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GENERALIZED BICONVEX FUNCTIONS AND DIRECTIONAL BIVARIATIONAL INEQUALITIES

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Abstract. Some new classes of generalized biconvex sets and biconvex functions with respect to an arbitrary function k and the bifunction $\beta(.-.)$ are introduced and studied. These new biconvex functions are nonconvex functions and include the convex functions and k-convex as special cases. We study some basic properties of generalized biconvex functions. It is shown that the minimum of generalized biconvex functions on the generalized biconvex sets can be characterized by a class of variational inequalities, which is called the directional bivariational inequalities. Using the auxiliary technique, several new inertial type methods for solving the bivariational inequalities are proposed and analyzed. Convergence analysis of the proposed methods is considered under suitable pseudomonotonicity, which are weaker conditions than monotonicity. Some open problems are also suggested for future research.

1. Introduction

In recent years, several extensions and generalizations of the convex sets and convex functions have been considered and investigated. Noor and Noor [22] introduced the concepts of the biconvex set and biconvex functions, which are playing part in various branches of pure and applied sciences. It have been shows that the biconvex set and biconvex functions are classical convex set and convex functions as special cases. Noor [17] and Noor et al [17, 21, 22, 23, 24, 25, 26, 27, 28, 29, 32] proved that the minimum of the differentiable biconvex functions on the biconnvex set can be characterized by a class of variational inequalities, which is known as the bivariational inequality. It is worth mentioning that bivariational inequalities include variational inequalities, which were introduced and studies by Stamapcchia [38]. Variational inequalities can be viewed as a novel and significant extension of the variational principles, the origin of which can be traced back to Euler, Hamilton, Newton, Lagnarange and Bernoulli brothers. These results have inspired a great deal of subsequent work, which has expanded the role and applications of the convexity in nonlinear optimization and engineering sciences. For the recent developments in bivariational inequalities and biequilibrium problems, see [17, 21, 22, 24, 25, 26, 27, 28, 29, 32, 35] and the references therein.
In a many problems, a set may not be a convex set. To overcome this drawback, the underlying set can be made a k-convex set with respect to an arbitrary function k. Cristescu et al [1], Micherda et al [7] and Hazy [4] defined the so-called

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(h,k) convex function involving two arbitrary functions, which is a natural generalization of the usual convexity, the s-convexity in the first and second sense. Noor [16, 18] and Noor et al [19] introduced the k-convex functions and studied their characterizations. It is worth mentioning that for $k(t) = te^{i\varphi}$, the φ-convex functions were introduced and studied by Noor [16].

We would like to point out that biconvex set, biconvex functions, k-convex set and k-convex functions are distinctly different generalizations of convex sets and convex functions in various directions. These type of functions have played a leading role in the developments of various branches of pure and applied sciences. It is natural to unify these classes and investigate their characterizations. Motivated and inspired by the recent activities in these fields, we introduce some new classes of biconvex sets and biconvex functions which are called modified generalized biconvex set and generalized biconvex functions. These new class of generalized biconvex set and generalized biconvex functions include the φ-biconvex sets, φ-biconvex and Toader type k-convex sets and k-convex functions. The new class of generalized biconvex functions can be viewed as modified refinement of the (h,k) convex functions of Hazy [4]. Several new concepts are introduced and their properties have been studied. We prove that the minimum of the differential generalized biconvex functions on the generalized biconvex set can be characterized by a class of variational inequalities, which are called directional bivariational inequalities. This results inspired us to consider the directional bivariational inequalities. It is well known that the projection methods, resolvent methods and their variant forms can not be used to solve the directional bivariational inequalities due to their nature. To overcome this drawback, one usually use the auxiliary principle technique, which is mainly due to Lions et al [6] and Glowinski et al [3], which has been used [9, 11, 12, 14, 15, 20, 26, 27, 28, 29, 30, 32, 33, 34, 35] to suggest and analyze several new iterative methods for solving a wide class of unrelated problems arising in pure and applied sciences. We apply the auxiliary principle technique to suggest some inertial proximal iterative methods for solving directional bivariational inequalities. Convergence analysis of the new proposed methods is considered under pseudomonontoncity and partially strongly monotonicity. Our method of convergence proof is very simple as compared with other methods. We have tried to convey the basic characterizations of these new classes of generalized biconvex functions and their applications in optimization theory along with some open problems.

2. Preliminaries

Let K be a nonempty closed set in a normed space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and norm, respectively. Let $\beta(. - .) : K_{k\beta} \times K_{k\beta} \longrightarrow R$, be a bifunction. We now recall the following basic concepts and results of the generalized biconvex sets and biconvex functions, which are introduced and studied in [17, 21, 24, 24, 25].

Definition 1. *The set $K_{k\beta}$ is said to be generalized biconvex set with respect to arbitrary function k and the bifunction $\beta(.-.)$, if*

$$u + k(t)\beta(v-u) \in K_{k\beta}, \qquad \forall u, v \in K_{k\beta}, \quad t \in [0,1].$$

Clearly, for $k(t) = t$, the set $K_{k\beta}$ is an biconvex set K_β.

If $k(t) = t^s, \quad s \in [0,1]$, then the generalized biconvex set $K_{k\beta}$ reduces to:

$$u + t^s\beta(v-u) \in K_{k\beta}, \quad \forall u, v \in K_{k\beta}, \quad t \in [0,1],$$

which is known as Toader type $k\beta$-biconvex set and appears to be a new one. If $\beta(v-u) = v-u$, then the sets $K_{k\beta}$ reduces to the set k-convex sets K_k which have been introduced and studied in [1, 4, 7].

From now onwards, the set $K_{k\beta}$ is a generalized biconvex set, unless otherwise specified.
We now introduce the concept of generalized biconvex function with respect to an arbitrary function k and bifunction $\beta(.-.)$.

Definition 2. *The function f on $K_{k\beta}$ is called generalized biconvex function, if there exists an arbitrary function k and bifunction $\beta(.-.)$, such that*

$$f(u + k(t)\beta(v-u)) \leq (1-k(t))f(u) + k(t)f(v), \quad \forall u, v \in K_{k\beta}, \ t \in [0,1].$$

Obviously every biconvex function with $k(t) = t$ is a generalized biconvex function, but the converse may not be true.
Also for $t = 1$, the generalized biconvex function reduces to:

$$f(u + k(1)\beta(v-u)) \leq f(v), \qquad \forall u, v \in K_{k\beta}. \tag{1}$$

If $k(t) = t^s, \quad s \in [0,1]$, then we have a new class of biconvex functions, which is called Toader's type biconvex functions.

Definition 3. *The function f on $K_{k\beta}$ is said to be generalized quasi biconvex function , if there exist a function k and the bifunction $\beta(.-.)$, such that*

$$f(u + k(t)\beta(v-u)) \leq \max\{f(u), f(v)\}, \qquad \forall u, v \in K_{k\beta}, \quad t \in [0,1].$$

Definition 4. *The function f on $K_{k\beta}$ is said to be logarithmic generalized biconvex function, if there exist a function k and the bifunction $\beta(.-.)$, such that*

$$f(u + k(t)\beta(v-u)) \leq (f(u))^{1-k(t)}(f(v))^{k(t)}, \quad \forall u, v \in K_{k\beta}, \quad t \in [0,1],$$

where $f(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} f(u + k(t)\beta(v-u)) &\leq (f(u))^{1-k(t)}(f(v))^{k(t)} \\ &\leq (1-k(t))f(u) + k(t)f(v) \\ &\leq \max\{f(u), f(v)\}, \quad \forall u, v \in K_{k\beta}, \quad t \in [0,1], \end{aligned}$$

Logarithmic generalized biconvex function $\Longrightarrow$ generalized biconvex functions and generalized biconvex functions $\Longrightarrow$ quasi generalized biconvex functions, but the converse is not true.

We also need the following assumption regarding the bifunction $\beta(\cdot - \cdot)$ and the function $k(t)$.

Condition M. Let $\beta(\cdot,\cdot) : K_{k\beta} \times K_{k\beta} \to H$ satisfy assumptions

$$\beta(-k(t)\beta(v-u)) = -k(t)\beta(v-u)$$
$$\beta(v-u-k(t)\beta(v-u)) = (1-k(t))\beta(v-u), \quad \forall u,v \in K_{k\beta}, t \in [0,1].$$

3. Properties of generalized biconvex functions

In this section, we discuss the properties of generalized biconvex functions and their variant forms.

Lemma 1. *Let f be a generalized biconvex function. Then any local minimum of f on $K_{k\beta}$ is a global minimum.*

Proof. Let the generalized biconvex function f have a local minimum at $u \in K_{k\beta}$. Assume the contrary, that is, $f(v) < f(u)$ for some $v \in K_{k\beta}$. Since f is a $k\beta$-biconvex function, so

$$f(u + k(t)\beta(v-u)) \le f(u) + k(t)(f(v) - f(u)),$$

which implies that

$$f(u + k(t)\beta(v-u)) - f(u) < 0,$$

for arbitrary small $k(t) > 0$, contradicting the local minimum. □

Essentially using the technique and ideas of the classical convexity [2, 8, 36], one can easily prove the following results.

Theorem 1. *If f is a generalized biconvex function on the generalized biconvex set $K_{k\beta}$, then the level set $L_\alpha = \{u \in K_{k\beta} : f(u) \le \alpha, \alpha \in \mathbb{R}\}$ is a generalized biconvex set with respect to the function k and bifunction $\eta(.-.)$.*

Theorem 2. *The function f is a generalized biconvex function, if and only if, $epi(f) = \{(u,\alpha) : u \in K_{k\beta}, \alpha \in \mathbb{R}, f(u) \le \alpha\}$ is a generalized biconvex set with respect to the function k and bifunction $\beta(.-.)$.*

Theorem 3. *The function f is a quasi generalized biconvex function, if and only if, the level set $L_\alpha = \{u \in K_{k\beta} : f(u) \le \alpha, \alpha \in \mathbb{R}\}$ is a generalized biconvex set with respect to the function k and the bifunction $\beta(.-.)$.*

Definition 5. *The function f is said to be a pseudo generalized biconvex function with respect to the function k and the bifunction $\beta(.,.)$, if there exists a strictly positive bifunction $W(\cdot - \cdot)$ such that*

$$\begin{aligned} f(v) &< f(u) \Rightarrow \\ f(u + k(t)\beta(v,u)) &\le f(u) + k(t)(k(t)-1)W(u-v), \qquad \forall u,v \in K_{k\beta}, \quad t \in (0,1). \end{aligned}$$

Theorem 4. *If the function f is a generalized biconvex function, then f is pseudo generalized biconvex function.*

Proof. Without loss of generality, we assume that $f(v) < f(u)$, $\forall u, v \in K_{k\beta}$. For every $t \in [0,1]$, we have

$$\begin{aligned} f(u+k(t)\beta(v-u)) &\leq (1-k(t))f(u)+k(t)f(v) \\ &< f(u)+k(t)(k(t)-1)\{f(u)-f(v)\} \\ &= f(u)+k(t)(k(t)-1)W(u-v), \end{aligned}$$

where $W(u,v) = f(u) - f(v) > 0$. Thus, it follows the function f is a pseudo generalized biconvex function, which is the required result. □

Theorem 5. *Let f be a generalized biconvex function. If $g : L \to \mathbb{R}$ is a nondecreasing function, then $g \circ f$ is a generalized biconvex function.*

Proof. Since f is a generalized biconvex function and g is decreasing, we have, $\forall u, v \in K_{k\beta}, \quad t \in [0,1]$

$$\begin{aligned} g \circ f(u+k(t)\beta(v-u)) &\leq g[(1-k(t))f(u)+k(t)f(v)] \\ &\leq (1-k(t))g \circ f(u)+k(t)g \circ f(v), \end{aligned}$$

from which it follows that $g \circ f$ is a generalized biconvex function. □

We now introduce the concept of k-directional derivative.

Definition 6. *We define the k-directional derivative of f at a point $u \in K_{k\beta}$ in the direction $v \in K_{k\beta}$ by*

$$\begin{aligned} Df(u,\beta(v-u)) : &= f'_{k\beta}(u;\beta(v-u)) \\ &= \lim_{k(t)\to 0^+}\{\frac{f(u+k(t)\beta(v-u))-f(u)}{k(t)}\}. \end{aligned}$$

Note that for $k(t) = t$ and $\beta(v-u) = v$, the k-directional derivative of f at $u \in K$ in the direction $v \in K$ coincides with the usual directional derivative of f at u in a direction v given by

$$Df(u,v) := f'(u;v) = \lim_{t\to 0^+}\frac{f(u+tv)-f(u)}{t}.$$

It is well known that the function $v \to f'_{k\beta}(u;\beta(v-u))$ is subadditive, positively homogeneous.

Definition 7. *The differentiable function f on $K_{k\beta}$ is said to be k-biconvex, if*

$$f(v)-f(u) \geq f'_{k\beta}(u;\beta(v-u)), \quad \forall u, v \in K_{k\beta},$$

where $f'_{k\eta}(u;\beta(v-u))$ is the k-directional derivative of f at $u \in K_{k\beta}$.

Theorem 6. *Let f be a k-differential generalized biconvex function on the generalized biconvex set $K_{k\beta}$. Then the function $v \to f'_{k\beta}(u;\beta(v-u))$ is positively homogeneous and generalized biconvex function.*

Proof. It is follow from the definition of the k-directional derivative that $f'_{k\beta}(u;\lambda\beta(v-u)) = \lambda f'_{k\beta}(u;\beta(v-u))$, whenever $v \in K_{k\beta}$ and $\lambda \geq 0$. Thus the function $v \to f'_{k\beta}(u;\beta(v-u))$ is positively homogeneous.

To prove the generalized biconvexity of the function $v \to f'_{k\beta}(u;\beta(v-u))$, we consider $\forall u, v, z \in K_{k\beta}, \quad k(t) \geq 0,\ \lambda \in (0,1)$,

$$\begin{aligned}
& \frac{1}{t}[f(u + k(t)(\lambda v + (1-\lambda)\beta(v-z))) - f(u)] \\
= \ & \frac{1}{k(t)}[f(\lambda(u + k(t)\beta(v-u)) + (1-\lambda)(u + k(t)\beta(z-u))) - f(u)] \\
\leq \ & \frac{1}{k(t)}[\lambda f(u + k(t)\beta(v,u)) + (1-\lambda)f(u + k(t)\beta(z-u)) - f(u)] \\
= \ & \lambda \frac{f(u + k(t)\beta(v-u)) - f(u)}{k(t)} + (1-\lambda)\frac{f(u + k(t)\beta(z-u)) - f(u)}{k(t)}. \quad (2)
\end{aligned}$$

Taking the limit as $k(t) \to 0^+$ in (2), we have

$$f'_{k\beta}(u;\lambda\beta(v-u) + (1-\lambda)z) \leq \lambda f'_{k\beta}(u;\beta(v-u)) + (1-\lambda)f'_{k\beta}(u;\beta(z-u)),$$

which shows that the function $v \to f'_{k\beta}(u;\beta(v-u))$ is generalized biconvex, which is the required result. □

For $k(t) = t$, the generalized biconvex function f becomes the biconvex function and the generalized biconvex set K_k is an biconvex set.

Theorem 7. *Let the function $f : K_{k\beta} \to \mathbb{R}$ be a k-differentiable generalized biconvex function such that $k(0) = 0$, and (1) holds. If Condition M holds, then the following statements are equivalent.*

(1) f is a generalized biconvex function.
(2) $f(v) - f(u) \geq f'(u;\beta(v-u)), \quad \forall u, v \in K_{k\beta}$.
(3) $k\beta$-directional derivative $f'_{k\beta}(\cdot - \cdot)$ of f is $k\beta$-monotone, that is,

$$f'_{k\eta}(u;\beta(v-u)) + f'_{k\beta}(v;\beta(u-v)) \leq 0, \quad \forall u, v \in K_{k\beta}.$$

Proof. Let f be a generalized biconvex function. Then

$$f(u + k(t)\beta(v,u)) \leq f(u) + k(t)\{f(v) - f(u)\} \quad \forall u, v \in K_{k\beta}, \quad t \in [0,1],$$

which can be written as

$$(f(v) - f(u)) \geq \{\frac{f(u + k(t)\beta((v-u)) - f(u)}{k(t)}\}. \quad (3)$$

Taking the limit as $k(t) \to 0^+$ in (3), we have

$$f(v) - f(u) \geq f'_{k\beta}(u;\beta(v-u)), \quad \forall u, v \in K_{k\beta}, \quad (4)$$

showing that the generalized biconvex function f is a generalized biconvex function.

Changing the role of u and v in (4), we have

$$f(u) - f(v) \geq f'_{k\beta}(v;\beta(u-v)), \quad \forall u, v \in K_{k\beta}, \quad (5)$$

Adding (4) and (5), we have

$$f'_{k\beta}(u;\beta(v-u)) + f'_{k\beta}(v;\beta(u-v)) \leq 0, \quad \forall u, v \in K_{k\beta}, \quad (6)$$

which shows that the k-directional derivative $f'_{k\beta}(\cdot - \cdot)$ is $k\beta$-monotone.

Conversely, let (6) hold. Since $K_{k\beta}$ is a $k\beta$-biconvex set, so

$$\forall u, v \in K_{k\beta}, \quad t \in [0,1], \quad v_t = u + k(t)\beta(v-u) \in K_{k\beta}.$$

Replacing v by v_t in (6) and simplifying, we have

$$f'_{k\beta}(v_t; \beta(v-u)) \geq f'_{k\beta}(u; \beta(v-u)), \quad \forall u, v \in K_{k\beta}. \tag{7}$$

Consider the auxiliary function

$$\begin{aligned} \zeta(t) &= f(u + k(t)\beta(v-u))) - f(u) \\ &\quad + t f'_{k\beta}(u; \beta(v-u)), \forall u, v \in K_{k\beta}. \end{aligned} \tag{8}$$

Using $k(0) = 0$, we have

$$\begin{aligned} \zeta(0) &= 0 \quad , \\ \zeta(1) &= f(u + k(1)\beta(v-u)) - f(u) + f'_{k\beta}(u : \beta(v-u)). \end{aligned} \tag{9}$$

Since f is differentiable, so the function $\zeta(t)$ is also differentiable. Hence, using (7), we have

$$\begin{aligned} \zeta'(t) &= f'(u + k(t)\beta(v-u)), \beta(v-u)) \\ &\geq f'_{k\beta}(u; \beta(v-u)). \end{aligned} \tag{10}$$

Integrating the inequality (10) on the interval $[0,1]$ and using (9), we have

$$\begin{aligned} f(u + k(1)\beta(v-u)) - f(u) + f'_{k\beta}(u : \beta(v-u)) &= \zeta(1) - \zeta(0) \\ &\geq \int_0^1 f'_{k\beta}(u; \beta(v-u)) dt \\ &= f'_{k\beta}(u; \beta(v-u)), \end{aligned}$$

from which, using (1), we obtain

$$f'_{k\beta}(u; \beta(v-u)) \leq f(u + k(1)\beta(v-u)) - f(u) \leq f(v) - f(u).$$

which is the required(4).
Now from (4), we have

$$\begin{aligned} f(v) - f(u + k(t)\beta(v-u)) &\geq f'_{k\beta}(u + k(t)\beta(v-u)); \beta(v - (u + k(t))\beta(v-u))) \\ &= (1 - k(t)) f'_{k\beta}(u + k(t)\beta(v-u)); \beta(v-u))). \end{aligned} \tag{11}$$

$$\begin{aligned} f(u) - f(u + k(t)\beta(v-u)) &\geq f'_{k\beta}(u + k(t)\beta(v-u)); \beta(u - (u + k(t))\beta(v-u))) \\ &= -k(t) f'_{k\beta}(u + k(t)\beta(v-u)); \beta(v-u))). \end{aligned} \tag{12}$$

Multiplying (11) by $k(t)$, (12) by $(1 - k(t))$ and adding the resultant, we have

$$f(u + k(t)\eta(v,u)) \leq f(u) + k(t)\{f(v) - f(u)\} \quad \forall u, v \in K_{k\eta}, \quad \in [0,1],$$

which shows that the function f is a generalized biconvex function. $\square$

Theorem 8. *Let the differential $f'_{k\beta}(.-.)$ of the generalized biconvex function f be Lipschitz continuous with constant $\xi \geq 0$. If $k(0) = 0$, then*

$$\begin{aligned} & f(u + k(1)\beta(v-u)) - f(u) \\ & \leq f'_{k\beta}(u;\beta(v-u)) + \xi\|\beta(v-u)\|^2 \int_0^{\prime} k(t)dt, \quad \forall u, v \in K_{k\beta}. \end{aligned} \tag{13}$$

Proof. Since $K_{k\beta}$ is a generalized biconvex set, $\forall u, v \in K_{k\beta}, \quad t \in [0,1]$, we consider the function

$$\varphi(t) = f(u + k(t)\beta(v-u)) - f(u) - tf'_{k\beta}(u;\beta(v-u)).$$

Using $k(0) = 0$, we obtain

$$\varphi(0) = 0, \quad \varphi(1) = f(u + k(1)\beta(v-u)) - f(u) - f'_{k\beta}(u;\beta(v-u)).$$

Also

$$\varphi'(t) = f'_k(u + k(t)\beta(v-u);\beta(v-u)) - f'{}_k(u;\beta(v-u)). \tag{14}$$

Integrating (14) on the interval $[0,1]$ and using the Lipschitz continuity of $f'_k(.-.)$ with constant $\beta \geq 0$, we have

$$\begin{aligned} \varphi(1) &= f(u + k(1)\beta(v-u)) - f(u) - f'_k(u;\beta(v-u)) \\ &\leq \int_0^1 |\varphi'(t)|dt \\ &= \int_0^{\prime} |f'_k(u + k(t)\beta(v-u));\beta(v-u)) - f'{}_k(u;\beta(v-u))|dt \\ &\leq \xi \int_0^{\prime} k(t)\|\beta(v-u)\|^2 dt = \xi\|\beta(v-u)\|^2 \int_0^{\prime} k(t)dt, \end{aligned}$$

□

4. Directional bivariational inequalities

In this section, we introduce and consider a new class of bivariational inequalities, which is called directional bivariational inequality.

For given bifunctions $D(.,.), \beta(.-.) : K_{k\beta} \times K_{k\beta} \longrightarrow R$, we consider the problem of finding $u \in K_{k\beta}$ such that

$$D(u, \beta(v-u)) \geq 0, \quad \forall v \in K_{k\beta}, \tag{15}$$

which is called the *directional bivariational inequality.*

We now show that the inequality (15) naturally arises as a minimum of the k-differentiable generalized biconvex functions on the generalized biconvex sets. This is the main motivation of our next result.

Theorem 9. *Let f be a k-differentiable generalized biconvex function on the generalized biconvex set $K_{k\beta}$. Then the $u \in K_{k\beta}$ is the minimum of the k-differentiable generalized biconvex function f on the generalized biconvex set $K_{k\beta}$, if and only if, $u \in K_{k\beta}$ satisfies the inequality*

$$f'_{k\beta}(u;\beta(v-u)) \geq 0, \quad \forall u, v \in K_{k\beta}. \tag{16}$$

Proof. Let $u \in K_{k\beta}$ be a minimum of the generalized biconvex function f. Then

$$f(u) \leq f(v), \quad \forall v \in K_{k\beta}. \tag{17}$$

Since $K_{k\beta}$ is generalized biconvex set, so, $\forall u, v, \in K_{k\beta}, \quad t \in [0,1]$, $v_t = u + k(t)\beta(v-u) \in K_{k\beta}$. Taking $v = v_t$ in (17), we have

$$f(u) \leq f(v_t) = f(u + k(t)\beta(v-u)),$$

which implies that

$$\frac{f(u + k(t)\beta(v-u)) - f(u)}{k(t)} \geq 0.$$

Taking the limit as $t \to 0^+$ in the above inequality, we have

$$f'_k(u; \beta(v-u)) \geq 0 \quad \forall v \in K_{k\beta},$$

the required (16).
Conversely, let $u \in K_{k\beta}$ be a solution of (16). Since f is a generalized biconvex function, it follows that

$$f(v) - f(u) \geq f'_{k\beta}(u; \beta(v-u)) \geq 0,$$

which implies that

$$f(u) \leq f(v), \quad \forall v \in K_{k\beta},$$

showing that $u \in K_{k\beta}$ is the minimum of the generalized biconvex function f, the required result. □

The inequality of the type (16) is called the directional bivariational inequality, which is a special case of directional bivariational inequality (15).

For $k(t) = t$, the generalized biconvex functions reduces to biconvex function, then the problem (16) coincides with classical directional variational inequalities. It is worth mentioning that even the directional variational inequalities have not been studied in the literature.

We now discuss some important special cases of directional bivariational inequalities.

Special Cases

(I). We note that, if $K_{k\beta} \equiv K_\beta$, the biconvex set in H, then problem (15) is equivalent to finding $u \in K_\beta$ such that

$$D(u, \beta(v-u)) \geq 0, \qquad \forall v \in K_\beta. \tag{18}$$

Inequality of type (18) is called the *directional bivariational inequality,* which appears to be a new one.

(II). If $D(u, \beta(v-u)) = \langle Tu, \beta(v-u) \rangle, \quad K_{k\beta} = K_\beta$, where T is a nonlinear operator, then problem (15) is equivalent to finding $u \in K_\beta$ such that

$$\langle Tu, \beta(v-u) \rangle \geq 0, \quad \forall v \in K_\beta, \tag{19}$$

which is called the bivariational inequality.

(III). If $D(u, \beta(v, u)) = \langle Tu, v - u)\rangle$, where T is a nonlinear operator and $K_{k\beta} = K$, the convex set, then problem (19) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K, \tag{20}$$

which is called t variational inequality, introduced and studied by Stampachia [38]. It has been shown a wide class of obstacle boundary value and initial value problems which arise in various branches of pure and applied sciences can be studied in the general framework of variational inequalities (37). For the applications, numerical methods, sensitivity analysis, dynamical system, merit functions and other aspects of variational inequalities, see [3, 5, 6, 10, 13, 14, 15, 17, 18, 19, 21, 22, 24, 25, 26, 27, 28, 29, 32, 33, 34, 35, 38, 39, 40] and the references therein.

It is worth mentioning that for suitable and appropriate choice of the operators, generalized biconnvex sets and spaces, one can obtain a wide class of variational inequalities and optimization programming. This shows that the directional bivariational inequalities are quite flexible and unified ones.

We now recall the following concepts and results.

Definition 8. *A bifunction $D(.,.) : H \times H \to H$ is said to be:*
(i) $k\beta$-monotone, if and only if,

$$D(u, \beta(v - u)) + D(v, \beta(u - v)) \leq 0, \quad \forall u, v \in H.$$

(ii) $k\beta$-pseudomonotone, if and only if,

$$D(u, \beta(v - u))) \geq 0 \quad \textit{implies that} \quad - D(v, \beta(u - v)) \geq 0, \quad \forall u, v \in H.$$

(iii) partially relaxed strongly $k\beta$-monotone, if and only if, there exists a constant $\alpha > 0$ such that

$$D(u, \beta(v - u))) + D(v, \beta(z - v)) \leq \alpha \|\beta(z - u)\|^2, \quad \forall u, v, z \in H.$$

Note that for $z = u$, partially relaxed strongly $k\beta$-monotonicity reduces to β-monotonicity. It is known that β-monotonicity implies kbeta-pseudomonotonicity; but the converse is not true.

We also recall the well-known result.

$$2\langle u, v\rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2, \quad \forall u, v \in H. \tag{21}$$

Theorem 10. *Let the bifucntion $D(.,.)$ be $k\beta$-pseudo-monotone, hemicontinuous and $\lim_{t \longrightarrow 0} k(t) = 0$. If Condition M holds, then the directional bivariational inequality is equivalent to finding $u \in K_{k\beta}$ such that*

$$D(v, \beta(u - v)) \geq 0, \quad \forall v \in K_{k\beta}. \tag{22}$$

Proof. Let $u \in K_{k\beta}$ be a solution of inequality (15). Then, using the $k\beta$-pseudo monotonicity of the bifunction $D(.,.)$, we have

$$-D(v, \beta(u-v)) \geq 0, \quad \forall v \in K_{k\beta}. \tag{23}$$

Since $K_{k\beta}$ is a generalized biconvex set, so, $\forall u, v \in K_{k\beta}, \quad t \in [0,1]$,
$v_t = u + k(t)\beta(v-u) \in K_{k\beta}$.
Replacing v by v_t in (23) and using Condition M, we obtain

$$\begin{aligned} -D(v_t, \beta(u-v_t)) &= -D(u+k(t)\beta(v-u); \beta(u-(u+k(t))\beta(v-u))) \\ &= k(t)D(u+k(t)\beta(v-u); \beta(v-u)) \geq 0, \end{aligned}$$

which implies that

$$D(u+k(t)\beta(v-u), \beta(v-u)) \geq 0, \quad \forall v \in K_{k\beta}$$

Using the hemicontinuity of the bifunction $D(.,.)$ and taking the limit, we obtain the inequality (15), since $\lim_{t \longrightarrow 0} k(t) = 0$. □

Remark 1. *We would like to mention that the inequality of the type* (23) *is known as the Minty directional bivariational inequality or dual directional bivariational inequality. Using this equivalent result, one can show that the solution set of the directional bivariational inequalities is a closed generalized biconvex set.*

Due to the inherent nonlinearity, the projection method and its variant form can not be used to suggest the iterative methods for solving these directional bivariational inequalities. To overcome these drawback, we now use the auxiliary principle technique of Glowinski et al.[3] as developed in [15, 33, 34, 40] to suggest and analyze some iterative methods for solving the directional bivariational inequality (15). This technique does not involve the concept of the projection and the resolvent, which is the main advantage of this technique.

For a given $u \in K_{k\beta}$ satisfying (15), consider the problem of finding $w \in\in K_{k\beta}$ such that

$$\rho D((w+\zeta(u-w), \beta(v-w)) + \langle w-u, v-w \rangle \geq 0, \forall v \in K_{k\beta}, \tag{24}$$

where $\rho > 0$ and $\zeta \in [0.1]$ are constants. Inequality of type (24) is called the auxiliary directionally bivariational inequality. Note that if $w = u$, then w is a solution of (15). This simple observation enables us to suggest the following iterative method for solving the problem(15).

Algorithm 1. *For a given $u_0 \in K_{k\beta}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \rho D(u_{n+1}+\zeta(u_n-u_{n+1})), \beta(u-u_{n+1})) \quad &+ \quad \langle u_{n+1}-u_n, v-u_{n+1} \rangle \\ &\geq 0, \quad \forall v \in K_{k\beta}. \end{aligned} \tag{25}$$

Algorithm 2 is called the hybrid proximal point algorithm for solving the problem(15).
If $\zeta = 0$, then Algorithm 1 reduces to:

Algorithm 2. *For a given $u_0 \in K_\beta$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho D(u_{n+1}, \beta(v - u_{n+1})) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K_{k\beta},$$

which is known as the proximal point algorithm for solving directional bivariational inequalities (15).
We now consider the convergence criteria of Algorithm 2 and this is the main motivation of our next result.

Theorem 11. *Let the operator $D(.,.) : K_{k\beta} \times K_{k\beta} \longrightarrow H$ be $k\beta$-pseudomonotone. If u_{n+1} is the approximate solution obtained from Algorithm 2 and $u \in K_{k\beta}$ is a solution of (15), then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2. \tag{26}$$

Proof. Let $u \in K_{k\beta}$ be a solution of (15). Then

$$-D(g(v, \beta(u - v)) \geq 0, \quad \forall v \in K_{k\beta}, \tag{27}$$

since $D(.,.)$ is $k\beta$-pseudomonotone.
Taking $v = u_{n+1}$ in (27), we have

$$-D(g(u_{n+1}), \beta(u, u_{n+1})) \geq 0. \tag{28}$$

Setting $v = u$ in (16), and using (37), we have

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq -\rho D(u_{n+1}, \eta(u, u_{n+1})) \geq 0. \tag{29}$$

Setting $v = u - u_{n+1}$ and $u = u_{n+1} - u_n$ in (21), we obtain

$$\begin{aligned} 2\langle u_{n+1} - u_n, u - u_{n+1}\rangle &= \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2 \\ &\quad -\|u - u_{n+1}\|^2. \end{aligned} \tag{30}$$

From (29) and (30), we obtain (26), which is the required result. □

Theorem 12. *Let H be a finite dimension subspace and let u_{n+1} be the approximate solution obtained from Algorithm 2. If $u \in K_{k\beta}$ is a solution of (15), then* $\lim_{n\longrightarrow\infty} u_n = u$.

Proof. Let $u \in K_{k\beta}$ be a solution of(15). Then it follows from (26) that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \|u_n - u_{n+1}\|^2 \leq \|u_0 - u\|^2,$$

which implies that

$$\lim_{n\longrightarrow\infty} \|u_n - u_{n+1}\| = 0. \tag{31}$$

Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$ and let the subsequence $\{u_j\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in K_{k\beta}$. replacing u_n by u_{n_j} in (31) and taking the limit $n_j \longrightarrow \infty$ and using (37), we have

$$D(\hat{u}, \beta(v - \hat{u})) \geq 0, \quad \forall v \in K_{k\beta},$$

which implies that $\hat{u}$ solves the directional bivariational inequality (15) and

$$\|u_n - u_{n+1}\|^2 \leq \|\hat{u} - u_n\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\longrightarrow\infty} u_n = \hat{u}$. the required result. □

It is well-known that to implement the proximal point methods, one has to calculate the approximate solution implicitly, which is in itself a difficult problem. To overcome this drawback, we suggest another iterative method, the convergence of which requires only partially relaxed strongly monotonicity, which is a weaker condition that monotonicity.
If $\zeta = 1$, then Algorithm 1 reduces to:

Algorithm 3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho D(u_n, \beta(v - u_{n+1})) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \forall v \in K_{k\beta}. \tag{32}$$

We now study the convergence of Algorithm 3 and this is the main motivation of our next result.

Theorem 13. *Let the operator $D(.,.)$ be partially relaxed strongly $k\beta$-monotone with constant $\alpha > 0$. If u_{n+1} is the approximate solution obtained from Algorithm 3 and $u \in K_{k\beta}$ is a solution of (15), then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \{1 - 2\rho\alpha\}\|u_n - u_{n+1}\|^2. \tag{33}$$

Proof. Let $u \in K_{k\beta}$ be a solution of (15). Then

$$D(u, \beta(v - u)) \geq 0, \quad \forall v \in K_{k\beta}. \tag{34}$$

Taking $v = u_{n+1}$ in (34), we have

$$D(u, \beta(u_{n+1} - u)) \geq 0. \tag{35}$$

Letting $v = u$ in (32), we obtain

$$\rho D(u_n, \beta(u - u_{n+1})) + \langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq 0,$$

which implies that

$$\begin{aligned}\langle u_{n+1} - u_n, u - u_{n+1}\rangle &\geq -\rho D(u_n, \beta(u - u_{n+1})) \\ &\geq -\rho\{D(u_n, \beta(u, u_{n+1})) + D(u, \beta(u_{n+1} - u)\} \\ &\geq -\alpha\rho\|u_n - u_{n+1}\|^2. \end{aligned} \tag{36}$$

since $D(.,.)$ is partially relaxed strongly monotone with constant $\alpha > 0$.

Combining (35) and (36), we obtain the required result (33). □

If $\zeta = \frac{1}{2}$, then Algorithm 1 reduces to:

Algorithm 4. *For a given $u_0 \in K_{k\beta}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho D(\frac{u_{n+1} + u_n}{2})), \beta(u - u_{n+1})) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \forall v \in K_{k\beta}.$$

which is called midpoint proximal method. Using essentially the technique of Theorem 12, one can study the convergence analysis of Algorithm 4.

Recently inertial methods are being developed for solving the variational inequalities and related optimization problems. Polyak [37] introduced these inertial methods to speed up the fast convergence criteria of the iterative methods. For more details and numerical implementation of these methods, see [15, 31, 33, 34] and the references therein. Using again the auxiliary principle technique, we can suggest some inertial iterative methods.

For a given $u \in K_{k\beta}$ satisfying (15), consider the problem of finding $w \in K_{k\beta}$ such that

$$\rho D((w+\zeta(u-w)),\beta(v-w)) \quad + \quad \langle w-u+\alpha(u-u), v-w\rangle \geq 0, \quad \forall v \in K_{k\beta}, \tag{37}$$

where $\rho > 0, \alpha$ and ζ are constants. Note that, if $w = u$, then w is a solution of (15). Consequently, one can suggest and analyze the following iterative method for solving the directional bivariational inequality (15).

Algorithm 5. *For a given $u_0, u_1 \in K_{k\eta}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho D((u_{n+1}+\zeta(u_n-u_{n+1})),\beta(v-u_{n+1})) + \langle u_{n+1}-u_n+\alpha(u_n-u_{n-1}), v-u_{n+1}\rangle \geq 0, \quad \forall v \in K_{k\beta}.$$

Algorithm 5 is called the inertial proximal point algorithm for solving directional bivariational inequality (15).

If $\zeta = 1$, then Algorithm 5 reduces to:

Algorithm 6. *For a given $u_0, u_1 \in K_{k\beta}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho D(u_n,\beta(v,-u_{n+1})) + \langle u_{n+1}-u_n+\alpha(u_n-u_{n-1}), v-u_{n+1}\rangle \geq 0, \quad \forall v \in K_{k\beta}.$$

Algorithm 7 is called the inertial explicit algorithm for solving directional bivariational inequality (15).
If $\zeta = 0$, then Algorithm 5 reduces to:

Algorithm 7. *For a given $u_0, u_1 \in K_{k\beta}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho D(u_{n+1},\beta(v,-u_{n+1})) \quad + \quad \langle u_{n+1}-u_n+\alpha(u_n-u_{n-1}), v-u_{n+1}\rangle \geq 0, \quad \forall v \in K_{k\beta}.$$

Algorithm 7 is called the inertial implicit algorithm for solving directional bivariational inequality (15). Convergence analysis of Algorithm 5 and Algorithm 7 can be studied using the above ideas and techniques of Noor [15] and Noor at al. [31, 33].
We again the apply the auxiliary principle technique to suggest another inertial

type algorithm for solving the problem(15).
For a given $u \in K_{k\beta}$ satisfying (15), consider the problem of finding $w \in K_{k\beta}$ such that

$$\begin{aligned} &\rho D((u-\zeta(u-u)), \beta(v-w)) \\ &+\langle w-(u-\alpha(u-u)), v-w\rangle \geq 0, \quad \forall v \in K_{k\beta}, \end{aligned} \tag{38}$$

where α is a constant. Note that, if $w=u$, then w is a solution of (15). Consequently, one can suggest and analyze the following iterative method for solving the directional bivariational inequality (15).

Algorithm 8. *For a given $u_0, u_1 \in K_{k\beta}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} &\rho D((u_n-\alpha(u_n-u_{n-1})), \beta(v-u_{n+1})) \\ + \quad &\langle u_{n+1}-(u_n-\alpha(u_n-u_{n-1})), v-u_{n+1}\rangle \geq 0, \quad \forall v \in K_{k\beta}. \end{aligned}$$

which is known as the inertial iterative method.

Algorithm 9) is equivalent to the following two-step method.

Algorithm 9. *For a given $u_0, u_1 \in K_{k\beta}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (u_n-\alpha(u_n-u_{n-1}) \\ \rho D(y_n, \beta(v-u_{n+1})) &+ \langle u_{n+1}-y_n, v-u_{n+1}\rangle \geq 0, \quad \forall v \in K_{k\beta}. \end{aligned}$$

which is known as the two-step (predictor-corrector) inertial iterative method.

Remark 2. *For $k(t)=k$, the directional bivariational inequalities reduce to bivariational inequalities. Interested readers may explore the applications and other aspects such as gap functions, error bounds, well-posedness, sensitivity of directional bivariational inequalities in various branches of pure and applied sciences.*

Conclusion

In this paper, we have introduced and studied some new classes biconvex functions, which is called the generalized biconvex functions. These concepts are more general and unifying ones than the previous ones. Several new properties of these generalized biconvex functions are considered and their relations with previously known results are highlighted. It is shown that the optimality conditions of the differentiable generalized biconvex functions can be characterised by a class of directional bivariational inequalities. This result is used to introduce some new classes of directional bivariational inequalities (15). Some new inertial type proximal methods are proposed using the auxiliary principle technique for solving the bivariational inequalities. Convergence analysis is considered under some suitable pseudomonotone conditions. It is itself an interesting problem to develop some efficient numerical methods for solving directional bivariational inequalities along with their applications in pure and applied sciences. Despite the current activity, much clearly remains to be done in these fields. It is expected that the ideas and techniques of this paper may be starting point for future research activities.

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