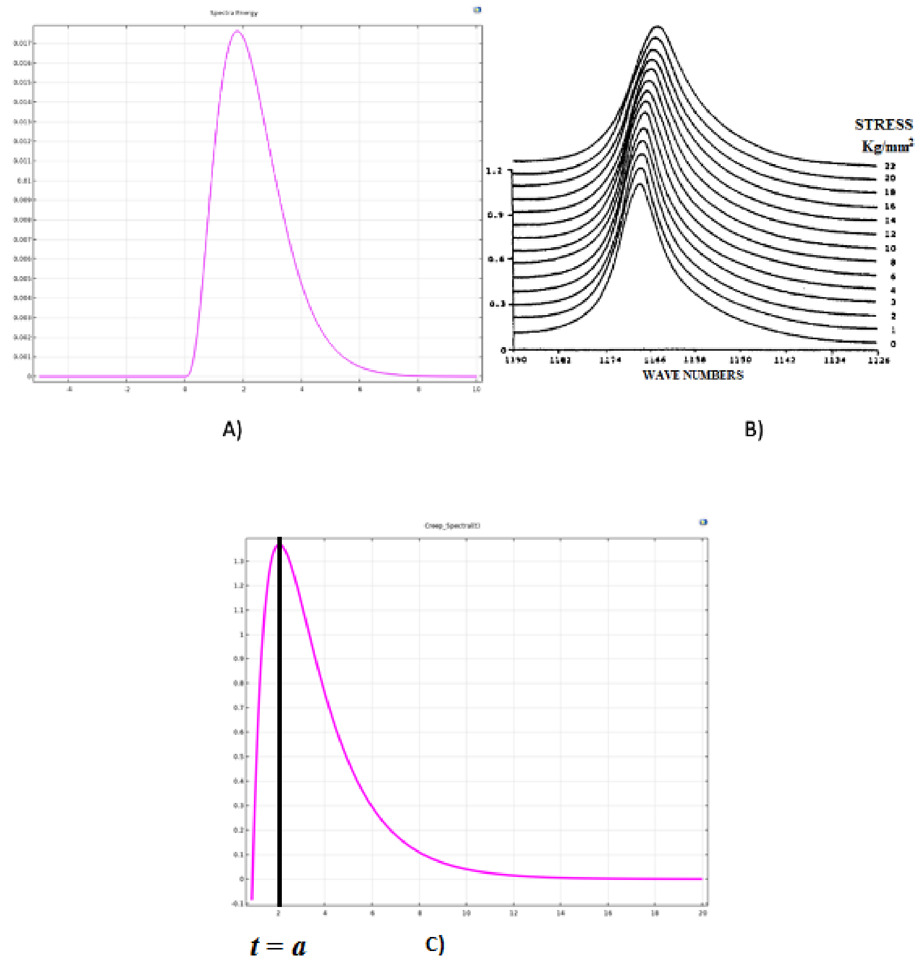

Figure 5. A). Inverse quasi-relaxation function or spectra curve $\phi_c(t)$. B). Molecular stress distribution function determining the stress relaxation modulus or creep compliances [14]. C). Another curve of the inverse quasi-relaxation function with a load $\delta(t-a)$.

6. Conclusions

The use of the hereditary integrals is fundamental in the construction of these transforms pair, since in the process to obtain the corresponding quasirelaxation transforms, both direct and inverse, the inheritance of the relaxation and creep functions respectively remains and contribute to the characterization of the corresponding quasi-relaxation transforms, considering also the energy contributions of the stress and strain tensors which are present in the analytical process of the hereditary integrals.

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