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ON THE SOLUTION OF A LINEAR MATRIX DIFFERENCE EQUATION

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Abstract. In this paper we present the closed-form expression for the solution of the linear matrix difference equation

$$X_{n+1} = AX_n + X_nB, \quad X(0) = C, \quad n = 0, 1, 2, ...$$

which we obtained with the aid of the Kronecker sum, Kronecker product and Vec operator.

1. Introduction

Matrix equations are important due to their application in many science fields. When dealing with many problems for descriptor linear systems, a matrix equation is often encountered. Since matrix equation is a very hot topic and many relating results can be found in the literature in recent years, a large amount of prior work has been devoted to solving matrix equations. For related references, see [1], [2], [3] and the references given there.

The remainder of this paper is organized as follows. In Section 2, we introduce some definition and properties related to the Kronecker sum, Kronecker product and Vec operator are given for reader's connivance. By using the Kronecker sum, Kronecker product and Vec operator, closed form solution to the linear matrix difference equation

$$X_{n+1} = AX_n + X_nB, \quad X(0) = C, \quad n = 0, 1, 2, ... \tag{1}$$

for $A \in M_r, B \in M_s$ and $X_n \in M_{r,s}$ is presented in Section 3.

2. Preliminaries and assumptions

In this section, we will recall some basic definitions and notations subsequently used in this paper.

Definition 1 ([4])**.** *If A and B are matrices in $\mathbb{R}_{m\times n}$ and in $\mathbb{R}_{k\times s}$ respectively, then the Kronecker product of $A = [a_{ij}]$ and $B = [b_{ij}]$ is denoted by $A \otimes B$ and defined as*

$$\begin{aligned} A \otimes B &= [a_{ij}B]_{mk\times ns} \\ &= \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}_{mk\times ns}. \end{aligned}$$

Definition 2 ([5])**.** *The Kronecker sum of matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$, denoted by $A \oplus B \in \mathbb{R}^{mn\times mn}$, is defined as:*

$$A \oplus B = A \otimes I_m + I_n \otimes B$$

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where I_m and I_n are identity matrices with mth and nth dimensions.

Definition 3 (([1])). *Vec of $A \in \mathbb{R}_{m\times n}$ is denoted by $VecA$. If we define $A_{.j} = (a_{1j}, a_{2j}, \ldots, a_{mj})^{\mathrm{T}}$, where $1 \leqslant j \leqslant n$. Then $VecA = (A_{.1}, A_{.2}, \ldots, A_{.n})^{\mathrm{T}}$, $(.)^{\mathrm{T}}$ denotes the transpose of the row vector $(.)$.*

Note that if $A = [A_1, A_2]$, then $VecA = \begin{pmatrix} VecA_1 \\ VecA_2 \end{pmatrix}$.

Now, we give some basic properties of the Kronecker product. If A, X, B are matrices of given orders then

$$\begin{aligned} &(A+B)\otimes C = A\otimes C + B\otimes C, \\ &A\otimes(B+C) = A\otimes B + A\otimes C, \\ &Vec(AXB) = \left(B^{\mathrm{T}}\otimes A\right) VecX. \end{aligned} \tag{2}$$

Let M_n denote the set of real matrices of order n. Throughout the paper, for $A \in M_n$ we will follow the following notation $A^0 = I_n$, where I_n is the identitiy matrix.

3. Main Results

We give a theorem which will provide us closed form solution to the linear matrix difference equation (1).

Theorem 1. *Let $A, I_n \in M_n$ and $B, I_m \in M_m$. Then*

$$\left(I_m\otimes A + B^T\otimes I_n\right)^k = \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} \left(B^T\right)^{k-r}\otimes A^r. \tag{3}$$

Proof. Since

$$\begin{aligned} \sum_{r=0}^{1} \begin{pmatrix} 1 \\ r \end{pmatrix} \left(B^T\right)^{1-r}\otimes A^r &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left(B^T\right)^{1-0}\otimes A^0 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(B^T\right)^{1-1}\otimes A^1 \\ &= B^T\otimes I_n + I_m\otimes A \\ &= I_m\otimes A + B^T\otimes I_n \\ &= \left(I_m\otimes A + B^T\otimes I_n\right)^1 \end{aligned}$$

and

$$\begin{aligned} \left(I_m\otimes A + B^T\otimes I_n\right)^2 &= \left(I_m\otimes A + B^T\otimes I_n\right)\left(I_m\otimes A + B^T\otimes I_n\right) \\ &= (I_m\otimes A)(I_m\otimes A) + (I_m\otimes A)\left(B^T\otimes I_n\right) \\ &\quad + \left(B^T\otimes I_n\right)(I_m\otimes A) + \left(B^T\otimes I_n\right)\left(B^T\otimes I_n\right) \\ &= (I_m\otimes A)^2 + B^T\otimes A + B^T\otimes A + \left(B^T\otimes I_n\right)^2 \\ &= (I_m\otimes A)^2 + 2B^T\otimes A + \left(B^T\otimes I_n\right)^2 \\ &= I_m\otimes A^2 + 2B^T\otimes A + \left(B^T\right)^2\otimes I_n \\ &= \left(B^T\right)^2\otimes I_n + 2B^T\otimes A + I_m\otimes A^2 \\ &= \begin{pmatrix} 2 \\ 0 \end{pmatrix}\left(B^T\right)^{2-0}\otimes A^0 + \begin{pmatrix} 2 \\ 1 \end{pmatrix}\left(B^T\right)^{2-1}\otimes A^1 + \begin{pmatrix} 2 \\ 2 \end{pmatrix}\left(B^T\right)^{2-2}\otimes A^2 \\ &= \sum_{r=0}^{2}\begin{pmatrix} 2 \\ r \end{pmatrix}\left(B^T\right)^{2-r}\otimes A^r, \end{aligned}$$

the result is true when $n = 1$ and $n = 2$. Now, assume that is is true for an arbitrary integer $k \geq 1$; that is:

$$\left(I_m \otimes A + B^T \otimes I_n\right)^k = \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} \left(B^T\right)^r \otimes A^{k-r}.$$

Then,

$$\begin{aligned}
\left(I_m \otimes A + B^T \otimes I_n\right)^{k+1} &= \left(I_m \otimes A + B^T \otimes I_n\right)^k \left(I_m \otimes A + B^T \otimes I_n\right) \\
&= \left[\sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} \left(B^T\right)^{k-i} \otimes A^i\right] \left(I_m \otimes A + B^T \otimes I_n\right) \\
&= \left[\sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} \left(B^T\right)^{k-i} \otimes A^i\right] \left(B^T \otimes I_n\right) + \left[\sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} \left(B^T\right)^{k-i} \otimes A^i\right] \left(I_m \otimes A\right) \\
&= \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} \left(B^T\right)^{k+1-i} \otimes A^i + \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} \left(B^T\right)^{k-i} \otimes A^{i+1} \\
&= \left[\begin{pmatrix} k \\ 0 \end{pmatrix} \left(B^T\right)^{k+1} \otimes I_n + \sum_{i=1}^{k} \begin{pmatrix} k \\ i \end{pmatrix} \left(B^T\right)^{k+1-i} \otimes A^i\right] \\
&+ \left[\sum_{i=0}^{k-1} \begin{pmatrix} k \\ i \end{pmatrix} \left(B^T\right)^{k-i} \otimes A^{i+1} + \begin{pmatrix} k \\ k \end{pmatrix} I_m \otimes A^{i+1}\right] \\
&= \begin{pmatrix} k+1 \\ 0 \end{pmatrix} \left(B^T\right)^{k+1} \otimes I_n + \sum_{i=1}^{k} \begin{pmatrix} k \\ i \end{pmatrix} \left(B^T\right)^{k+1-i} \otimes A^i \\
&+ \sum_{i=1}^{k} \begin{pmatrix} k \\ i-1 \end{pmatrix} \left(B^T\right)^{k+1-i} \otimes A^i + \begin{pmatrix} k+1 \\ k+1 \end{pmatrix} I_m \otimes A^{i+1} \\
&= \begin{pmatrix} k+1 \\ 0 \end{pmatrix} \left(B^T\right)^{k+1} \otimes I_n + \sum_{i=1}^{k} \left[\begin{pmatrix} k \\ i \end{pmatrix} + \begin{pmatrix} k \\ i-1 \end{pmatrix}\right] \left(B^T\right)^{k+1-i} \otimes A^i \\
&+ \begin{pmatrix} k+1 \\ k+1 \end{pmatrix} I_m \otimes A^{i+1} \\
&= \begin{pmatrix} k+1 \\ 0 \end{pmatrix} \left(B^T\right)^{k+1} \otimes I_n + \sum_{i=1}^{k} \begin{pmatrix} k+1 \\ i \end{pmatrix} \left(B^T\right)^{k+1-i} \otimes A^i \\
&+ \begin{pmatrix} k+1 \\ k+1 \end{pmatrix} I_m \otimes A^{i+1} \\
&= \sum_{i=0}^{k+1} \begin{pmatrix} k+1 \\ i \end{pmatrix} \left(B^T\right)^{k+1-i} \otimes A^i.
\end{aligned}$$

Thus, by induction, the result is true for every $k \geq 1$. □

We are ready to extract a formula for the exact solution of equation (1). The following theorem gives this formula.

Theorem 2. *The solution of* (1) *is given by the following formula*

$$X_n = \sum_{r=0}^{n} \left(\begin{pmatrix} n \\ r \end{pmatrix} A^r C B^{n-r}\right), n = 0, 1, 2, \ldots.$$

Proof. Now from (1) and by using the Definition (3) we obtain

$$VecX_{n+1} = Vec\left(AX_n + X_nB\right)$$

or an easy calculation shows that

$$VecX_{n+1} = Vec\left(AX_n\right) + Vec\left(X_nB\right). \tag{4}$$

From (4) we get

$$VecX_{n+1} = Vec\left(AX_nI_s\right) + Vec\left(I_rX_nB\right). \tag{5}$$

Substituting the given relation in (2) at (5) yields:

$$VecX_{n+1} = \left(I_s \otimes A\right)VecX_n + \left(B^{\mathrm{T}} \otimes I_r\right)VecX_n$$

or an easy calculation shows that

$$VecX_{n+1} = \left(B^{\mathrm{T}} \otimes I_r + I_s \otimes A\right)VecX_n. \tag{6}$$

From Definition (2) and (6) we have

$$VecX_{n+1} = \left(B^T \oplus A\right)VecX_n. \tag{7}$$

Now, let $G = B^T \oplus A$ and $x_n = vecX_n$. Thus, from (7) we write

$$x_{n+1} = Gx_n. \tag{8}$$

Let $c = vecC$. Hence, from (1) and (8), we obtain the following equation

$$x_{n+1} = Gx_n, x(0) = c. \tag{9}$$

The solution of (9) is given by the following formula

$$x_n = G^n c, n = 0, 1, 2, \ldots. \tag{10}$$

From (10) we obtain

$$VecX_n = \left(B^T \oplus A\right)^n VecC, n = 0, 1, 2, \ldots. \tag{11}$$

By substituting the relation $B^T \oplus A = B^{\mathrm{T}} \otimes I_r + I_s \otimes A$ in (11) we get

$$VecX_n = \left(B^{\mathrm{T}} \otimes I_r + I_s \otimes A\right)^n VecC, n = 0, 1, 2, \ldots$$

or an easy calculation shows that

$$vecX_n = \left(I_s \otimes A + B^T \otimes I_r\right)^n VecC, n = 0, 1, 2, \ldots. \tag{12}$$

elde edilir. From (3) and (12) we have

$$VecX_n = \left[\sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \left(B^T\right)^{n-r} \otimes A^r\right] VecC, n = 0, 1, 2, \ldots.$$

Therefore, we can write

$$VecX_n = \sum_{r=0}^{n} \left[\left(\begin{pmatrix} n \\ r \end{pmatrix} \left(B^T\right)^{n-r} \otimes A^r\right) VecC\right], n = 0, 1, 2, \ldots. \tag{13}$$

Now we use relation (2) in (13) and obtain

$$VecX_n = Vec \sum_{r=0}^{n} \left(\begin{pmatrix} n \\ r \end{pmatrix} A^r C B^{n-r}\right), n = 0, 1, 2, \ldots. \tag{14}$$

Thus, we obtain the solution of (1) by the following formula

$$X_n = \sum_{r=0}^{n} \left(\begin{pmatrix} n \\ r \end{pmatrix} A^r C B^{n-r}\right), n = 0, 1, 2, \ldots.$$

This completes the proof. □

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