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SOME UNIFORM CONVERGENCE RESULTS FOR THE HENSTOCK-KURZWEIL-STIELTJES-$\Diamond$-DOUBLE INTEGRALS OF INTERVAL-VALUED FUNCTIONS ON TIME SCALES

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Abstract. In this paper, we prove some uniform convergence theorems for interval Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrals on Time Scales. An example is included.

1. Introduction

The study of double integrals arise in a natural way in connection with statistical problems. This study is interesting from the point of view of measure and integration theory (see [9]). The theory of interval analysis has a long history which can be traced back to the celebrated book of Moore et al. [8]. Interesting applications of interval analysis include; in automatic error analysis, computer graphics, rental network optimization and many others. We refer to [8] for details about this concept of interval analysis.
Time scales version of integration are related to the usual form that exist in literature. This relation shows that most of the properties of a time scale integral can be realized by using the techniques tailored to the time scale settings. Henstock-Kurzweil integral on time scales was first studied by Thompson [11]. Yoon [12] presented some properties of interval-valued Henstock-Stieltjes integral on time scales. In 2020, some properties of Henstock-Kurzweil-Stieltjes integral are given by Afariogun and Olaoluwa [1] for topological vector space-valued functions on time scales. The work of Yoon [12] also motivated Afariogun et al [2] on their work on Henstock-Kurzweil-Stieltjes-$\Diamond$-integrals of interval-valued functions on time scales. Interesting readers are referred to see [13] for details on concepts of two dimensional time scales theory. Some properties of metric space with applications by Hussein [5] is of interest in this paper. Fuzzy continuous function and its properties by Guang-Quan [4] plays important role in the study of interval-valued functions. For some of researchers that have worked on Henstock-Kurzweil integrals, see [3, 6, 7, 10].
Recently, Thange and Gangane [10] published an article on generalised Henstock-Kurzweil integral with multiple point. It is therefore the purpose of this paper to prove some uniform convergence results of Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrals for interval-valued functions with example.

A time scale $\mathbb{T}$ is any closed non-empty subset of $\mathbb{R}$, with the topology inherited from the standard topology on the real numbers $\mathbb{R}$.
Let $a, b \in \mathbb{T}_1, c, d \in \mathbb{T}_2$, where $a < d, c < d$, and a rectangle $\mathcal{R} = [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} = \{(t, s) :$

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$t \in [a,b), s \in [c,d), t \in \mathbb{T}_1, s \in \mathbb{T}_2\}$. Let $g_1, g_2 : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be two non-decreasing functions on $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$, respectively. Let $F : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be bounded on $\mathcal{R}$. Let P_1 and P_2 be two partitions of $[a,b]_{\mathbb{T}_1}$ and $[c,d]_{\mathbb{T}_2}$ such that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, ..., s_n\} \subset [c,d]_{\mathbb{T}_2}$. Let $\{\xi_1, \xi_2, ..., \xi_n\}$ denote an arbitrary selection of points from $[a,b]_{\mathbb{T}_1}$ with $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}_1}, i = 1, 2, ..., n$. Similarly, let $\{\zeta_1, \zeta_2, ..., \zeta_n\}$ denote an arbitrary selection of points from $[c,d]_{\mathbb{T}_2}$ with $\zeta_j \in [s_{j-1}, s_j)_{\mathbb{T}_2}, j = 1, 2, ..., k$.

Definition 1. [1] *A pair $\delta = (\delta_L, \delta_R)$ of real-valued functions defined on $[a,b]_{\mathbb{T}}$ is said to be a Δ-gauge for $[a,b]_{\mathbb{T}}$ if*

$$\begin{cases} \delta_L(t) > 0 \quad on \quad (a,b]_{\mathbb{T}} \\ \delta_R(t) > 0 \ on \ \ [a,b)_{\mathbb{T}} \\ \delta_L(a) \geq 0, \ \delta_R(b) \geq 0 \\ \delta_R(t) \geq \sigma(t) - t \ \ for\ all\ t \in [a,b)_{\mathbb{T}}. \end{cases}$$

A pair $\gamma = (\gamma_L, \gamma_R)$ of real-valued functions defined on $[a,b]_{\mathbb{T}}$ is said to be a $\bigtriangledown$-gauge for $[a,b]_{\mathbb{T}}$ if

$$\begin{cases} \gamma_L(t) > 0 \quad on \quad (a,b]_{\mathbb{T}} \\ \gamma_R(t) > 0 \ on \ \ [a,b)_{\mathbb{T}} \\ \gamma_L(a) \geq 0, \ \gamma_R(b) \geq 0 \\ \gamma_R(t) \geq t - \rho(t) \ \ for\ all\ t \in (a,b]_{\mathbb{T}}, \end{cases}$$

where the function $\sigma : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and the function $\rho(t)$ defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ are respectively the forward and backward jump operators.
Given a Δ-gauge δ and a $\bigtriangledown$-gauge γ for $[a,b]_{\mathbb{T}}$, the partition $P = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}}$ and is said to be:

- *δ-fine if $g(\xi_i) - \delta_L(\xi_i) \leq g(t_{i-1}) < g(t_i) \leq g((\xi_i) + \delta_R(\xi_i)$ and*
- *γ-fine if $g(\xi_i) - \gamma_L(\xi_i) \leq g(t_{i-1}) < g(t_i) \leq g((\xi_i) + \gamma_R(\xi_i)$.*

It can be shown that such partitions always exist.

Throughout this paper, the notation $\Diamond$ will be used to denote dynamic combination of (delta and nabla) Δ and $\bigtriangledown$ derivatives on time scales.

Let I an interval. The left and right endpoints of I will be denoted by $\underline{I}$ and $\overline{I}$ respectively and it is represented by $I = [\underline{I}, \overline{I}]$. Denoted $I_{\mathbb{R}}$ the set of all closed bounded intervals $\mathbb{R}$ such that

$$I_{\mathbb{R}} = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}.$$

For an interval $I = [a,b] \in I_{\mathbb{R}}$, let $\underline{I} = a$ and $\overline{I} = b$, so that $\underline{I}$ represents the least element of I and $\overline{I}$ represents the greatest element of I.

Definition 2. *Let $(I_{\mathbb{R}}, D)$ be a complete metric space. The Hausdorff distance between I_1 and I_2 is defined by*

$$D(I_1, I_2) = \max\{|\underline{I_1} - \underline{I_2}|, |\overline{I_1} - \overline{I_2}|\}.$$

Definition 3. [2] *Let $F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to I_{\mathbb{R}}$ be an interval-valued function on $\mathcal{R}$ such that $F = [\underline{F}, \overline{F}]$ and let g_1, g_2 be non-decreasing functions defined on $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively with partitions $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i =$*

$1,2,...,n$ and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$ for $j = 1,2,...,k$. Then

$$S(P_1, P_2, F, g_1, g_2) = \sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i, \zeta_j)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))$$

is defined as Henstock-Kurzweil-Stieltjes-$\Diamond$-double sum of F with respect to functions g_1 and g_2, and partitions P_1 and P_2. We also adopt the notation $S(F,P,g)$ where $P = P_1 \times P_2$ and $g = g_1 \times g_2 : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \longrightarrow \mathbb{R}^2;\ (t,s) \mapsto (g_1(t), g_2(s))$.

Now, the Henstock-Kurzweil-Stieltjes-$\Diamond$-double sum of F with respect to increasing functions g_1 and g_2 can be taken as

$$S(P,F,g) = \sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i, \zeta_j)\Diamond g_{1_i} \Diamond g_{2_j}.$$

Let $P = P_1 \times P_2$, then the Henstock-Kurzweil-Stieltjes-$\Diamond$-double sum of F with respect to functions g_1 and g_2 is denoted by $S(P, F, g_1, g_2)$.

Definition 4. [2] *Let $F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to I_{\mathbb{R}}$ be an interval-valued function on $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ such that $F = [\underline{F}, \overline{F}]$. We say that F is Henstock-Kurzweil-Stieltjes-$\Diamond$-integrable with respect to non-decreasing functions $g_1 : [a,b)_{\mathbb{T}_1} \longrightarrow \mathbb{R}$ and $g_2 : [c,d)_{\mathbb{T}_2} \longrightarrow \mathbb{R}$ if there exists an interval $I_0 \in I_{\mathbb{R}}$ such that for every $\varepsilon > 0$, there are $\Diamond$-gauges δ_1 and δ_2 (or γ_1 and γ_2) for $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively such that*

$$D\left(S(P,F,g), I_0\right) < \varepsilon,$$

provided that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a,b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1,...,n$ is a δ_1-fine (or γ_1) and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1,2,...,k$ is a δ_2-fine (or γ_2) are δ-fine (or γ) partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively.

Then, I_0 is the Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral of F with respect to g_1 and g_2 defined on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$, and write

$$\int\int_{\mathcal{R}} F(t,s)\Diamond g_1(t)\Diamond g_2(s) = I_0.$$

It follows from the continuity of F that there two continuous real-valued functions $\underline{F}$ and $\overline{F}$ such that, for $t,s \in \mathbb{R}$,

$$F(t,s) = [\underline{F(t,s)}, \overline{F(t,s)}].$$

Moreover,

$$\int\int_{\mathcal{R}} F(t,s)\Diamond g_1(t)\Diamond g_2(s) = \left[\int\int_{\mathcal{R}} \underline{F(t,s)}\Diamond g_1(t)\Diamond g_2(s), \int\int_{\mathcal{R}} \overline{F(t,s)}\Diamond g_1(t)\Diamond g_2(s)\right].$$

2. Main results

Firstly, the following definition is important:

Definition 5. *A sequence $F_n \in IVHKS[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ is said to be interval-valued uniformly convergent to F in $\mathcal{R}$ if for each $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that $D(F_n(t,s), F(t,s)) < \varepsilon$ for all $n \geq n_0$ and for all $t,s \in [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$.*

It is quite natural to expect that $F \in IVHKS[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. That is to say that set of all Interval-Valued Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable functions on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$.

The following Theorems are useful in the main results.

Theorem 1. [4] *Whenever* $\overline{a}, \overline{b}, \overline{c}, \overline{d} \in I_{\mathbb{R}}, k \in \mathbb{R}$, *we have*
(i) $D(\overline{a} + \overline{b}, \overline{a} + \overline{c}) = D(\overline{b}, \overline{c})$;
(ii) $D(\overline{a} - \overline{b}, \overline{a} - \overline{c}) = D(\overline{b}, \overline{c})$;
(iii) $D(\overline{b} - \overline{a}, \overline{c} - \overline{a}) = D(\overline{b}, \overline{c})$;
(iv) $D(k \cdot \overline{a}, k \cdot \overline{b}) = |k| \cdot D(\overline{a}, \overline{b})$;
(v) if $\overline{a} \leq \overline{b} \leq \overline{c}$, *then* $D(\overline{a}, \overline{b}) \leq D(\overline{a}, \overline{c})$ *and* $D(\overline{b}, \overline{c}) \leq D(\overline{a}, \overline{c})$;
(vi) if $\overline{a} \leq \overline{b} \leq \overline{c}$, *and* $\overline{a} \leq \overline{d} \leq \overline{b}$, *then* $D(\overline{c}, \overline{d}) \leq D(\overline{a}, \overline{b})$.

Theorem 2. [2] *(Bolzano Cauchy Criterion) Let* $F : [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \to I_{\mathbb{R}}$ *be an interval-valued function over a rectangle* $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ *and consider two non-decreasing functions* $g_1 : [a, b)_{\mathbb{T}_1} \longrightarrow \mathbb{R}$ *and* $g_2 : [c, d)_{\mathbb{T}_2} \longrightarrow \mathbb{R}$. *Then,* F *is Henstock-Kurzweil-Stieltjes-*$\Diamond$*-double integrable on* $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ *with respect to* g_1 *and* g_2 *if and only if for each* $\varepsilon > 0$ *there exists* $\Diamond$*-gauges* δ_1 *and* δ_2 *(or* γ_1 *and* γ_2*) for* $[a, b)_{\mathbb{T}_1}$ *and* $[c, d)_{\mathbb{T}_2}$ *respectively, such that* $D(S(P^1, F, g), S(P^2, F, g), F, g)) < \varepsilon$ *for all pairs* $P^1 = P_1^1 \times P_2^1$ *and* $P^2 = P_1^2 \times P_2^2$ *of* δ_1 *(or* γ_1*)-fine partitions of* $[a, b)_{\mathbb{T}_1}$ *and* δ_2 *(or* γ_2*)-fine partitions of* $[c, d)_{\mathbb{T}_2}$.

The following results are for conditions of uniform integrability of a sequence of interval-valued functions on time scales.

Theorem 3. *Let* F *be an interval-valued Henstock-Kurzweil-Stieltjes-*$\Diamond$*-integrable function, consider a sequence of interval-valued functions* $F_n : [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \to I_{\mathbb{R}}, n \in \mathbb{N}$ *in* $I_{\mathbb{R}}$ *and a monotone increasing functions* $g_1, g_2 : [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \to I_{\mathbb{R}}$. *Then* $\{F_n\}$ *is uniformly integrable with respect to increasing functions* g_1, g_2, *if the following two conditions are satisfied:*

(i) F *is Henstock-Kurzweil-Stieltjes-*$\Diamond$*-integrable on* $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$.

$$(ii) \qquad \lim_{n\to\infty} \int\int_{\mathcal{R}} F_n(t, s)\Diamond g_1(t)\Diamond g_2(s) = \int\int_{\mathcal{R}} F(t, s)\Diamond g_1(t)\Diamond g_2(s).$$

Proof. Firstly show that $\int\int_{\mathcal{R}} F_n(t, s)\Diamond g_1(t)\Diamond g_2(s)$ is a Cauchy sequence in $\mathcal{R}$. Let $\varepsilon > 0$. Since for each $n \in \mathbb{N}$, there exists a $\delta_n \in \mathcal{R}$ such that

$$D\left(\sum_{i=1}^{n}\sum_{j=1}^{k} F_n(\xi_i, \zeta_j)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1})), \int\int_{\mathcal{R}} F_n(t, s)\Diamond g_1(t)\Diamond g_2(s)\right) < \frac{\varepsilon}{3}$$

provided that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, ..., n$ and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c, d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$, $j = 1, 2, ..., k$ are δ-fine (or γ) partitions of $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$. Again since $\{F_n\}$ in $\mathcal{R}$ is interval-valued uniformly convergent to F, there exists a $n_0 \in \mathbb{N}$ such that

$$D(F_n(\xi_i, \zeta_j), F_m(\xi_i, \zeta_j)) < \frac{\varepsilon}{3((b - a)(d - c))}$$

for all $n \geq n_0$ and for all $i, j = 1, 2, ..., n$.

So, for all $n, m \geq n_0$, taking an arbitrary partition (ξ, ζ) simultaneously δ_n-fine and δ_m-fine, we have

$$\begin{aligned}
& D\left(\int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s), \int\int_{\mathcal{R}} F_m(t,s)\Diamond g_1(t)\Diamond g_2(s)\right) \\
\leq \; & D\left(\sum_{i=1}^{n}\sum_{j=1}^{k} F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})), \int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s)\right) \\
& + D[\sum_{i=1}^{n}\sum_{j=1}^{k} F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})), \\
& \sum_{i=1}^{n}\sum_{j=1}^{k} F_m(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})] \\
& + D\left(\sum_{i=1}^{n}\sum_{j=1}^{k} F_m(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})), \int\int_{\mathcal{R}} F_m(t,s)\Diamond g_1(t)\Diamond g_2(s)\right) \\
< \; & \frac{\varepsilon}{3} + \sum_{i=1}^{n}\sum_{j=1}^{k} D\left(F_n(\xi_i,\zeta_j), F_m(\xi_i,\zeta_j)\right) + \frac{\varepsilon}{3} \\
& < \frac{\varepsilon}{3} + \frac{\varepsilon}{3((b-a)(d-c))}((b-a)(d-c)) + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Thus, $\int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s))$ is a Cauchy sequence in the metric space $(I_{\mathbb{R}}, D)$.

Since by Theorem 1, $(I_{\mathbb{R}}, D)$ is a complete metric space, $\int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s))$ converges in $(I_{\mathbb{R}}, D)$.

Suppose

$$\lim_{n\to\infty}\int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s)) = I_0.$$

Let $\varepsilon > 0$ be given. By the condition of Theorem 2, there exists a $n_0 \in \mathbb{N}$ such that

$$D(F_n(\xi_i,\zeta_j), F(\xi_i,\zeta_j)) < \frac{\varepsilon}{3((b-a)(d-c))}$$

for all $n \geq n_0$ and for all $t, s \in [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. Now for any tagged partition (ξ, ζ) of $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ and $n \geq n_0$, by Theorem 1, we get

$$\begin{aligned}
& D[\sum_{i=1}^{n}\sum_{j=1}^{k} F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})), \\
& \sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})] \\
\leq \; & \sum_{i=1}^{n}\sum_{j=1}^{k}[(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})]D(F_n(\xi_i,\zeta_j), F(\xi_i,\zeta_j)) \\
& \leq (b-a)(d-c)\sum_{i=1}^{n}\sum_{j=1}^{k} D(F_n(\xi_i,\zeta_j), F(\xi_i,\zeta_j)) < \frac{\varepsilon}{3}.
\end{aligned}$$

Now, since $\int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s)) = I_0$, there exists a $p(\geq n_0) \in \mathbb{N}$ such that

$$D\left(\int\int_{\mathcal{R}} F_p(t,s)\Diamond g_1(t)\Diamond g_2(s)), I_0\right) < \frac{\varepsilon}{3}.$$

Since $\int\int_{\mathcal{R}} F_p(t,s)\Diamond g_1(t)\Diamond g_2(s))$ exists, there exists a $\delta_p \in \mathcal{R}$ such that every δ_p-fine division (ξ,ζ) of $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ satisfies

$$D\left(\sum_{i=1}^{n}\sum_{j=1}^{k} F_p(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})), \int\int_{\mathcal{R}} F_p(t,s)\Diamond g_1(t)\Diamond g_2(s)\right) < \frac{\varepsilon}{3}.$$

Thus,

$$\begin{aligned}
& D\left(\sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})), I_0\right) \\
\leq\ & D(\sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})), \\
& \sum_{i=1}^{n}\sum_{j=1}^{k} F_p(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})) \\
& +D\left(\sum_{i=1}^{n}\sum_{j=1}^{k} F_p(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})), \int\int_{\mathcal{R}} F_p(t,s)\Diamond g_1(t)\Diamond g_2(s)\right) \\
& +D\left(\int\int_{\mathcal{R}} F_p(t,s)\Diamond_1 g_1(t)\Diamond_2 g_2(s), I_0\right) < \varepsilon
\end{aligned}$$

for every δ_p-fine division (ξ,ζ) of $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. Hence $\int\int_{\mathcal{R}} F(t,s)\Diamond g_1(t)\Diamond g_2(s)$ exists and

$$\int\int_{\mathcal{R}} F(t,s)\Diamond g_1(t)\Diamond g_2(s) = I_0.$$

Next is to define a sequence of interval-valued Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable functions as follows:

Definition 6. *A sequence $\{F_n(t,s)\}$ of interval-valued Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable functions in $I_{\mathcal{R}}$ with respect to increasing functions g_1, g_2 is called interval-valued uniformly Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ if for each $\varepsilon > 0$, there exists a $\delta(t,s) \in \mathcal{R}$ such that every δ-fine division (ξ,ζ) of $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ satisfies*

$$D\left(\sum_{i=1}^{n}\sum_{j=1}^{k} F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})), \int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s)\right)$$

$< \varepsilon$, *for all $n \in \mathbb{N}$.*

Theorem 4. *Let $\{F_n(t,s)\}$ be a sequence of interval-valued uniformly Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable functions in $I_{\mathcal{R}}$ with respect to increasing functions g_1, g_2 and $F \in I_{\mathcal{R}}$ be such that for each $t,s \in [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$, $\{F_n(t,s)\}$ converges to $F(t,s)$ in the metric space $(I_{\mathcal{R}}, D)$. Then*

(i) F is Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$ and;
(ii)

$$\int\int_{\mathcal{R}} F(t,s)\Diamond g_1(t)\Diamond g_2(s)=\lim_{n\to\infty}\int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s).$$

Proof. Let $\varepsilon>0$ be given. Since $\{F_n\}$ is an interval-valued uniformly Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable sequence in a rectangle $\mathcal{R}$, there exists a $\delta(t,s)$-fine division (ξ,ζ) of $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$ that satisfies

$$D\left(\sum_{i=1}^{n}\sum_{j=1}^{k}F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}),\int\int_{\mathcal{R}}F_n(t,s)\Diamond g_1(t)\Diamond g_2(s)\right)$$
$$<\frac{\varepsilon}{3},\quad \text{for all } n\in\mathbb{N}.$$

By the condition of the Theorem 2, there exists a $n_0\in\mathbb{N}$ such that

$$D(F_n(\xi_i,\zeta_j),F_l(\xi_i,\zeta_j))<\frac{\varepsilon}{3[(b-a)(d-c)]}$$

for all $n,l>n_0$ and so

$$\begin{aligned}
&D[\sum_{i=1}^{n}\sum_{j=1}^{k}F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})),\\
&\sum_{i=1}^{n}\sum_{j=1}^{k}F_l(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})]\\
\leq\ &\sum_{i=1}^{n}\sum_{j=1}^{k}[(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})]D(F_n(\xi_i,\zeta_j),F_l(\xi_i,\zeta_j))\\
\leq\ &(b-a)(d-c)\sum_{i=1}^{n}\sum_{j=1}^{k}D(F_n(\xi_i,\zeta_j),F_l(\xi_i,\zeta_j))<\frac{\varepsilon}{3}.
\end{aligned}$$

for all $n,l>n_0$. Then

$$\begin{aligned}
&D\left(\int\int_{\mathcal{R}}F_n(t,s)\Diamond g_1(t)\Diamond g_2(s),\int\int_{\mathcal{R}}F_l(t,s)\Diamond g_1(t)\Diamond g_2(s)\right)\\
\leq\ &D\left(\int\int_{\mathcal{R}}F_n(t,s),\sum_{i=1}^{n}\sum_{j=1}^{k}F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})\right)\\
+\ &D(\sum_{i=1}^{n}\sum_{j=1}^{k}F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}),\\
&\sum_{i=1}^{n}\sum_{j=1}^{k}F_l(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}))\\
+\ &D\left(\sum_{i=1}^{n}\sum_{j=1}^{k}F_l(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}),\int\int_{\mathcal{R}}F_l(t,s)\Diamond g_1(t)\Diamond g_2(s)\right)
\end{aligned}$$
$$<\varepsilon,\quad \text{for all } n,l>n_0.$$

So $\{\int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s)\}$ is a Cauchy sequence in the complete metric space $(I_{\mathcal{R}}, D)$. Therefore $\{\int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s)\}$ converges in the metric space $(I_{\mathcal{R}}, D)$. Suppose

$$\lim_{n\to\infty}\int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s) = I_0. \int\int_{\mathcal{R}} F(t,s)\Diamond g_1(t)\Diamond g_2(s) = I_0.$$

To show this, let $\varepsilon > 0$ be given. Since $\{F_n(t,s)\}$ is an interval-valued uniformly Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable sequence on $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$, there exists a positive $\delta\in\mathbb{R}$ such that δ-fine division (ξ,ζ) of $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$ satisfies

$$D\left(\sum_{i=1}^{n}\sum_{j=1}^{k} F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}), \int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s)\right)$$

$$< \frac{\varepsilon}{3}, \text{ for all } n\in\mathbb{N}.$$

Since $\lim_{n\to\infty}\int\int_{\mathcal{R}} F_n(t,s) = I_0$, there exists a $n_1\in\mathbb{N}$ such that

$$D\left(\int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s), I_0\right) < \frac{\varepsilon}{3}$$

for all $n \geq n_1$. Again, by the conditions of Theorem 2, there exists a $n_2(\geq n_1)\in\mathbb{N}$ such that

$$\begin{aligned}
& D[\sum_{i=1}^{n}\sum_{j=1}^{k} F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})),\\
& \sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})]\\
\leq\ & \sum_{i=1}^{n}\sum_{j=1}^{k}[(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1})]D(F_n(\xi_i,\zeta_j),F(\xi_i,\zeta_j))\\
\leq\ & (b-a)(d-c)\sum_{i=1}^{n}\sum_{j=1}^{k} D(F_n(\xi_i,\zeta_j),F(\xi_i,\zeta_j)) < \frac{\varepsilon}{3}.
\end{aligned}$$

Thus

$$\begin{aligned}
& D\left(\sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}), I_0\right)\\
\leq\ & D(\sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}),\\
& \sum_{i=1}^{n}\sum_{j=1}^{k} F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}))\\
+\ & D\left(\sum_{i=1}^{n}\sum_{j=1}^{k} F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}), \int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s)\right)\\
+\ & D\left(\int\int_{\mathcal{R}} F_n(t,s)\Diamond g_1(t)\Diamond g_2(s), I_0\right) < \varepsilon
\end{aligned}$$

for every δ-fine division (ξ,ζ) of $[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}$. Hence $\int\int_{\mathcal{R}} F(t,s)\Diamond g_1(t)\Diamond g_2(s)$ exists and

$$\int\int_{\mathcal{R}} F(t,s)\Diamond g_1(t)\Diamond g_2(s) = I_0.$$

Example 1. *Let $F_n:[a,b)_{\mathbb{T}_1}\times[c,d)_{\mathbb{T}_2}\to I_{\mathbb{R}}$ be a sequence of Dirichlet interval-valued function defined by:*

$$F_n(t,s)=\begin{cases}[0,1] & if\quad (t,s)\in\{r_1,r_2,\cdots,r_n\}\\ [-1,0] & if\quad (t,s)\notin\{q_1,q_2,\cdots,q_n\}\end{cases}$$

with respect to non-decreasing functions $g_1:[a,b)_{\mathbb{T}_1}\longrightarrow\mathbb{R}$ and $g_2:[c,d)_{\mathbb{T}_2}\longrightarrow\mathbb{R}$. Therefore $F_n(t,s)$ converges uniformly to F and F is interval-valued Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable with

$$\int_a^b\int_c^d F(t,s)\Diamond g_1(t)\Diamond g_2(s)=0.$$

To show this, let $\varepsilon>0$. Enumerate the rational numbers in $[a,b]_{\mathbb{T}_1}$ as $\{r_1,r_2,...,\}$ and enumerate the rational numbers in $[a,b]_{\mathbb{T}_2}$ as $\{q_1,q_2,...,\}$. Now, define a Δ-gauge $\delta_{1\varepsilon}$-fine for $[a,b]_{\mathbb{T}_1}$, the partition $P_1=\{t_0,t_1,...,t_n\}\subset[a,b]_{\mathbb{T}_1}$ with tag points $\xi_i\in[t_{i-1},t_i]_{\mathbb{T}_1}$ for $i=1,...,n$ is a $\delta_{1\varepsilon}$-fine (or $\gamma_{1\varepsilon}$) and $P_2=\{s_0,s_1,...,s_k\}\subset[c,d]_{\mathbb{T}_2}$ with tag points $\zeta_j\in[s_{j-1},s_j]_{\mathbb{T}_2}$, $j=1,2,...,k$ is a $\delta_{2\varepsilon}$-fine (or $\gamma_{2\varepsilon}$) are δ-fine (or γ) partitions of $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively.

Now define $F_1(t,s)=1$ if $(t,s)=r_1$ and $F_1(t,s)=0$ otherwise. Next, define $F_2(t,s)=1$ if $(t,s)=r_1$ or $(t,s)=r_2$ and $F_2(t,s)=0$ otherwise. By continuing in this process, we get

$$F_n(t,s)=\begin{cases}[0,1] & if\quad (t,s)\in\{r_1,r_2,\cdots,r_n\}\\ [-1,0] & if\quad (t,s)\notin\{q_1,q_2,\cdots,q_n\}\end{cases}$$

From Definition 1:
A pair $\delta=(\delta_L,\delta_R)$ of real-valued functions defined on $[a,b]_{\mathbb{T}}$ is a Δ-gauge for $[a,b]_{\mathbb{T}}$ and a pair $\gamma=(\gamma_L,\gamma_R)$ of real-valued functions defined on $[a,b]_{\mathbb{T}}$ is a $\bigtriangledown$-gauge for $[a,b]_{\mathbb{T}}$.
Let $\delta_1=(\delta_{1L},\delta_{1R})$ and $\delta_2=(\delta_{2L},\delta_{2R})$, then

$$\delta_{1R}(t)=\delta_{1L}(t)=\begin{cases}\dfrac{\sqrt{\varepsilon}}{2^i}, & if\quad t=r_i\\ 1, & if\quad t\notin\{r_1,r_2,...\}.\end{cases}$$

Similarly,

$$\delta_{2R}(s)=\delta_{2L}(s)=\begin{cases}\dfrac{\sqrt{\varepsilon}}{2^j}, & if\quad s=q_j\\ 1, & if\quad s\notin\{q_1,q_2,...\}.\end{cases}$$

Let $g_1(t)=\frac{t}{2}$, $\quad t\in[a,b]_{\mathbb{T}_1}$ and $g_2(s)=2s$, $\quad s\in[a,b]_{\mathbb{T}_2}$. Then,

$$\Diamond_\varepsilon(t,s)=\begin{cases}\left(\dfrac{\sqrt{\varepsilon}}{2^i},\dfrac{\sqrt{\varepsilon}}{2^j}\right), & if\quad (t,s)=r_i,q_j\\ 1, & if\quad t\notin\{r_1,r_2,...\}, s\notin\{q_1,q_2,...\}.\end{cases}$$

If $\xi_i\in\{r_1,r_2,...\}$ is a tag on $[t_{i-1},t_i]$ and $\zeta_i\in\{q_1,q_2,...\}$ is a tag on $[s_{j-1},s_j]$, then $F_n(\xi_i,\zeta_j)=[0,1]$ and

$$(g_1(t_i)-g_1(t_{i-1}))\,(g_2(s_j)-g_2(s_{j-1}))\le\Diamond_\varepsilon(\xi_i,\zeta_j)=\frac{\sqrt{\varepsilon}}{2^i}.\frac{\sqrt{\varepsilon}}{2^j}$$

where g_1 and g_2 are monotone non-decreasing functions. Thus we have:

$$F_n(\xi_i,\zeta_j)\left(g_1(\frac{1}{2}t_i)-g_1(\frac{1}{2}t_{i-1})\right)(g_2(2s_j)-g_2(2s_{j-1}))\leq \Diamond_\varepsilon(\xi_i,\zeta_j)=\frac{\sqrt{\varepsilon}}{2^i}.\frac{\sqrt{\varepsilon}}{2^j}\leq\frac{\varepsilon}{2^{i+j}}.$$

If $\xi_i\notin\{r_1,r_2,...\}$ and $\zeta_j\notin\{q_1,q_2,...\}$ are tags on $[t_{i-1},t_i]$ and $[s_{j-1},s_j]$ respectively, then $F_n(\xi_i,\zeta_j)=[-1,0]$ and

$$(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}))\leq\Diamond_\varepsilon(\xi_i,\zeta_j)=[0,1].$$

Therefore,

$$F_n(\xi_i,\zeta_j)\left(g_1(\frac{1}{2}t_i)-g_1(\frac{1}{2}t_{i-1})\right)[(g_2(2s_j)-g_2(2s_{j-1})]\leq\Diamond_\varepsilon(\xi_i,\zeta_j)=\{0\}=I_0.$$

So subintervals with tags $\xi_i\in\{r_1,r_2,...\}$ and $\zeta_j\in\{q_1,q_2,...\}$ do not contribute to the Henstock-Kurzweil-Stieltjes sum $S(F_n,p,g)$. Let α be the set of indices i,j such that $\xi_i\in\{r_1,r_2,...\}$ and $\zeta_j\in\{q_1,q_2,...\}$ and β be the set of indices i,j such that $\xi_i\in\{r_1,r_2,...\}$ and $\zeta_j\in\{q_1,q_2,...\}$. We conclude that:

$$\begin{aligned}
|S(F_n,P,g)-I_0| &= |S(F_n,P,g)-\{0\}|\\
&= \left|\sum_{i=1}^{n}\sum_{j=1}^{k}F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}))\right|\\
&\leq \left|\sum_{i\in\alpha}\sum_{j\in\alpha}F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}))\right|\\
&+ \left|\sum_{i\in\beta}\sum_{j\in\beta}F_n(\xi_i,\zeta_j)(g_1(t_i)-g_1(t_{i-1}))(g_2(s_j)-g_2(s_{j-1}))\right|\\
&\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\sqrt{\varepsilon}}{2^i}\cdot\frac{\sqrt{\varepsilon}}{2^j}\\
&\leq \sum_{i=1}^{\infty}\frac{\sqrt{\varepsilon}}{2^i}\sum_{j=1}^{\infty}\frac{\sqrt{\varepsilon}}{2^j}\\
&\leq \varepsilon\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{2^{i+j}}<\varepsilon.
\end{aligned}$$

Conclusion

We proved some uniform convergence results of Henstock-Kurzweil-Stieltjes-◊-double integrals for interval-valued functions in which an example is given to support our results.

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Conflict of Interests

The authors declare that there is no competing interests between them during the time of writing this paper.

References

[1] Afariogun, D.A., Olaoluwa, H.O., *Refinement of Henstock-Kurzweil-Stieltjes-$\Diamond$-Integral for topological Vector Space-valued functions on time scales*, J. Sci. Research and Develop., **19**(2020), No.2, 1–12.

[2] Afariogun, D.A., Mogbademu, A.A. and Olaoluwa, H.O., *On Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrals of interval-valued functions on time scales*, Annals of Maths. and Comp. Sci., **2**(2021), 29–40.

[3] Cunanan, A.F., Benitez, J., *Simple Properties and Existence Theorem for the Henstock-Kurzweil-Stieltjes integral of functions taking values on $C[a, b]$ Space-valued functions*, Eur. J. Pure Appl. Math., **13**(2020), No. 1, 130–143.

[4] Guang-Quan, Z., *Fuzzy continuous function and its properties*, Fuzzy Sets and Systems, **43**(1991), 159–171.

[5] Hussein, I.H., Abed, S.S., *Fixed Point Theorems in General Metric Space with an Application*, Baghdad Sci. J., **18**(2021), No.1 (Suppl.).

[6] Mahanta, S., Ray, S., *On the Generalization of Henstock-Kurzweil Fourier Transform*, arX:2202.10394 [math. CA], (2022), 1–14.

[7] Maly, J., Kuncova, K., *On a Generalization of Henstock-Kurzweil Integrals*, Math Bohem, **144**(2019), No.4, 1–30.

[8] Moore, R.E., Kearfott, R.B. and Cloud, M.J., *Introduction to Interval Analysis*, Society for Industrial and Applied Mathematics, University City Science Center Philadelphia, PA, United State (2009).

[9] Taylor, M.E., *Measure Theory and Integration*, American Mathematical Society, Graduate Studies in Mathematics, **76**,(20060; 157–305).

[10] Thange, T.G., Gangane, S.S., *Generalization of Henstock-Kurweil Integral with Multiple Point*, Baghdad Sci. J., **20**(2023), No.1 (Special Issue) ICAAM, 388–393.

[11] Thompson, B., *Henstock-Kurzweil integrals on time scales*, Panamer. Math. J., **18**(2008), No.1, 1–19.

[12] Yoon, J.H., *On Henstock-Stieltjes Integrals of Interval-Valued Functions on Time Scales*, Journal of the Chungcheong Mathematical Society, **29**(2016), 109–115.

[13] Zareen, A.K., Fahd, J., Aziz, K., Hasib, K., *Derivation of dynamical integral inequalities based on two-dimensional time scales theory*, J. Ineq. Appl., (2020), 209, 1–17.

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