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GENERALIZED FRESNEL INTEGRALS AND THE DIRAC REPRESENTATIVE SEQUENCES GENERATED BY THEM

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Abstract. The Fresnel cosine and sine integrals are generalized in the Euclidean space $\mathbb{R}^n$, $n \geq 2$. Two families of functions are associated with them and it is shown that they converge in the sense of distributions towards Dirac's distribution. The properties of these Dirac representative sequences are established, and the obtained results are exemplified in cases $n = 1, 2, 3$.

1. Introduction

The theory of distributions (generalized functions) is used not only in many mathematical disciplines, such as functional analysis and applied mathematics, but also in physics and engineering sciences. It represents a general and unitary background regarding the mathematical representation of some physical quantities and the analysis of some discontinuous phenomena.

The Dirac representative sequences have many applications not only in the solving of boundary value problems from engineering sciences, but also in the representation of physical quantities with punctual support, such as force and moment concentrated in a point [1], [2].

We shall denote by $\mathcal{D}(\mathbb{R}^n)$ Schwartz' space of indefinitely differentiable functions with compact support, and by $\mathcal{D}'(\mathbb{R}^n)$ the set of linear continuous functionals defined on $\mathcal{D}(\mathbb{R}^n)$.

With the help of these representative Dirac sequences we can express the solutions of some boundary problems, such as the ones regarding the transverse vibrations of elastic bars [3].

Thus, in this case, we consider the families of functions

$$f_t(x) = \frac{1}{\sqrt{2\pi ct}} \cos\frac{x^2}{4ct},\ g_t(x) = \frac{1}{\sqrt{2\pi ct}} \sin\frac{x^2}{4ct},\ x \in \mathbb{R},\ t > 0,\ c = const. \tag{1}$$

which have the proprieties

$$I_1 = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi ct}} \cos\frac{x^2}{4ct}\mathrm{d}x = \int_{\mathbb{R}} f_t(x)\mathrm{d}x = 1, \tag{2}$$

$$I_2 = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi ct}} \sin\frac{x^2}{4ct}\mathrm{d}x = \int_{\mathbb{R}} g_t(x)\mathrm{d}x = 1. \tag{3}$$

Hence $I_1 = I_2 = 1$.

Consequently, two families of sequences correspond to the Fresnel integrals, namely $(f_t)_{t>0},\ (g_t)_{t>0}$ given by (1).

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The families of functions (1) are Dirac representative sequences, since we have

$$\lim_{t\to+0} f_t(x) = \lim_{t\to+0}\frac{1}{\sqrt{2\pi ct}}\cos\frac{x^2}{4ct} = \delta(x), \tag{4}$$

$$\lim_{t\to+0} g_t(x) = \lim_{t\to+0}\frac{1}{\sqrt{2\pi ct}}\sin\frac{x^2}{4ct} = \delta(x), \tag{5}$$

where the limit is considered in the sense of distributions from $\mathcal{D}'(\mathbb{R})$.

From here it follows

$$\lim_{t\to+0}\frac{1}{2\sqrt{2\pi ct}}\left(\cos\frac{x^2}{4ct}+\sin\frac{x^2}{4ct}\right) = \delta(x), \tag{6}$$

hence, the distribution depending on the parameter $t > 0$,

$$h_t(x) = \frac{1}{2\sqrt{2\pi ct}}\left(\cos\frac{x^2}{4ct}+\sin\frac{x^2}{4ct}\right),\ t>0$$

represents a Dirac representative sequence.

2. Fresnel cosine and sine integrals from $\mathbb{R}^n$, $n \geq 2$

Let be the functions f, $g : \mathbb{R} \to \mathbb{R}$, where

$$f(x) = \cos(ax^2),\ g(x) = \sin(ax^2),\ a > 0. \tag{7}$$

The integrals

$$I_1^* = \int_0^\infty \cos(ax^2)\mathrm{d}x, \tag{8}$$

$$I_2^* = \int_0^\infty \sin(ax^2)\mathrm{d}x, \tag{9}$$

are called Fresnel's integrals in cosine and sine [8], [6], and we have

$$I_1^* = I_2^* = \frac{1}{2}\sqrt{\frac{\pi}{2a}},\ a > 0. \tag{10}$$

Due to the parity of the functions (7) we have

$$I_1 = I_2 = \int_{\mathbb{R}} \cos(ax^2)\mathrm{d}x = \int_{\mathbb{R}} \sin(ax^2)\mathrm{d}x = \sqrt{\frac{\pi}{2a}},\ a > 0. \tag{11}$$

We will call these integrals Fresnel's integrals in cosine and sine from $\mathbb{R}$. Fresnel integrals and Cornu's spiral occurred originally in the analysis of the diffraction of light. These integrals occur in a variety of physical applications, and also in engineering sciences, for example in the study of transverse vibrations of elastic bars [4].

Let be the functions f, $g : \mathbb{R}^n \to \mathbb{R}^n$, where

$$f(x) = \cos\left(a\|x\|^{2n}\right),\ g(x) = \sin\left(a\|x\|^{2n}\right),\ \|x\| = \sqrt{x_1^2 + ... + x_n^2},\ a > 0. \tag{12}$$

Definition 1. *The integrals*

$$I_1 = \int_{\mathbb{R}^n} \cos\left(a\|x\|^{2n}\right)\mathrm{d}x, \tag{13}$$

$$I_2 = \int_{\mathbb{R}^n} \sin\left(a\|x\|^{2n}\right)\mathrm{d}x, \tag{14}$$

will be named, Fresnel cosine and sine integrals from $\mathbb{R}^n$.

Due to the parity of the functions (12) we can write

$$I_1 = 2\int_0^\infty ... \int_0^\infty \cos\left(a\|x\|^{2n}\right) \mathrm{d}x,\ I_2 = 2\int_0^\infty ... \int_0^\infty \sin\left(a\|x\|^{2n}\right) \mathrm{d}x, \tag{15}$$

where

$$I_1^* = \int_0^\infty ... \int_0^\infty \cos\left(a\|x\|^{2n}\right) \mathrm{d}x,\ I_2^* = \int_0^\infty ... \int_0^\infty \sin\left(a\|x\|^{2n}\right) \mathrm{d}x, \tag{16}$$

represents the Fresnel integrals from $\mathbb{R}_+^n = \{x \in \mathbb{R}^n,\ x_1 \geq 0, ..., x_n \geq 0\}$.

Proposition 1. *The Fresnel integrals (13), (14) are convergent, and we have*

$$I_1 = I_2 = \frac{1}{n\sqrt{2a}} \frac{\pi^{(n+1)/2}}{\Gamma\left(\dfrac{n}{2}\right)}, \tag{17}$$

where $\Gamma(x)$ *represets gamma function.*

Proof. In the n-dimensional Euclidean space $\mathbb{R}^n$ we shall use the spherical coordinates $(r, \theta_1, \theta_2, ..., \theta_{n-1}) \in \mathbb{R}^n$, whose connection with the Cartesian coordinates $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is expressed by the relations

$$\begin{aligned} x_1 &= r\sin\theta_1\sin\theta_2 ... \sin\theta_{n-2}\sin\theta_{n-1}, \\ x_2 &= r\sin\theta_1\sin\theta_2 ... \sin\theta_{n-2}\cos\theta_{n-1}, \\ x_3 &= r\sin\theta_1\sin\theta_2 ... \sin\theta_{n-3}\cos\theta_{n-2}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n-2} &= r\sin\theta_1\sin\theta_2\cos\theta_3, \\ x_{n-1} &= r\sin\theta_1\cos\theta_2, \\ x_n &= r\cos\theta_1, \end{aligned} \tag{18}$$

where

$$r \geqslant 0,\ \theta_i \in [0, \pi],\ i = \overline{1, n-2},\ \theta_{n-1} \in [0, 2\pi). \tag{19}$$

The Jacobian of the transformation (18) is

$$\begin{aligned} J(r, \theta_1, \theta_2, ..., \theta_{n-1}) &= \frac{\partial(x_1, x_2, ..., x_n)}{\partial(r, \theta_1, \theta_2, ..., \theta_{n-1})} \\ &= r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2 ... \sin\theta_{n-2}. \end{aligned} \tag{20}$$

The volume element $\mathrm{d}x = \mathrm{d}x_1...\mathrm{d}x_n$ in spherical coordinates (18) has the expression

$$\mathrm{d}x = \mathrm{d}v = J(r, \theta_1, ..., \theta_{n-1})\mathrm{d}r\,\mathrm{d}\theta_1...\mathrm{d}\theta_{n-1} = r^{n-1}\mathrm{d}r\,\mathrm{d}S_1. \tag{21}$$

where $\mathrm{d}S_1$ represents the area element of the unit radius sphere from $\mathbb{R}^n$, centered at the origin, $S_1 = \{x \in \mathbb{R}^n,\ \|x\| = 1\}$.

It results that

$$\mathrm{d}S_1 = \frac{J}{r^{n-1}}\mathrm{d}r\,\mathrm{d}\theta_1...\mathrm{d}\theta_{n-1} = \sin^{n-2}\theta_1\sin^{n-3}\theta_2...\sin\theta_{n-2}\mathrm{d}\theta_1...\mathrm{d}\theta_{n-1}. \tag{22}$$

Regarding the area of the unit radius sphere S_1 it has the expression

$$|S_1| = area S_1 = \frac{2\pi^{n/2}}{\Gamma\left(\dfrac{n}{2}\right)}. \tag{23}$$

Consequently, we can write for the integrals from (13) and (14)

$$I_1 = \int_{\mathbb{R}^n} \cos\left(a\|x\|^{2n}\right) \mathrm{d}x = \int_0^\infty \int_{S_1} \cos\left(ar^{2n}\right) r^{n-1} \mathrm{d}r \mathrm{d}S_1 = \int_0^\infty r^{n-1} \cos\left(ar^{2n}\right) \mathrm{d}r \int_{S_1} \mathrm{d}S_1 = |S_1| \int_0^\infty r^{n-1} \cos\left(ar^{2n}\right) \mathrm{d}r. \tag{24}$$

Taking into account (23) we obtain

$$I_1 = \frac{|S_1|}{n} \int_0^\infty \cos\left(ar^{2n}\right) \mathrm{d}\left(r^n\right) = \frac{2\pi^{n/2}}{n\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \cos\left(ar^{2n}\right) \mathrm{d}\left(r^n\right). \tag{25}$$

Making the change of variable $r^n = y \geq 0$, we have

$$I_1 = \frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)} \frac{1}{2}\sqrt{\frac{\pi}{2a}} = \frac{1}{n\sqrt{2a}} \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n}{2}\right)}, \tag{26}$$

hence, the formula (17).

Proceeding analogously, we have

$$I_2 = \int_{\mathbb{R}^n} \sin\left(a\|x\|^{2n}\right) \mathrm{d}x = \int_0^\infty \int_{S_1} \sin\left(ar^{2n}\right) r^{n-1} \mathrm{d}r \mathrm{d}S_1 = = \frac{|S_1|}{n} \int_0^\infty \sin\left(ar^{2n}\right) \mathrm{d}\left(r^n\right) = \frac{1}{n\sqrt{2a}} \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n}{2}\right)}, \tag{27}$$

hence, the formula (17), which proves proposition 1.1. □

Remark 1. *Although the relation (17) was proved for $n \geq 2$, it is valid for $n = 1$ as well.*

Remark 2. *From the formula (17), for $n = 1$, we will find (11). Indeed,*

$$\int_{\mathbb{R}} \cos\left(ax^2\right) \mathrm{d}x = \int_{\mathbb{R}} \sin\left(ax^2\right) \mathrm{d}x = \sqrt{\frac{\pi}{2a}}, \tag{28}$$

because $\|x\| = |x|$ and $\Gamma\left(\frac{n}{2}\right) = \sqrt{\pi}$.

In $\mathbb{R}^2$, i.e. for $n = 2$, from (11) we obtain

$$\int_{\mathbb{R}^2} \cos\left(a\|x\|^4\right) \mathrm{d}x = \int_{\mathbb{R}^2} \sin\left(a\|x\|^4\right) \mathrm{d}x = \frac{\pi}{2}\sqrt{\frac{\pi}{2a}}. \tag{29}$$

In $\mathbb{R}^3$, i.e. for $n = 3$, from (11) we obtain

$$\int_{\mathbb{R}^3} \cos\left(a\|x\|^6\right) \mathrm{d}x = \int_{\mathbb{R}^3} \sin\left(a\|x\|^6\right) \mathrm{d}x = \frac{2\pi}{3}\sqrt{\frac{\pi}{2a}}. \tag{30}$$

3. Representative Dirac sequences generated by Fresnel integrals from $\mathbb{R}^n$

Definition 2. [1] *Let $f_i : \mathbb{R}^n \to \mathbb{C},\ i \in \mathbb{N}$, be a sequence of locally integrable functions. We say that $(f_i)_{i\geq 1}$ is a Dirac representative sequence if on the space $\mathcal{D}'(\mathbb{R}^n)$ we have $\lim_{i\to\infty} f_i(x) = \delta(x)$, that is*

$$\forall\varphi \in \mathcal{D}(\mathbb{R}^n) \quad \Rightarrow \quad \lim_{i\to\infty} (f_i(x), \varphi(x)) = (\delta(x), \varphi(x)) = \varphi(0). \tag{31}$$

Let be the functions $F,\ G : \mathbb{R}^n \to \mathbb{R}$ having the expressions

$$F(x) = \frac{n\sqrt{2a}\,\Gamma\left(\dfrac{n}{2}\right)}{\pi^{\frac{n+1}{2}}} \cos\left(a\|x\|^{2n}\right), \tag{32}$$

$$G(x) = \frac{n\sqrt{2a}\,\Gamma\left(\dfrac{n}{2}\right)}{\pi^{\frac{n+1}{2}}} \sin\left(a\|x\|^{2n}\right). \tag{33}$$

Taking into account (17), the functions $F,\ G$ have the properties

$$F,\ G \in C^0(\mathbb{R}^n) \quad \text{and} \quad \int_{\mathbb{R}^n} F(x)\mathrm{d}x = \int_{\mathbb{R}^n} G(x)\mathrm{d}x = 1. \tag{34}$$

We consider the following family of functions $F_\varepsilon(x),\ G_\varepsilon(x),\ \varepsilon > 0$, having the expression

$$F_\varepsilon(x) = \frac{1}{\varepsilon^n} F\left(\frac{x}{\varepsilon}\right),\ \ G_\varepsilon(x) = \frac{1}{\varepsilon^n} G\left(\frac{x}{\varepsilon}\right), \tag{35}$$

hence

$$\begin{aligned} F_\varepsilon(x) &= \frac{n\sqrt{2a}}{\pi^{\frac{n+1}{2}}}\Gamma\left(\frac{n}{2}\right)\frac{1}{\varepsilon^n}\cos\left(\frac{a\|x\|^{2n}}{\varepsilon^{2n}}\right), \\ G_\varepsilon(x) &= \frac{n\sqrt{2a}}{\pi^{\frac{n+1}{2}}}\Gamma\left(\frac{n}{2}\right)\frac{1}{\varepsilon^n}\sin\left(\frac{a\|x\|^{2n}}{\varepsilon^{2n}}\right). \end{aligned} \tag{36}$$

Proposition 2. *The families of functions (36) are Dirac representative sequences, hence, we have*

$$\lim_{\varepsilon\to+0} F_\varepsilon(x) = \lim_{\varepsilon\to+0} G_\varepsilon(x) = \delta(x). \tag{37}$$

Proof. For any function $\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ we have

$$(F_\varepsilon(x),\ \varphi(x)) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} F\left(\frac{x}{\varepsilon}\right)\varphi(x)\mathrm{d}x,\ \ \varepsilon > 0. \tag{38}$$

Performing the change of variable $x = \varepsilon u,\ x_k = \varepsilon u_k,\ k = \overline{1,n}$ the Jacobian of the transformations is

$$J(u) = \frac{\partial(x_1, x_2, ..., x_n)}{\partial(u_1, u_2, ..., u_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & ... & \frac{\partial x_1}{\partial u_n} \\ ... & ... & ... & ... \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & ... & \frac{\partial x_n}{\partial u_n} \end{vmatrix}$$
$$= \begin{vmatrix} \varepsilon & 0 & ... & 0 & 0 \\ 0 & \varepsilon & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \varepsilon & 0 \\ 0 & 0 & ... & 0 & \varepsilon \end{vmatrix} = \varepsilon^n; \tag{39}$$

thus we can write

$$(F_\varepsilon(x),\ \varphi(x)) = \int_{\mathbb{R}^n} F(u)\varphi(\varepsilon u)\mathrm{d}u = \int_{\mathbb{R}^n} F(u)\left[\varphi(\varepsilon u) - \varphi(0)\right]\mathrm{d}u + \varphi(0). \tag{40}$$

On the other hand, because $\int_{\mathbb{R}^n} |F(u)|\mathrm{d}u = M$ is finite, we have

$$\left|\int_{\mathbb{R}^n} F(u)\left[\varphi(\varepsilon u) - \varphi(0)\right]\mathrm{d}u\right| \le \sup_{\mathbb{R}^n} |\varphi(\varepsilon u) - \varphi(0)| \int_{\mathbb{R}^n} |F(u)|\,\mathrm{d}u, \tag{41}$$

hence

$$\left|\int_{\mathbb{R}^n} F(u)\left[\varphi(\varepsilon u) - \varphi(0)\right]\mathrm{d}u\right| \le M \sup_{\mathbb{R}^n} \left|\varphi(\varepsilon u) - \varphi(0)\right|. \tag{42}$$

On the basis of the continuity of the function $\varphi(\varepsilon u) \in \mathcal{D}(\mathbb{R}^n)$, we obtain

$$\lim_{\varepsilon\to+0}\left|\int_{\mathbb{R}^n} F(u)\left[\varphi(\varepsilon u) - \varphi(0)\right]\mathrm{d}u\right| \le M \lim_{\varepsilon\to 0}\sup_{\mathbb{R}^n} |\varphi(\varepsilon u) - \varphi(0)| \le 0, \tag{43}$$

hence

$$\lim_{\varepsilon\to+0}\int_{\mathbb{R}^n} F(u)\left[\varphi(\varepsilon u) - \varphi(0)\right]\mathrm{d}u = 0. \tag{44}$$

Consequently, from (40) it results

$$\lim_{\varepsilon\to+0}\left(F_\varepsilon(x),\ \varphi(x)\right) = \varphi(0) = \left(\delta(x),\ \varphi(x)\right), \tag{45}$$

thus

$$\lim_{\varepsilon\to+0} F_\varepsilon(x) = \delta(x). \tag{46}$$

Proceeding analogously, for the family of sequences $G_\varepsilon,\ \varepsilon > 0$, we obtain the relations (37). □

Remark 3. *Based on the result of the above proposition, it is shown that the family of functions*

$$h_\varepsilon(x) = \frac{1}{2}\frac{n\sqrt{2a}}{\pi^{\frac{n+1}{2}}}\Gamma\left(\frac{n}{2}\right)\frac{1}{\varepsilon^n}\left[\cos\left(\frac{a\|x\|^{2n}}{\varepsilon^{2n}}\right) + \sin\left(\frac{a\|x\|^{2n}}{\varepsilon^{2n}}\right)\right],\ \ \varepsilon > 0, \tag{47}$$

represents a Dirac representative sequence.

Indeed, from (37) we have

$$\begin{gathered} \lim_{\varepsilon\to+0}\frac{1}{2}\left(F_\varepsilon(x) + G_\varepsilon(x)\right) = \delta(x), \\ \lim_{\varepsilon\to+0}\left(F_\varepsilon(x) - G_\varepsilon(x)\right) = 0, \end{gathered} \tag{48}$$

hence

$$\begin{aligned} &\lim_{\varepsilon\to+0} \frac{1}{2} \frac{n\sqrt{2a}}{\pi^{\frac{n+1}{2}}} \Gamma\left(\frac{n}{2}\right) \frac{1}{\varepsilon^n} \left[\cos\left(\frac{a\|x\|^{2n}}{\varepsilon^{2n}}\right) + \sin\left(\frac{a\|x\|^{2n}}{\varepsilon^{2n}}\right)\right] = \delta(x), \\ &\lim_{\varepsilon\to+0} \frac{n\sqrt{2a}}{\pi^{\frac{n+1}{2}}} \Gamma\left(\frac{n}{2}\right) \frac{1}{\varepsilon^n} \left[\cos\left(\frac{a\|x\|^{2n}}{\varepsilon^{2n}}\right) - \sin\left(\frac{a\|x\|^{2n}}{\varepsilon^{2n}}\right)\right] = 0. \end{aligned} \tag{49}$$

Thus $h_\varepsilon(x),\quad \varepsilon > 0$ is a representative Dirac sequence.

4. Conclusions

The Fresnel cosine and sine integrals are generalized in the Euclidean space $\mathbb{R}^n$, $n \geq 2$. Two families of functions are associated with them and it is shown the they are Dirac representative sequences.

Considering a new parameter $t > 0$, according to the relation $\varepsilon = \sqrt{4ct},\ c > 0$, relations (36) become

$$\begin{aligned} F_\varepsilon(x) = f_t(x) &= \frac{n\sqrt{2a}}{\pi^{\frac{n+1}{2}}} \Gamma\left(\frac{n}{2}\right) \frac{1}{(4ct)^{\frac{n}{2}}} \cos\left(\frac{a\|x\|^{2n}}{(4ct)^n}\right), \\ G_\varepsilon(x) = g_t(x) &= \frac{n\sqrt{2a}}{\pi^{\frac{n+1}{2}}} \Gamma\left(\frac{n}{2}\right) \frac{1}{(4ct)^{\frac{n}{2}}} \sin\left(\frac{a\|x\|^{2n}}{(4ct)^n}\right). \end{aligned} \tag{50}$$

Taking into account (37) we obtain

$$\lim_{t\to 0} f_t(x) = \lim_{t\to 0} g_t(x) = \delta(x). \tag{51}$$

From (51) for $n = 1$, we find the formulas (4), (5) and (6).

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