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PERIODIC AND SOLITARY WAVE SOLUTIONS FOR A MODIFIED KADOMTSEV PETVIASHVILI MODEL

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Abstract. Using a similar approach as Korteweg and de Vries we obtain periodic solutions expressed in terms of the Jacobi elliptic function cn for a modified Kadomtsev–Petviashvili (MKP) equation, and we will call them internal cnoidal waves. As well, we will show that internal solitary wave solutions are recovered through a limiting process after the elliptic modulus of the Jacobi elliptic function cn that describes the cnoidal waves for the MKP equation.

1. Introduction

Internal waves oscillate within a stratified fluid, and they are gravity dominated waves. We will refer to them as internal gravity waves. They exist on the interface between two stratified moving fluids of different densities. When low density fluid overlies high density fluid, the internal gravity waves will propagate horizontally along the interface, just as surface gravity waves, with a speed determined by the difference between the above and below densities at the internal interface. Following [14] ”we ensuing waves are a simple model of gravity waves that exist in the interior of the atmosphere and, perhaps especially, the ocean, in which we idealize the continuous stratification of the real fluid by supposing that the fluid comprises two (or conceivably more) layers of immiscible fluids of different densities stacked on top of each other. We will consider only the hydrostatic case in which case the layers form a ‘stacked shallow water’ system.” We will limit ourselves to two moving layers.

Due to the density change at the internal interface, the internal gravity waves will have a slight deviation in the second space-dimension, i.e., wave-refraction, thus becoming quasi two-dimensional internal gravity waves.

To derive an appropriate model for internal gravity wave propagation at the fluid interface we seek solutions in the form

$$u = e^{i(k_1 x + k_2 y - \omega t)}, 0 < k_2 \ll k_1. \tag{1}$$

satisfying the dispersion relation

$$\omega = k_1^2 - 6 - \lambda \left(\frac{k_2}{k_1}\right)^2, \lambda > 0 \tag{2}$$

Observations on the dispersion relation (2):

- The term k_1^2 shows the dispersive nature of the internal gravity waves.
- The term 6 addresses the nonlinearity nature of the internal gravity waves. The term 6 is chosen to aid in the integration of the resulting partial differential model.
- λ is related to the unit force applied per unit mass of water that refracts in shallow layer water by a small angular velocity, and such, the waves becoming quasi two-dimensional.
  $\lambda > 0$ signifies that we have only internal gravity waves at the internal interface (we have immiscible fluids), i.e., there is no surface tension at the internal interface as in the

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case of air-water interface. As well λ signifies that the unit for the force applied per unit mass in unit time is 1 Newton. Therefore we will use $\lambda = 1 \left[\frac{\mathrm{N}}{\mathrm{kg}} \cdot \mathrm{s}\right]$, and we will work in the International System of Units (SI Units), i.e., the units of length-weight-time used will be measured in meter-kilogram-second.

- The term $\left(\frac{k_2}{k_1}\right)^2 \ll 0$ shows the quasi-two-dimensionality of the internal gravity waves.

The dispersion relation (2) suggests that u satisfies the partial differential equation (PDE)

$$(iu_{xt} + 6|u|^m u_x + u_{xxx})_x + \lambda u_{yy} = 0, m \in \mathbb{R} \tag{3}$$

where u is a complex-valued function.

In (3) we will use $m = 2$, following the model of describing deepwater waves given by the nonlinear Schrödinger equation (NLS).

$$iu_t \pm \frac{\alpha}{2} u_{xx} - \alpha u|u|^2 = 0, \ \alpha > 0 \text{ constant.} \tag{4}$$

The NLS model (4) is suggested by the works [12] and [17]. It is important to note that the function u in the model (4) is complex-valued as in the proposed model (3) too.

The proposed model (3) is a "combination," justified by the dispersion relation (2), of the NLS model (4) and the renowned Kadomtsev-Petviashvili model [5]

$$(u_t + 6uu_x + u_{xxx})_x + \lambda u_{yy} = 0. \tag{5}$$

proposed in 1970 by Kadomtsev and Petviashvili as a two-dimensional dispersive wave equation for studying the stability of the one-dimensional soliton solution of the Korteweg de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{6}$$

under the influence of weakly transverse perturbations.

Hence, the model proposed for our analysis is

$$\left(iu_{xt} + 6|u|^2 u_x + u_{xxx}\right)_x + \lambda u_{yy} = 0 \tag{7}$$

where u is a complex-valued function. The model describes the propagation of quasi-two-dimensional nonlinear and dispersive internal gravity waves in a fluid (i.e., water) comprised of two layers of immiscible fluids of different densities stacked on top of each other, where we consider the hydrostatic case where the layers form a 'stacked shallow water' system.

The interfacial waves described by the model (7) are similar to surface waves. But, because the density difference between the two underwater layers is not comparable with the density difference between air and water (it is much less), the gravitational acceleration that a displaced fluid parcel is subject to is reduced by about a factor thousand. Thus, internal gravity waves can have

- larger amplitudes,
- longer periods,
- shorter wavelengths than the surface gravity waves of same period.

In this paper we will follow a similar type of thinking as Korteweg and de Vries in 1895 [7]. We will obtain periodic solutions expressed in terms of the Jacobi elliptic function cn [3] for the MKP equation (7). Also, we will derive solitary wave solutions for the MKP equation (7) from the periodic solutions by considering a limiting process onto the elliptic modulus of the Jacobi elliptic function *cn*.

The results obtained in this paper align with the continuous endeavour of studying the periodic wave nature and solitary wave nature described by the KP model, two relevant examples being [6] and [11].

The novelty of this paper is showing the periodic wave nature and solitary wave nature of internal waves described by a modified KP model.

2. Periodic Solutions for the MKP Equation (7)

Using separation of variables in (7)

$$u(x, y, t) = r(x, y)T(t), \tag{8}$$

such that $r(x, y)$ is a real-valued function and $T(t)$ is a complex-valued function such that $|T| = 1$, we obtain

$$\frac{i\,T'}{T} = \frac{(6r^2 r_x)_x + r_{xxxx} + \lambda r_{yy}}{-r_{xx}} = \mu,\ \mu \text{ constant}, \tag{9}$$

and we will be interested to analyze the case when $\mu < 0$.

From (9) we obtain

$$T(t) = e^{-i\mu t}, \tag{10}$$

and the following second order nonlinear partial differential equation for $r = r(x, y)$

$$\mu r_{xx} + (6r^2 r_x)_x + r_{xxxx} + \lambda r_{yy} = 0. \tag{11}$$

We look for solutions for the equation (11) of the form

$$r(x, y) = h\eta(k_1 x + k_2 y),\ h > 0,\ k_1 > 0,\ k_2 > 0,\ k_1 \gg k_2. \tag{12}$$

where h is the height of the lower layer underneath the interface.

Substituting (12) into the equation (11), and simplifying, we obtain

$$\mu k_1^2 \eta'' + 6h^2 k_1^2 \left(\eta^2 \eta'\right)' + k_1^4 \eta^{iv} + \lambda k_2^2 \eta'' = 0. \tag{13}$$

Integrating (13) and choosing the integration constant zero, and simplifying, we obtain

$$\mu k_1^2 \eta' + 2h^2 k_1^2 \left(\eta^3\right)' + k_1^4 \eta''' + \lambda k_2^2 \eta' = 0. \tag{14}$$

Integrating (14), we obtain

$$\mu k_1^2 \eta + 2h^2 k_1^2 \eta^3 + k_1^4 \eta'' + \lambda k_2^2 \eta = C_1,\ C_1 \text{ integration constant}. \tag{15}$$

Multiplying (15) by η' and integrating, we readily obtain

$$\mu k_1^2 \eta^2 + h^2 k_1^2 \eta^4 + k_1^4 (\eta')^2 + \lambda k_2^2 \eta^2 = 2C_1 \eta + C_2,\ C_2 \text{ integration constant}. \tag{16}$$

We write the ordinary differential equation (16) as follows

$$\frac{k_1}{h}\eta' = \pm\sqrt{P(\eta)},\ P(\eta) = -\eta^4 - \frac{\mu k_1^2 + \lambda k_2^2}{h^2 k_1^2}\eta^2 + \frac{2C_1}{h^2 k_1^2}\eta + \frac{C_2}{h^2 k_1^2}. \tag{17}$$

We are interested in the case when $C_1 = 0$ and the polynomial P factors as follows

$$\begin{aligned} P(\eta) = (\eta^2 + \eta_1)(\eta_2 - \eta^2),\ 0 \le \eta_1 < \eta_2,\ \eta_2 - \eta_1 &= -\frac{\mu k_1^2 + \lambda k_2^2}{h^2 k_1^2}, \\ C_2 &= h^2 k_1^2 \eta_1 \eta_2. \end{aligned} \tag{18}$$

From (17) and (18), we obtain

$$\frac{k_1}{h}\frac{d\eta}{\sqrt{(\eta^2 + \eta_1)(\eta_2 - \eta^2)}} = \pm d(k_1 x + k_2 y) \tag{19}$$

Integrating (19), we obtain

$$\frac{k_1}{h}\int_{\sqrt{\eta_2}}^{\eta} \frac{dw}{\sqrt{(w^2 + \eta_1)(\eta_2 - w^2)}} = \pm(k_1 x + k_2 y + C),\ C \text{ integration constant}. \tag{20}$$

Making the substitution

$$w = \sqrt{\eta_2}\cos\theta, \tag{21}$$

and performing all the calculations, the equation (20) becomes

$$\mp\frac{h}{k_1}\sqrt{\eta_1+\eta_2}(k_1x+k_2y+C)=\int_0^{\phi}\frac{1}{\sqrt{1-m\sin^2\theta}}d\theta,\ m=\frac{\eta_2}{\eta_1+\eta_2},$$

$$\Downarrow$$

$$\mathrm{cn}\left(\frac{h}{k_1}\sqrt{\eta_1+\eta_2}(k_1x+k_2y+C)\middle|\,m\right)=\cos\phi, \tag{22}$$

where cn is the Jacobi elliptic function defined as follows [3]

$$\mathrm{cn}(\tau|m)=\cos\phi,\ \tau=\int_0^{\phi}\frac{d\theta}{\sqrt{1-m\sin^2\theta}},\ m\in[0,\ 1]. \tag{23}$$

As well, we used the fact that the cn function is an even function. Then, referring back to the substitution (21), we finally obtain

$$\eta=\sqrt{\eta_2}\cos\phi=\sqrt{\eta_2}\mathrm{cn}\left(\frac{h}{k_1}\sqrt{\eta_1+\eta_2}(k_1x+k_2y+C)\middle|\,m\right). \tag{24}$$

Thus, the solution we were looking for the equation (11) is

$$\begin{gathered}r(x,y)=h\eta(k_1x+k_2y)=h\sqrt{\eta_2}\mathrm{cn}\left(\frac{h}{k_1}\sqrt{\eta_1+\eta_2}(k_1x+k_2y+C)\middle|\,m\right),\\ m=\frac{\eta_2}{\eta_1+\eta_2},\ 0\le\eta_1<\eta_2,\ \eta_2-\eta_1=-\frac{\mu k_1^2+\lambda k_2^2}{h^2k_1^2},\ C\in\mathbb{R}.\end{gathered} \tag{25}$$

From (8), (10), and (25), the MKP equation (7) has the following periodic solution

$$\begin{gathered}u(x,y,t)=r(x,y)e^{-i\mu t},\\ r(x,y)=h\sqrt{\eta_2}\mathrm{cn}\left(\frac{h}{k_1}\sqrt{\eta_1+\eta_2}(k_1x+k_2y+C)\middle|\,m\right),\\ m=\frac{\eta_2}{\eta_1+\eta_2},\ 0\le\eta_1<\eta_2,\ \eta_2-\eta_1=-\frac{\mu k_1^2+\lambda k_2^2}{h^2k_1^2},\ C\in\mathbb{R},\\ h>0,\ k_1>0,\ k_2>0,\ k_1\gg k_2.\end{gathered} \tag{26}$$

The solution (26) is periodic due to the fact that the cn function is periodic with period $4K$ [16] where K is the constant given by the following integral

$$K=\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}},\ m\in[0,\ 1]. \tag{27}$$

Figure 1 illustrates the time evolution of a periodic solution for the MKP (7). Physically speaking, the cnoidal wave depicted in Figure 1 is nothing else but the envelope of the modulated carrier waves (i.e., the modulated oscillatory components) of the solution (26). The envelope of the modulated carrier waves is given by the graphs of the functions $\pm A=\pm|u|=\pm\sqrt{u\bar{u}}$. We will call the upper part of the envelope *the profile of a cnoidal wave*, given by the formula below,

$$A(x,t)=\sqrt{\eta_2}\left|\mathrm{cn}\left(\frac{h\sqrt{\eta_1+\eta_2}}{k_1}(k_1x+k_2y+C)|\frac{\eta_2}{\eta_1+\eta_2}\right)\right|. \tag{28}$$

Properties of the solution (26):

- Standing wave; it oscillates in time, but the envelope of the carrier waves, i.e., the profile of the cnoidal wave, does not propagate in space.
- The peak amplitude of the cnoidal wave at any point in space is constant with respect to time.
- The oscillations at different points throughout the cnoidal wave are in phase.

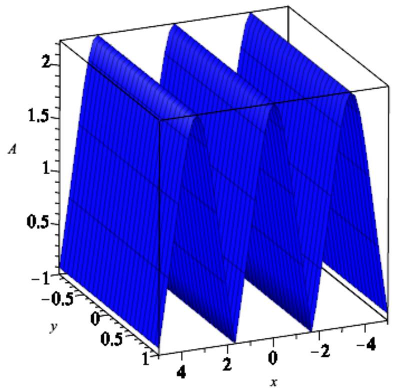

FIGURE 1. Cnoidal wave of the MKP (7) for $h = 0.5$, $\lambda = 1$, $\mu = -1$, $\eta_1 = 1$, $k_1 = 0.2$, $k_2 = 0.00001$, and $C = 0$.

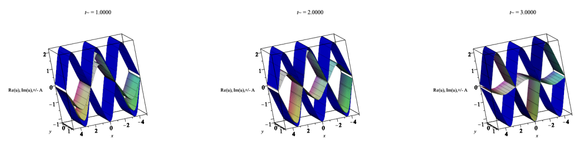

FIGURE 2. Cnoidal wave of the MKP (7) for $h = 0.5$, $\lambda = 1$, $\mu = -1$, $\eta_1 = 1$, $k_1 = 0.2$, $k_2 = 0.00001$, and $C = 0$.

Figure 2 illustrates the properties mentioned above about the solution (26).

Applying scale symmetry [9] onto the solution (26), we obtain the following scaled solution for the MKP (7)

$$u(x,y,t) \longmapsto u(x,y,t|\delta) = \delta u(\delta x, \delta y, \delta^2 t), \delta > 0. \tag{29}$$

The MKP (7) is Galilean invariant as follows: If $u(x,y,t)$ is a solution of the MKP (7) then we can obtain a new solution by changing the inertial reference frame, and adding a phase factor as follows

$$\begin{aligned} u(x,y,t) \longmapsto u(x,t|v) = u(x-vt, y-vt, t)e^{-\frac{i}{2}\alpha^2 v\left(k_1 x + k_2 y + C - \frac{vt}{2}\right)}, \\ \alpha = \frac{\sqrt{-(\mu k_1^2 + \lambda k_2^2)}}{h k_1},\ v \in \mathbb{R}. \end{aligned} \tag{30}$$

Applying the Galilean invariance (30) onto the scaled solution (29), we obtain the following complex-valued solution of the MKP (7)

$$\begin{aligned} u(x,t) = \delta r(\delta(x-vt), \delta(y-vt))e^{-\frac{i}{4}\left(2\alpha^2 v(k_1 x + k_2 y + C) - (\alpha^2 v^2 - 4\delta^2 \mu)t\right)}, \\ \mu < 0,\ \lambda > 0,\ \delta > 0,\ v \in \mathbb{R}, \\ r(x,y) = h\sqrt{\eta_2}\,\mathrm{cn}\left(\frac{h}{k_1}\sqrt{\eta_1+\eta_2}(k_1 x + k_2 y + C)\middle| m\right), \\ m = \frac{\eta_2}{\eta_1+\eta_2},\ 0 \le \eta_1 < \eta_2,\ \eta_2 - \eta_1 = \alpha^2 = -\frac{\mu k_1^2 + \lambda k_2^2}{h^2 k_1^2},\ C \in \mathbb{R}, \\ h > 0,\ k_1 > 0,\ k_2 > 0,\ k_1 \gg k_2. \end{aligned} \tag{31}$$

Properties of the solution (31):

- No longer standing wave; it oscillates in time and the envelope of the carrier waves, i.e., the profile of the cnoidal wave, propagate in space.
- The peak amplitude of the cnoidal wave at any point in space varies with respect to time.

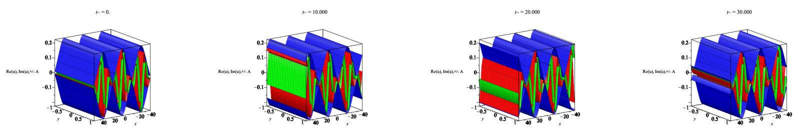

FIGURE 3. Cnoidal wave of the MKP (7) for $h = 0.5$, $\lambda = 1$, $\mu = -1$, $\delta = 0.1$, $v = 1$, $\eta_1 = 1$, $k_1 = 0.2$, $k_2 = 0.00001$, and $C = 0$.

Figure 3 illustrates the time evolution of a periodic solution for the MKP (7). The oscillations at different points throughout the cnoidal wave are not in phase, and the phase invariance allows the oscillatory components of the carrier waves of MKP to undergo modulation. The envelope of the modulated carrier waves is given by the formula below

$$A(x,t) = \delta\sqrt{\eta_2}\left|\mathrm{cn}\left(\frac{h\sqrt{\eta_1+\eta_2}}{k_1}\left(\delta\left(k_1x+k_2y-(k_1+k_2)vt\right)+C\right)\middle|\,\frac{\eta_2}{\eta_1+\eta_2}\right)\right|, \tag{32}$$

and it propagates with the group velocity

$$\begin{aligned} v_g = \nabla_{(k_1,k_2)}\Omega,\ \Omega = -\frac{1}{4}\alpha^2v^2+\delta^2\mu,\ \alpha^2 = -\frac{\mu k_1^2+\lambda k_2^2}{h^2k_1^2}&,\\ \mu<0,\ \lambda>0,\ \delta>0,\ v\in\mathbb{R}&,\\ h>0,\ k_1>0,\ k_2>0,\ k_1\gg k_2&. \end{aligned} \tag{33}$$

3. SOLITARY SOLUTIONS FOR THE KP EQUATION (7)

When a dispersive harmonic wave of the MKP model (7) is subject to the cubic nonlinearity $|u|^2u_x$, the wave will be subject to a "force" that will act against the dispersion process. In other words, the nonlinearity will cancel out the dispersive effect so that the wave will steepen its wavefront. When the wave reaches a "perfect" balance between dispersion and nonlinearity, its oscillatory components will become modulated waves with a localized shaped envelope that decays at infinity. In other words, they will become wave packets. The envelope of these modulated carrier waves is known as the profile of a solitary wave, or a soliton. In this section, we will obtain solitary wave solutions for the MKP model (7) through a limiting process for the elliptic modulus, m, in (31).

Taking the limiting process $\eta_1 \longrightarrow 0$ in (31), the elliptic modulus m will approach 1, and the solution (31) will have the profile of a solitary wave given by the formula below,

$$\begin{aligned} u(x,t) = \delta r(\delta(x-vt),\delta(y-vt))e^{-\frac{i}{4}\left(2\alpha^2v(k_1x+k_2y+C)-(\alpha^2v^2-4\delta^2\mu)t\right)}&,\\ \mu<0,\ \lambda>0,\ \delta>0,\ v\in\mathbb{R}&,\\ r(x,y) = \sqrt{\eta_2}\,\mathrm{sech}\left(\frac{h}{k_1}\sqrt{\eta_2}(k_1x+k_2y+C)\right)&,\\ \eta_2>0,\ \eta_2=\alpha^2=-\frac{\mu k_1^2+\lambda k_2^2}{h^2k_1^2},\ C\in\mathbb{R}&,\\ h>0,\ k_1>0,\ k_2>0,\ k_1\gg k_2&. \end{aligned} \tag{34}$$

The solitary wave described by formula (34), i.e., the profile of the complex-valued function u, $A = |u| = \sqrt{u\bar{u}}$, satisfies the expected boundary conditions mentioned below,

$$\begin{aligned}\nabla_{(x,y)}A(x,y,t) = 0,\ (x,y) \in \left\{k_1 x + k_2 y = (k_1 + k_2)vt - \frac{C}{\delta} \mid t \in R\right\},\\ A(x,y,t) \to 0 \text{ as } \|(x,y)\| \to \infty.\end{aligned} \tag{35}$$

The dispersion and the nonlinearity of the solitary wave described by the formula (34) are in "perfect" balance, and its oscillatory components are modulated carrier waves given by

$$\begin{aligned}\operatorname{Re}(u(x,y,t)) = \delta\sqrt{\eta_2}\frac{\cos\left(\frac{\alpha^2 v}{2}(k_1 x + k_2 y + C) + \left(-\frac{1}{4}\alpha^2 v^2 + \delta^2\mu\right)t\right)}{\cosh(\lambda(\delta(x - vt) + C))},\\ \operatorname{Im}(u(x,y,t)) = -\delta\sqrt{\eta_2}\frac{\sin\left(\frac{\alpha^2 v}{2}(k_1 x + k_2 y + C) + \left(-\frac{1}{4}\alpha^2 v^2 + \delta^2\mu\right)t\right)}{\cosh(\lambda(\delta(x - vt) + C))},\\ \alpha^2 = -\frac{\mu k_1^2 + \lambda k_2^2}{h^2 k_1^2},\ C \in \mathbb{R},\\ \mu < 0,\ \lambda > 0,\ \delta > 0,\ v \in \mathbb{R},\\ h > 0,\ k_1 > 0,\ k_2 > 0,\ k_1 \gg k_2.\end{aligned} \tag{36}$$

The envelope of the modulated carrier waves (36) propagates with the group velocity (33), and it is given by the graphs of the functions $\pm A = \pm|u| = \pm\sqrt{u\bar{u}}$. We will call the upper part of the envelope *the profile of a solitary wave*.

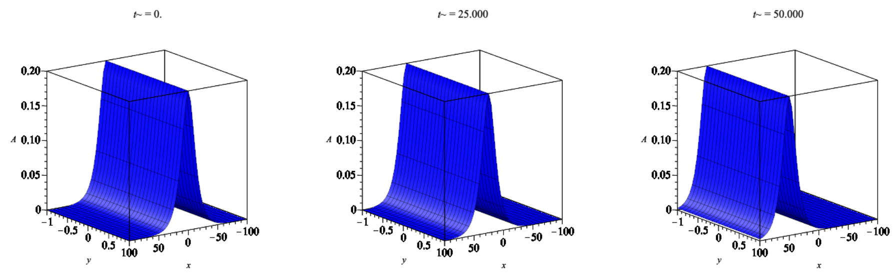

FIGURE 4. Solitary wave of the MKP (7) for $h = 0.5$, $\lambda = 1$, $\mu = -1$, $\delta = 0.1$, $v = 1$, $\eta_2 = 3.99$, $k_1 = 0.2$, $k_2 = 0.00001$, and $C = 0$.

Figure 4 illustrates the time evolution of a solitary wave of the MKP (7), and Figure 5 illustrates the time evolution of the modulated carrier waves (36) and their envelope for the MKP (7). They travel from left to right with the group velocity (33).

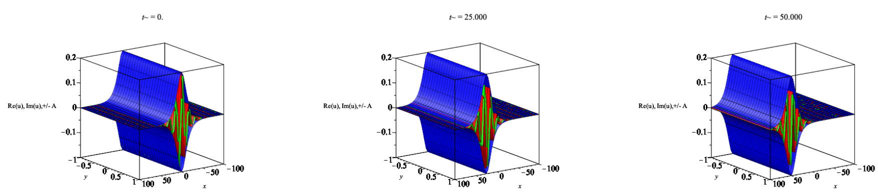

FIGURE 5. Modulated carrier waves and solitary wave for the MKP (7) for $h = 0.5$, $\lambda = 1$, $\mu = -1$, $\delta = 0.1$, $v = 1$, $\eta_2 = 3.99$, $k_1 = 0.2$, $k_2 = 0.00001$, and $C = 0$.

4. Summary

Periodic and solitary dispersive nonlinear waves are already a familiar topic in nonlinear dispersive waves literature, few examples being [1] [2] [18]. There are already numerous studies performed on the KP equation (5), and the equation has been studied in a modified or generalized way. An example of periodic and solitary wave solutions for the generalized KP equation is [11]. In this article, the KP model (5) describing dispersive nonlinear gravity waves in shallow water was synergized with the NLS model (4) describing dispersive nonlinear gravity waves in deep water to create a proposed model (7) to describe the propagation of quasi-two-dimensional nonlinear and dispersive internal gravity waves in a 'stacked shallow water' system.

We often think of waves as surface phenomena occurring at the air-water interface. When low-density water overlies high-density water, internal waves propagate along the interface between the two fluid layers. These internal waves usually occur along the continental shelf in oceans or at density boundaries between subsurface layers. They are responsible for creating "dead water" [15] first reported by the Norwegian polymath Fridtjof Nansen in 1893. The solutions obtained in this paper may explain the formation of such internal waves and their time evolution. The model proposed here refers to the hydrostatic case where the layers form a 'stacked shallow water' system, which can fit the scenario of a boat experiencing strong resistance, due to the energy-producing internal waves, to forward motion in apparently calm conditions occurring when the ship is sailing in a freshwater layer whose depth is comparable to the boat's draft. This strong resistance can also be experienced by submersibles [4] when they encounter dead water between subsurface layers.

The discussion regarding the existence of internal solitary waves started in the late twentieth century, and an excellent example of such debate is [10]. For instance, when underwater interfacial tides steepen, they may form internal solitary waves whose intense energy may affect the movement of a submersible. As well, when there is a proximity of the internal interface to the surface, the solitary waves may leave an imprint at the surface; an example of such phenomenon is presented in [13] at the transition between North Sea water (salt water) and the Rhine River water (freshwater) that flows out on top of the salty water. Internal solitary waves form at the internal interface between the two-water media, distinct from the short wind waves.

While internal gravity waves propagating in continuously stratified deep water may go unnoticed, the suggested model presented here is a conceptual, theoretical study for such phenomena, leaving the discovery of the space-time patterns created by such waves in nature for future work.

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