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TRIGONOMETRICALLY tgs-CONVEXITY

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Abstract. In this manuscript, we introduce and study the concept of trigonometrically tgs-convex function and prove two Hermite-Hadamard type integral inequalities for the newly introduced class of functions. Also, some applications to special means of real numbers are also given.

1. Preliminaries and fundamentals

Throughout the paper I is a non-empty interval in $\mathbb{R}$.

Definition 1. *A function $f : I \to \mathbb{R}$ is said to be convex if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

It is well known that convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. Also, inequalities present an attractive and active field of research. In recent years, various inequalities for convex functions and their variable forms have been developed by many researchers using different techniques. Inequalities have a lot of applications in the fields of applied mathematics, differential equations and the other branches of mathematics and sciences. See articles [1, 3, 5, 6, 7, 9, 10, 12] and the references therein.
Let $f : [a,b] \to \mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is known as the Hermite-Hadamard inequality (for more information, see [4]). Hermite-Hadamard integral inequalities for convex functions are used to find the estimates of the mean value of continuous convex function. In recently, inequality (1) has attracted much interest from many researchers and scientists, a considerable papers have been appeared on the generalizations, variants and extensions of inequality (1). Some refinements of the Hermite-Hadamard integral inequalities for convex functions have been obtained [1, 2, 13].

In [11], the authors define a new concept of the so-called tgs-convex function and establish the Hadamard-Hadamard type integral inequalities.

Definition 2 ([11])**.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a nonnegative function. We say that $f : I \to \mathbb{R}$ is tgs-convex function on I if the inequality*

$$f(tu + (1-t)v) \leq t(1-t)\left[f(u) + f(v)\right]$$

holds for all $u, v \in I$ and $t \in (0,1)$. We say that f is tgs-concave if $(-f)$ is tgs-convex.

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Theorem 1 ([11])**.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a tgs-convex function and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$2f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{6}.$$

In [8], Kadakal introduce the concept of trigonometrically convex function, which is a special case of h-convex functions and prove two Hermite-Hadamard type integral inequalities for the newly introduced class of functions. The author also obtain two refinements of the Hermite-Hadamard integral inequalities for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is trigonometrically convex.

Definition 3 ([8])**.** *A non-negative function $f : I \to \mathbb{R}$ is called trigonometrically convex function on interval $[a, b]$, if for each $x, y \in [a, b]$ and $t \in [0, 1]$,*

$$f(tx + (1-t)y) \leq \left(\sin\frac{\pi t}{2}\right) f(x) + \left(\cos\frac{\pi t}{2}\right) f(y).$$

Denoted by $TC(I)$ the class of all trigonometrically convex functions on interval I.

Theorem 2 ([8])**.** *Let the function $f : [a, b] \to \mathbb{R}$, $b > 0$, be an trigonometrically convex function. If $0 \leq a < b <$ and $f \in L[a, b]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{2}{\pi}\left[f(a) + f(b)\right].$$

Theorem 3. *Let the function $f : [a, b] \to \mathbb{R}$, $b > 0$,be an trigonometrically convex function. If $0 \leq a < b$ and $f \in L[a, b]$, then the following inequalities holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\sqrt{2}}{b-a}\int_a^b f(x)dx.$$

This article is organized as follows. In chapter 2, we introduce a new concept, which is called trigonometrically tgs-convex function and give by setting some algebraic properties of them. In chapter 3, we obtain the Hermite-Hadamard inequality for the trigonometrically tgs-convex function. In chapter 3, we give some applications to special means of real numbers.

2. Main results for trigonometrically tgs-convex functions

In this section we introduce a new concept, which is called trigonometrically tgs-convex function, as follows:

Definition 4. *A function $f : I \to \mathbb{R}$ is called trigonometrically tgs-convex function on interval $[a, b]$, if for each $x, y \in [a, b]$ and $t \in [0, 1]$,*

$$\begin{aligned} f(tx + (1-t)y) &\leq \left(\sin\frac{\pi t}{2}\right)\left(\cos\frac{\pi t}{2}\right)\left[f(x) + f(y)\right] \\ &= \frac{\sin(\pi t)}{2}\left[f(x) + f(y)\right]. \end{aligned} \tag{2}$$

We will denote by $TC(I)$ the class of all trigonometrically convex functions on interval I. Since $\sin(\pi t) \leq 1$, clearly, if f is a trigonometric tgs-convex function., we write

$$f(x) = f(tx + (1-t)x) \leq \frac{\sin(\pi t)}{2}\left[f(x) + f(x)\right] \leq \sin(\pi t) f(x),$$

i.e.

$$(\sin(\pi t) - 1) f(x) \geq 0.$$

Therefore, we say that every trigonometrically tgs-convex function is an negative function.

Example 1. *The function $f : [0, +\infty) \to \mathbb{R},\ f(x) = -\sqrt{x}$ is a tgs-convex function.*

Example 2. *The function $f : [1, +\infty) \to \mathbb{R},\ f(x) = -\ln x$ is a tgs-convex function.*

Example 3. *The function $f : [0, +\infty) \to \mathbb{R},\ f(x) = -c, c > 0$ is is a tgs-convex function.*

Example 4. *The function $f : [0, +\infty) \to \mathbb{R},\ f(x) = -x$ is a tgs-convex function.*

Theorem 4. *Let $f, g : [a, b] \to \mathbb{R}$. If f and g are trigonometrically tgs-convex functions, then $f + g$ is trigonometrically tgs-convex function, for $c \in \mathbb{R}$ $(c \geq 0)$ cf is trigonometrically tgs-convex function.*

Proof. Let f and g are trigonometrically tgs-convex functions, then

$$\begin{aligned}
(f+g)(tx+(1-t)y) &= f(tx+(1-t)y) + g(tx+(1-t)y) \\
&\leq \frac{\sin(\pi t)}{2}[f(x)+f(y)] + \frac{\sin(\pi t)}{2}[g(x)+g(y)] \\
&= \frac{\sin(\pi t)}{2}([f(x)+g(x)] + [f(y)+g(y)]).
\end{aligned}$$

Let f is a trigonometrically tgs-convex function, then

$$\begin{aligned}
(cf)(tx+(1-t)y) &\leq c\left[\frac{\sin(\pi t)}{2}[f(x)+f(y)]\right] \\
&= \frac{\sin(\pi t)}{2}[cf(x)+cf(y)] \\
&= \frac{\sin(\pi t)}{2}[(cf)(x)+(cf)(y)].
\end{aligned}$$

□

Theorem 5. *If $f : I \to J \subset (-\infty, 0)$ is convex and $g : J \to \mathbb{R}$ is trigonometrically tgs-convex and increasing, then $gof : I \to \mathbb{R}$ is trigonometrically tgs-convex function.*

Proof. For $x, y \in I$ and $t \in [0, 1]$, we get

$$\begin{aligned}
(gof)(tx+(1-t)y) &= g(f(tx+(1-t)y)) \\
&\leq g(tf(x)+(1-t)f(y)) \\
&\leq \frac{\sin(\pi t)}{2}[g(f(x)) + g(f(y))] \\
&= \frac{\sin(\pi t)}{2}[(g\circ f)(x) + (g\circ f)(y)].
\end{aligned}$$

This completes the proof of theorem. □

3. Hermite-Hadamard inequality for trigonometrically tgs-convex functions

The goal of this paper is to develop concepts of the trigonometrically tgs-convex functions and to establish some inequalities of Hermite-Hadamard type for these classes of functions.

In the sequel of this article, with the notation $L[a, b]$, we show the space of integrable functions on $[a, b]$.

Theorem 6. *Let the function $f:[a,b]\to\mathbb{R}$, be an trigonometrically tgs-convex function. If $f\in L[a,b]$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{\pi}. \tag{3}$$

Proof. By using trigonometrically tgs-convexity of the function f, if the variable is changed as $u=ta+(1-t)b$, then we get

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(u)du &= \int_0^1 f(ta+(1-t)b)\,dt\\ &\leq \frac{1}{2}\int_0^1 \sin(\pi t)\left[f(a)+f(b)\right]dt\\ &= \frac{f(a)+f(b)}{2}\int_0^1\sin(\pi t)\,dt\\ &= \frac{f(a)+f(b)}{\pi}.\end{aligned}$$

By the trigonometrically tgs-convexity of the function f, we have

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &= f\left(\frac{[ta+(1-t)b]+[(1-t)a+tb]}{2}\right)\\ &= f\left(\frac{1}{2}[ta+(1-t)b]+\frac{1}{2}[(1-t)a+tb]\right)\\ &\leq \frac{1}{2}\sin\frac{\pi}{2}f(ta+(1-t)b)+\frac{1}{2}\sin\frac{\pi}{2}f((1-t)a+tb)\\ &= \frac{1}{2}\left[f(ta+(1-t)b)+f((1-t)a+tb)\right].\end{aligned}$$

Now, if we take integral the last inequality on $t\in[0,1]$ and choose $x=ta+(1-t)b$ and $y=tb+(1-t)a$, we deduce

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[\int_0^1 f(ta+(1-t)b)+\int_0^1 f((1-t)a+tb)\,dt\right]\\ &\quad+\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)dx+\frac{1}{b-a}\int_a^b f(y)dy\right]\\ &= \frac{1}{b-a}\int_a^b f(x)dx.\end{aligned}$$

This completes the proof of theorem. □

4. Applications for special means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers r,s with $s>r$:

1. The arithmetic mean

$$A:=A(r,s)=\frac{r+s}{2},\quad r,s\geq 0.$$

2. The geometric mean

$$G:=G(r,s)=\sqrt{rs},\quad r,s\geq 0.$$

3. The harmonic mean

$$H := H(r,s) = \frac{2rs}{r+s}, \quad r,s > 0.$$

4. The logarithmic mean

$$L := L(r,s) = \begin{cases} \frac{s-r}{\ln s - \ln r}, & r \neq s \\ r, & r = s \end{cases} ; \quad r,s > 0.$$

5. The p-logarithmic mean

$$L_p := L_p(r,s) = \begin{cases} \left(\frac{s^{p+1}-r^{p+1}}{(p+1)(s-r)}\right)^{\frac{1}{p}}, & r \neq s, p \in \mathbb{R}\backslash\{-1,0\} \\ r, & r = s \end{cases} ; \quad r,s > 0.$$

6.The identric mean

$$I := I(r,s) = \frac{1}{e}\left(\frac{s^s}{r^r}\right)^{\frac{1}{s-r}}, \quad r,s > 0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also well known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 1. *Let $a,b \in [0,+\infty)$ with $a < b$. Then, the following inequalities are obtained:*

$$\frac{2}{\pi}A\left(\sqrt{a},\sqrt{b}\right) \leq \frac{2}{3}\left[A\left(a^2,b^2\right) + \frac{G^2(a,b)}{2}\right]A^{-1}\left(a\sqrt{a},b\sqrt{b}\right) \leq A^{\frac{1}{2}}(a,b).$$

Proof. The assertion follows from the inequalities (3) for the function

$$f(x) = -\sqrt{x}, \quad x \in [0,+\infty).$$

□

Proposition 2. *Let $a,b \in (0,\infty)$ with $a < b$. Then, the following inequalities are obtained:*

$$-\frac{a+b}{2} \leq -\frac{b+a}{2} \leq -\frac{2}{\pi}A(a,b) \Rightarrow \frac{2}{\pi} \leq 1$$

Proof. The assertion follows from the inequalities (3) for the function

$$f(x) = -x, \quad x \in (0,\infty).$$

□

Proposition 3. *Let $a,b \in [1,+\infty)$ with $a < b$. Then, the following inequalities are obtained:*

$$\frac{1}{\pi}\ln G^2(a,b) \leq \ln I(a,b) \leq \ln A(a,b).$$

Proof. The assertion follows from the inequalities (3) for the function

$$f(x) = -\ln x, \quad x \in [1,+\infty).$$

□

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