TJMM
15 (2023), No. 1-2, 51-64

SOME OSTROWSKI TYPE INEQUALITIES FOR TWO SIN-INTEGRAL TRANSFORMS OF ABSOLUTELY CONTINUOUS FUNCTIONS

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Abstract. For a Lebesgue integrable function $f : [a,b] \subset [-\pi/4, \pi/4] \to \mathbb{C}$ we consider *the sin-integral transforms*

$$S_f(x) := \int_a^b f(t) \sin(x-t)\, dt,\ x \in [a,b]$$

and

$$\widetilde{S}_f(x) := \int_a^x f(t) \sin(t-a)\, dt + \int_x^b f(t) \sin(b-t)\, dt,\ x \in [a,b].$$

We provide in this paper some upper bounds for the quantities

$$|f(b)\cos(b-x) - f(a)\cos(x-a) - S_f(x)|$$

and

$$\left|\widetilde{S}_f(x) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right)\cos\left(x - \frac{a+b}{2}\right)f(x)\right]\right|$$

for $x \in [a,b]$, in terms of the p-norms of the derivative f' for absolutely continuous functions $f : [a,b] \subset [-\pi/4, \pi/4] \to \mathbb{C}$.

1. Introduction

In 1938, A. Ostrowski [5], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\, dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\, dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right] \|f'\|_\infty (b-a), \tag{1}$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

The following result, which is an improvement on Ostrowski's inequality, holds.

2010 *Mathematics Subject Classification.* 26D15, 26D10, 44A15, 44A35.
Key words and phrases. Lebesgue integral, Ostrowski inequality, Integral transforms, Absolutely continuous functions.

Theorem 2 (Dragomir, 2002 [2])**.** *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ whose derivative $f'\in L_\infty[a,b]$. Then*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \qquad (2)\\&\le\frac{1}{2(b-a)}\left[\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2\right]\\&\le\|f'\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a);\end{aligned}$$

for all $x\in[a,b]$, where $\|\cdot\|_{[m,n],\infty}$ denotes the usual norm on $L_\infty[m,n]$, i.e., we recall that

$$\|g\|_{[m,n],\infty}=ess\sup_{t\in[m,n]}|g(t)|<\infty.$$

The case of 1-norm is as follows:

Theorem 3 (Dragomir, 2002 [1])**.** *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \qquad (3)\\&\le\frac{x-a}{b-a}\|f'\|_{[a,x],1}+\frac{b-x}{b-a}\|f'\|_{[x,b],1}\\&\le\left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_{[a,b],1}\end{aligned}$$

for all $x\in[a,b]$, where $\|\cdot\|_{[m,n],1}$ denotes the usual norm on $L_1[m,n]$ with $m<n$, i.e., we recall that

$$\|g\|_{[m,n],1}:=\int_m^n|g(t)|\,dt<\infty.$$

The following inequality for the p-norms also holds.

Theorem 4 (Dragomir, 2013 [3])**.** *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, then*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \qquad (4)\\&\le\frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\|f'\|_{[a,x],p}+\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}\|f'\|_{[x,b],p}\right](b-a)^{1/q}\\&\le\frac{1}{(q+1)^{1/q}}\\&\times\left(\|f'\|^{\alpha}_{[a,x],p}+\|f'\|^{\alpha}_{[x,b],p}\right)^{\frac{1}{\alpha}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}\beta}+\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}\beta}\right]^{\frac{1}{\beta}}(b-a)^{1/q}\end{aligned}$$

for all $x \in [a,b]$, *where* $\alpha > 1$ *and* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\|\cdot\|_{[m,n],p}$ *denotes the usual* p*-norm on* $L_p[m,n]$ *with* $m < n$, *i.e., we recall that*

$$\|g\|_{[m,n],p} := \left(\int_m^n |g(t)|\, dt\right)^{1/p} < \infty.$$

More related results are presented in recent survey paper [4].

For a Lebesgue integrable function $f : [a,b] \subset [-\pi/4, \pi/4] \to \mathbb{C}$ we consider *the sin-integral transforms*

$$S_f(x) := \int_a^b f(t) \sin(x-t)\, dt,\ x \in [a,b]$$

and

$$\widetilde{S}_f(x) := \int_a^x f(t) \sin(t-a)\, dt + \int_x^b f(t) \sin(b-t)\, dt,\ x \in [a,b].$$

Motivated by the above results, ee establish in this paper some upper bounds for the quantities

$$|f(b)\cos(b-x) - f(a)\cos(x-a) - S_f(x)|$$

and

$$\left|\widetilde{S}_f(x) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right)\cos\left(x - \frac{a+b}{2}\right)f(x)\right]\right|$$

for $x \in [a,b]$, in terms of the p-norms of the derivative f' for absolutely continuous functions $f : [a,b] \subset [-\pi/4, \pi/4] \to \mathbb{C}$.

2. Error Bounds for S_f

The first main result is as follows:

Theorem 5. *If* f *is absolutely continuous on* $[a,b] \subset [-\pi/4, \pi/4]$ *with* $f' \in L_\infty[a,b]$, *then*

$$\begin{aligned} &|f(b)\cos(b-x) - f(a)\cos(x-a) - S_f(x)| \\ &\leq 2\sin\left(\frac{b-a}{2}\right)\cos\left(x - \frac{a+b}{2}\right)\|f'\|_{[a,b],\infty} \end{aligned} \tag{5}$$

for all $x \in [a,b]$.

If p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $f' \in L_p[a,b]$, *then*

$$\begin{aligned} &|f(b)\cos(b-x) - f(a)\cos(x-a) - S_f(x)| \\ &\leq \left(\int_a^b \cos^q(t-x)\, dt\right)^{1/q} \|f'\|_{[a,b],p} \end{aligned} \tag{6}$$

for all $x \in [a,b]$.

Also, we have

$$|f(b)\cos(b-x) - f(a)\cos(x-a) - S_f(x)| \leq \|f'\|_{[a,b],1} \tag{7}$$

for all $x \in [a,b]$.

Proof. Using the integration by parts formula, we get

$$\begin{aligned}
&\int_a^b f'(t)\cos(x-t)\,dt \qquad (8)\\
&= f(t)\cos(x-t)|_a^b - \int_a^b f(t)\sin(x-t)\,dt\\
&= f(b)\cos(x-b) - f(a)\cos(x-a) - \int_a^b f(t)\sin(x-t)\,dt\\
&= f(b)\cos(b-x) - f(a)\cos(x-a) - S_f(x)
\end{aligned}$$

for all $x \in [a,b]$.

By taking the modulus we have, since $|t-x| \le \frac{\pi}{2}$,

$$\begin{aligned}
&|f(b)\cos(b-x) - f(a)\cos(x-a) - S_f(x)|\\
&\le \left|\int_a^b f'(t)\cos(x-t)\,dt\right| \le \int_a^b |f'(t)|\,|\cos(t-x)|\,dt\\
&= \int_a^b |f'(t)|\cos(t-x)\,dt \le \|f'\|_{[a,b],\infty}\int_a^b \cos(t-x)\,dt\\
&= \|f'\|_{[a,b],\infty}[\sin(b-x) - \sin(a-x)]\\
&= \|f'\|_{[a,b],\infty}[\sin(b-x) + \sin(x-a)]\\
&= 2\sin\left(\frac{b-a}{2}\right)\cos\left(x - \frac{a+b}{2}\right)\|f'\|_{[a,b],\infty}
\end{aligned}$$

for all $x \in [a,b]$, which proves (5).

Using Hölder's inequality for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\begin{aligned}
\int_a^b |f'(t)|\cos(t-x)\,dt &\le \left(\int_a^b |f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b \cos^q(t-x)\,dt\right)^{1/q}\\
&= \|f'\|_{[a,b],p}\left(\int_a^b \cos^q(t-x)\,dt\right)^{1/q}
\end{aligned}$$

for all $x \in [a,b]$, which proves (6).

Also

$$\begin{aligned}
\int_a^b |f'(t)|\cos(t-x)\,dt &\le \sup_{t\in[a,b]}\cos(t-x)\int_a^b |f'(t)|\\
&= \sup_{t\in[a,b]}\cos(t-x)\,\|f'\|_{[a,b],1} = \|f'\|_{[a,b],1}
\end{aligned}$$

for all $x \in [a,b]$, which proves (7). □

Remark 1. *In particular, if we take $x = \frac{a+b}{2}$, then we get from Theorem 5 that*

$$\left|[f(b) - f(a)]\cos\left(\frac{b-a}{2}\right) - S_f\left(\frac{a+b}{2}\right)\right| \le 2\sin\left(\frac{b-a}{2}\right)\|f'\|_{[a,b],\infty}, \qquad (9)$$

for $f' \in L_\infty [a, b]$

$$\left|[f(b) - f(a)] \cos\left(\frac{b-a}{2}\right) - S_f\left(\frac{a+b}{2}\right)\right| \leq \left(\int_a^b \cos^q \left(t - \frac{a+b}{2}\right) dt\right)^{1/q} \|f'\|_{[a,b],p} \tag{10}$$

for $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $f' \in L_p [a, b]$ *and*

$$\left|[f(b) - f(a)] \cos\left(\frac{b-a}{2}\right) - S_f\left(\frac{a+b}{2}\right)\right| \leq \|f'\|_{[a,b],1}. \tag{11}$$

Corollary 1. *If* f *is absolutely continuous on* $[a, b] \subset [-\pi/4, \pi/4]$ *with* $f' \in L_2 [a, b]$, *then*

$$|f(b) \cos(b - x) - f(a) \cos(x - a) - S_f(x)| \leq \frac{\sqrt{2}}{2} (b-a)^{1/2} \left(1 + \frac{\sin(b-a)}{b-a} \cos\left[2\left(x - \frac{a+b}{2}\right)\right]\right)^{1/2} \|f'\|_{[a,b],2} \tag{12}$$

for all $x \in [a, b]$.

In particular,

$$\left|[f(b) - f(a)] \cos\left(\frac{b-a}{2}\right) - S_f\left(\frac{a+b}{2}\right)\right| \leq \frac{\sqrt{2}}{2} (b-a)^{1/2} \left(1 + \frac{\sin(b-a)}{b-a}\right)^{1/2} \|f'\|_{[a,b],2}. \tag{13}$$

Proof. From (6) we have for $p = q = 2$ that

$$|f(b) \cos(b - x) - f(a) \cos(x - a) - S_f(x)| \leq \left(\int_a^b \cos^2 (t - x)\, dt\right)^{1/2} \|f'\|_{[a,b],2} \tag{14}$$

for all $x \in [a, b]$.

Observe that

$$\begin{aligned}\int_a^b \cos^2 (t - x)\, dt &= \frac{1}{2}(b-a) + \frac{1}{4}[\sin(2(b-x)) + \sin(2(x-a))] \\ &= \frac{1}{2}(b-a) + \frac{1}{2}\sin(b-a) \cos\left[2\left(x - \frac{a+b}{2}\right)\right] \\ &= \frac{1}{2}\left(b - a + \sin(b-a) \cos\left[2\left(x - \frac{a+b}{2}\right)\right]\right) \\ &= \frac{1}{2}(b-a)\left(1 + \frac{\sin(b-a)}{b-a} \cos\left[2\left(x - \frac{a+b}{2}\right)\right]\right),\end{aligned}$$

which by (14) gives the desired result (12). □

3. Error Bounds for $\widetilde{S}_f$

We also have:

Theorem 6. *If f is absolutely continuous on $[a,b] \subset [-\pi/4, \pi/4]$ with $f' \in L_\infty[a,b]$, then*

$$\left|\widetilde{S}_f(x) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right)\cos\left(x - \frac{a+b}{2}\right)f(x)\right]\right| \tag{15}$$
$$\leq \|f'\|_{[a,x],\infty}[\sin(x-a)] + \|f'\|_{[x,b],\infty}\sin(b-x)$$
$$\leq 2\|f'\|_{[a,b],\infty}\sin\left(\frac{b-a}{2}\right)\cos\left(x - \frac{a+b}{2}\right)$$

for $x \in [a,b]$.

If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $f' \in L_p[a,b]$, then

$$\left|\widetilde{S}_f(x) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right)\cos\left(x - \frac{a+b}{2}\right)f(x)\right]\right| \tag{16}$$
$$\leq \|f'\|_{[a,x],p}\left(\int_a^x \cos^q(t-a)\,dt\right)^{1/q} + \|f'\|_{[x,b],p}\left(\int_a^x \cos^q(t-b)\,dt\right)^{1/q}$$
$$\leq \|f'\|_{[a,b],p}\left(\int_a^x \cos^q(t-a)\,dt + \int_a^x \cos^q(t-b)\,dt\right)^{1/q}$$

for $x \in [a,b]$.

Also, we have

$$\left|\widetilde{S}_f(x) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right)\cos\left(x - \frac{a+b}{2}\right)f(x)\right]\right| \tag{17}$$
$$\leq \|f'\|_{[a,b],1}$$

for $x \in [a,b]$.

Proof. Using integration by parts, we have

$$\int_a^x f'(t)\cos(t-a)\,dt = f(t)\cos(t-a)\big|_a^x + \int_a^x f(t)\sin(t-a)\,dt$$
$$= f(x)\cos(x-a) - f(a) + \int_a^x f(t)\sin(t-a)\,dt$$

and

$$\int_x^b f'(t)\cos(t-b) = f(t)\cos(t-b)\big|_x^b + \int_x^b f(t)\sin(t-b)\,dt$$
$$= f(b) - f(x)\cos(b-x) + \int_x^b f(t)\sin(t-b)\,dt$$
$$= f(b) - f(x)\cos(b-x) - \int_x^b f(t)\sin(b-t)\,dt$$

for $x \in [a,b]$.

If we subtract from the first identity the second, then we get

$$\begin{aligned}
&\int_a^x f'(t)\cos(t-a)\,dt - \int_x^b f'(t)\cos(t-b) \\
&= f(x)\cos(x-a) - f(a) + \int_a^x f(t)\sin(t-a)\,dt \\
&- f(b) + f(x)\cos(b-x) + \int_x^b f(t)\sin(b-t)\,dt \\
&= f(x)\left[\cos(b-x)+\cos(x-a)\right] - f(a) - f(b) + \widetilde{S}_f(x) \\
&= \widetilde{S}_f(x) - \left[f(a)+f(b) - 2\cos\left(\frac{b-a}{2}\right)\cos\left(x-\frac{a+b}{2}\right)f(x)\right]
\end{aligned}$$

for $x \in [a,b]$.

If we take the modulus and take into account the fact that $\leq t-a, b-t \leq \frac{\pi}{2}$, we get

$$\begin{aligned}
&\left|\widetilde{S}_f(x) - \left[f(a)+f(b) - 2\cos\left(\frac{b-a}{2}\right)\cos\left(x-\frac{a+b}{2}\right)f(x)\right]\right| \\
&= \left|\int_a^x f'(t)\cos(t-a)\,dt + \int_x^b f'(t)\cos(t-b)\right| \\
&\leq \left|\int_a^x f'(t)\cos(t-a)\,dt\right| + \left|\int_x^b f'(t)\cos(t-b)\right| \\
&\leq \int_a^x |f'(t)|\cos(t-a)\,dt + \int_x^b |f'(t)|\cos(t-b)\,dt
\end{aligned}$$

$$\begin{aligned}
&\leq \|f'\|_{[a,x],\infty}\int_a^x \cos(t-a)\,dt + \|f'\|_{[x,b],\infty}\int_x^b \cos(b-t)\,dt \\
&= \|f'\|_{[a,x],\infty}\left[\sin(x-a)\right] + \|f'\|_{[x,b],\infty}\sin(b-x) \\
&\leq \max\left\{\|f'\|_{[a,x],\infty}, \|f'\|_{[x,b],\infty}\right\}\left[\sin(x-a)+\sin(b-x)\right] \\
&= \|f'\|_{[a,b],\infty}\left[2\sin\left(\frac{b-a}{2}\right)\cos\left(x-\frac{a+b}{2}\right)\right] \\
&= 2\|f'\|_{[a,b],\infty}\sin\left(\frac{b-a}{2}\right)\cos\left(x-\frac{a+b}{2}\right)
\end{aligned}$$

for $x \in [a,b]$, which proves (15)

Using Hölder's inequality for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\begin{aligned}
&\left|\widetilde{S}_f(x) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right)\cos\left(x - \frac{a+b}{2}\right) f(x)\right]\right| \\
&\leq \int_a^x |f'(t)| \cos(t-a)\, dt + \int_x^b |f'(t)| \cos(t-b)\, dt \\
&\leq \left(\int_a^x |f'(t)|^p\, dt\right)^{1/p} \left(\int_a^x \cos^q(t-a)\, dt\right)^{1/q} \\
&+ \left(\int_x^b |f'(t)|^p\, dt\right)^{1/p} \left(\int_x^b \cos^q(t-b)\, dt\right)^{1/q} \\
&\leq \left(\left[\left(\int_a^x |f'(t)|^p\, dt\right)^{1/p}\right]^p + \left[\left(\int_x^b |f'(t)|^p\, dt\right)^{1/p}\right]^p\right)^{1/p} \\
&\times \left[\left(\left(\int_a^x \cos^q(t-a)\, dt\right)^{1/q}\right)^q + \left(\left(\int_x^b \cos^q(t-b)\, dt\right)^{1/q}\right)^q\right]^{1/q} \\
&= \left(\int_a^x |f'(t)|^p\, dt + \int_x^b |f'(t)|^p\, dt\right)^{1/p} \\
&\times \left(\int_a^x \cos^q(t-a)\, dt + \int_x^b \cos^q(t-b)\, dt\right)^{1/q} \\
&= \left(\int_a^b |f'(t)|^p\, dt\right)^{1/p} \left(\int_a^x \cos^q(t-a)\, dt + \int_x^b \cos^q(t-b)\, dt\right)^{1/q}
\end{aligned}$$

for $x \in [a,b]$, which proves (16).

Finally, we also have

$$\begin{aligned}
&\left|\widetilde{S}_f(x) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right)\cos\left(x - \frac{a+b}{2}\right) f(x)\right]\right| \\
&\leq \int_a^x |f'(t)| \cos(t-a)\, dt + \int_x^b |f'(t)| \cos(t-b)\, dt \\
&\leq \sup_{t\in[a,x]} \cos(t-a) \int_a^x |f'(t)|\, dt + \sup_{t\in[x,b]} \cos(t-b) \int_x^b |f'(t)|\, dt \\
&= \int_a^x |f'(t)|\, dt + \int_x^b |f'(t)|\, dt = \int_a^b |f'(t)|\, dt,
\end{aligned}$$

for $x \in [a,b]$, which proves (17). □

Remark 2. *In particular, if we take* $x = \frac{a+b}{2}$*, then we get from Theorem 6 that*

$$\begin{aligned}
&\left|\widetilde{S}_f\left(\frac{a+b}{2}\right) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right) f\left(\frac{a+b}{2}\right)\right]\right| \qquad (18)\\
&\leq \left(\|f'\|_{[a,x],\infty} + \|f'\|_{[x,b],\infty}\right)\sin\left(\frac{b-a}{2}\right)\\
&\leq 2\|f'\|_{[a,b],\infty}\sin\left(\frac{b-a}{2}\right)
\end{aligned}$$

for $f' \in L_\infty[a,b]$.

If $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $f' \in L_p[a,b]$, *then*

$$\begin{aligned}
&\left|\widetilde{S}_f\left(\frac{a+b}{2}\right) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right) f\left(\frac{a+b}{2}\right)\right]\right| \qquad (19)\\
&\leq \|f'\|_{\left[a,\frac{a+b}{2}\right],p}\left(\int_a^{\frac{a+b}{2}}\cos^q(t-a)\,dt\right)^{1/q}\\
&+ \|f'\|_{\left[\frac{a+b}{2},b\right],p}\left(\int_{\frac{a+b}{2}}^b\cos^q(t-b)\,dt\right)^{1/q}\\
&\leq \|f'\|_{[a,b],p}\left(\int_a^{\frac{a+b}{2}}\cos^q(t-a)\,dt + \int_{\frac{a+b}{2}}^b\cos^q(t-b)\,dt\right)^{1/q}
\end{aligned}$$

and

$$\left|\widetilde{S}_f\left(\frac{a+b}{2}\right) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right) f\left(\frac{a+b}{2}\right)\right]\right| \leq \|f'\|_{[a,b],1}. \qquad (20)$$

Corollary 2. *If* f *is absolutely continuous on* $[a,b] \subset [-\pi/4, \pi/4]$ *with* $f' \in L_2[a,b]$, *then*

$$\begin{aligned}
&\left|\widetilde{S}_f(x) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right)\cos\left(x - \frac{a+b}{2}\right) f(x)\right]\right| \qquad (21)\\
&\leq \frac{\sqrt{2}}{2}\|f'\|_{[a,x],2}\left(x - a + \frac{1}{2}\sin(2(x-a))\right)^{1/2}\\
&+ \frac{\sqrt{2}}{2}\|f'\|_{[x,b],2}\left(b - x + \frac{1}{2}\sin(2(b-x))\right)^{1/2}\\
&\leq \frac{\sqrt{2}}{2}(b-a)^{1/2}\|f'\|_{[a,b],2}\left(1 + \frac{\sin(b-a)}{b-a}\cos\left[2\left(x - \frac{a+b}{2}\right)\right]\right)^{1/2}
\end{aligned}$$

for $x \in [a,b]$.

In particular,

$$\begin{aligned}
&\left|\widetilde{S}_f\left(\frac{a+b}{2}\right) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right) f\left(\frac{a+b}{2}\right)\right]\right| \qquad (22)\\
&\leq \frac{1}{2}(b-a)^{1/2}\left(\|f'\|_{[a,x],2} + \|f'\|_{[x,b],2}\right)\left(1 + \frac{\sin(b-a)}{b-a}\right)^{1/2}\\
&\leq \frac{\sqrt{2}}{2}(b-a)^{1/2}\|f'\|_{[a,b],2}\left(1 + \frac{\sin(b-a)}{b-a}\right)^{1/2}.
\end{aligned}$$

Proof. From (16) for $p = q = 2$ we get

$$\left|\widetilde{S}_f(x) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right)\cos\left(x - \frac{a+b}{2}\right)f(x)\right]\right| \tag{23}$$
$$\leq \|f'\|_{[a,x],2}\left(\int_a^x \cos^2(t-a)\,dt\right)^{1/2} + \|f'\|_{[x,b],2}\left(\int_x^b \cos^2(t-b)\,dt\right)^{1/2}$$
$$\leq \|f'\|_{[a,b],2}\left(\int_a^x \cos^2(t-a)\,dt + \int_x^b \cos^2(t-b)\,dt\right)^{1/2}$$

for $x \in [a,b]$.

Observe that

$$\int_a^x \cos^2(t-a)\,dt = \frac{1}{2}(x-a) + \frac{1}{4}\sin(2(x-a)),$$
$$\int_x^b \cos^2(t-b)\,dt = \frac{1}{2}(b-x) + \frac{1}{4}\sin(2(b-x))$$

and

$$\int_a^x \cos^2(t-a)\,dt + \int_x^b \cos^2(t-b)\,dt$$
$$= \frac{1}{2}(x-a) + \frac{1}{4}\sin(2(x-a)) + \frac{1}{2}(b-x) + \frac{1}{4}\sin(2(b-x))$$
$$= \frac{1}{2} + \frac{1}{4}\sin(2(b-x)) + \frac{1}{4}\sin(2(x-a))$$
$$= \frac{1}{2}(b-a) + \frac{1}{2}\sin(b-a)\cos\left[2\left(x - \frac{a+b}{2}\right)\right]$$
$$= \frac{1}{2}\left(b - a + \sin(b-a)\cos\left[2\left(x - \frac{a+b}{2}\right)\right]\right)$$
$$= \frac{1}{2}(b-a)\left(1 + \frac{\sin(b-a)}{b-a}\cos\left[2\left(x - \frac{a+b}{2}\right)\right]\right)$$

for $x \in [a,b]$, which proves (21). □

4. Some Examples

Consider the function $\ell_p(t) = t^p$, $p \geq 1$, $t \in [a,b] \subset [0, \pi/4]$. Then

$$S_{\ell_p}(x) := \int_a^b t^p \sin(x-t)\,dt,$$

and

$$\widetilde{S}_{\ell_p}(x) := \int_a^x t^p \sin(t-a)\,dt + \int_x^b t^p \sin(b-t)\,dt,$$

for $x \in [a,b]$.

Therefore by (5) we obtain

$$\left|b^p\cos(b-x) - a^p\cos(x-a) - S_{\ell_p}(x)\right| \tag{24}$$
$$\leq 2pb^{p-1}\sin\left(\frac{b-a}{2}\right)\cos\left(x - \frac{a+b}{2}\right)$$

for $x \in [a,b]$.

For $x = \frac{a+b}{2}$ we get

$$\left| (b^p - a^p) \cos\left(\frac{b-a}{2}\right) - S_{\ell_p}\left(\frac{a+b}{2}\right) \right| \leq 2pb^{p-1} \sin\left(\frac{b-a}{2}\right) \tag{25}$$

If we use (15), then we get

$$\begin{aligned} &\left| \widetilde{S}_{\ell_p}(x) - \left[a^p + b^p - 2\cos\left(\frac{b-a}{2}\right) \cos\left(x - \frac{a+b}{2}\right) x^p \right] \right| \\ &\leq 2pb^{p-1} \sin\left(\frac{b-a}{2}\right) \cos\left(x - \frac{a+b}{2}\right) \end{aligned} \tag{26}$$

for $x \in [a, b]$.

For $x = \frac{a+b}{2}$ we get

$$\begin{aligned} &\left| \widetilde{S}_{\ell_p}\left(\frac{a+b}{2}\right) - \left[a^p + b^p - 2\cos\left(\frac{b-a}{2}\right) \left(\frac{a+b}{2}\right)^p \right] \right| \\ &\leq 2pb^{p-1} \sin\left(\frac{b-a}{2}\right) \end{aligned} \tag{27}$$

Also, we consider the function $f(t) = \exp t^p = \exp \ell_p(t)$, $t \in [a, b] \subset [0, \pi/4]$, $p \geq 1$. We have $f'(t) = pt^{p-1} \exp \ell_p(t)$ and $\|f'\|_{[a,b],\infty} = pb^{p-1} \exp(b^p)$. Then

$$S_{\exp \ell_p}(x) := \int_a^b \exp \ell_p(t) \sin(x - t)\, dt,\ x \in [a, b]$$

and

$$\widetilde{S}_{\exp \ell_p}(x) := \int_a^x \exp \ell_p(t) \sin(t - a)\, dt + \int_x^b \exp \ell_p(t) \sin(b - t)\, dt,\ x \in [a, b].$$

From (5) we get

$$\begin{aligned} &\left| \exp \ell_p(b) \cos(b - x) - \exp \ell_p(a) \cos(x - a) - S_{\exp \ell_p}(x) \right| \\ &\leq 2pb^{p-1} \exp(b^p) \sin\left(\frac{b-a}{2}\right) \cos\left(x - \frac{a+b}{2}\right) \end{aligned} \tag{28}$$

for $x \in [a, b]$.

If we take $x = \frac{a+b}{2}$, then we get

$$\begin{aligned} &\left| [\exp \ell_p(b) \cos - \exp \ell_p(a)] \left(\frac{b-a}{2}\right) - S_{\exp \ell_p}\left(\frac{a+b}{2}\right) \right| \\ &\leq 2pb^{p-1} \exp(b^p) \sin\left(\frac{b-a}{2}\right) \end{aligned} \tag{29}$$

If we use (15), then we get

$$\begin{aligned} &\left| \widetilde{S}_{\exp \ell_p}(x) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right) \cos\left(x - \frac{a+b}{2}\right) \exp x^p \right] \right| \\ &\leq 2pb^{p-1} \exp(b^p) \sin\left(\frac{b-a}{2}\right) \cos\left(x - \frac{a+b}{2}\right) \end{aligned} \tag{30}$$

for $x \in [a, b]$.

If we take $x = \frac{a+b}{2}$, then we get

$$\left|\widetilde{S}_{\exp \ell_p}\left(\frac{a+b}{2}\right) - \left[f(a) + f(b) - 2\cos\left(\frac{b-a}{2}\right)\exp\left(\frac{a+b}{2}\right)^p\right]\right| \tag{31}$$
$$\leq 2pb^{p-1}\exp(b^p)\sin\left(\frac{b-a}{2}\right).$$

5. Some Numerical Experiments

Let $a \in [0, \pi/4)$ and ε small with$[a, a+\varepsilon] \subset [0, \pi/4]$. Consider

$$S_{\ell_p}(x) := \int_a^b t^p \sin(x-t)\, dt.$$

Then by (25) we have that

$$B(a,\varepsilon) := \left|((a+\varepsilon)^p - a^p)\cos\left(\frac{\varepsilon}{2}\right) - S_{\ell_p}\left(\frac{2a+\varepsilon}{2}\right)\right| \tag{32}$$
$$\leq 2p(a+\varepsilon)^{p-1}\sin\left(\frac{\varepsilon}{2}\right).$$

Table 1 details the numerical values of $S_{\ell_2}\left(\frac{2a+\varepsilon}{2}\right)$, $B(a,\varepsilon)$, and its upper bound from (32) for some choices of a and ε.

		$\mathbf{a_1} = 0.0$	$\mathbf{a_2} = 0.1$	$\mathbf{a_3} = 0.2$	$\mathbf{a_4} = 0.3$	$\mathbf{a_5} = 0.4$	$\mathbf{a_6} = 0.5$
$S_{\ell_2}\left(\frac{2a+\varepsilon}{2}\right)$ $B(a,\varepsilon)$ Upper Bound	$\varepsilon = 0.1$	$-8.331E-06$ 0.009996 0.019992	$-2.4996E-05$ 0.029988 0.039983	$-4.166E-05$ 0.049979 0.059975	$-5.832E-05$ 0.069971 0.079967	$-7.498E-05$ 0.089963 0.099958	-0.000092 0.109954 0.119950
	$\varepsilon = 0.01$	$-8.333E-10$ 0.000100 0.000200	$-1.750E-08$ 0.002100 0.002200	$-3.417E-08$ 0.004100 0.004200	$-5.083E-08$ 0.006100 0.006200	$-6.750E-08$ 0.008100 0.008200	$-8.417E-08$ 0.010100 0.010200
	$\varepsilon = 0.001$	$-8.333E-14$ $1.000E-06$ $2.000E-06$	$-1.675E-11$ 0.000201 0.000202	$-3.342E-11$ 0.000401 0.000402	$-5.008E-11$ 0.000601 0.000602	$-6.675E-11$ 0.000801 0.000802	$-8.342E-11$ 0.001001 0.001002
	$\varepsilon = 0.0001$	$-8.333E-18$ $1.000E-08$ $2.000E-08$	$-1.668E-14$ $2.001E-05$ $2.002E-05$	$-3.335E-14$ $4.001E-05$ $4.002E-05$	$-5.000E-14$ $6.001E-05$ $6.002E-05$	$-6.670E-14$ $8.001E-05$ $8.002E-05$	$-8.337E-14$ $1.0001E-04$ $1.0002E-04$
	$\varepsilon = 0.00001$	$-8.33333E-22$ $1.0000E-10$ $2.0000E-10$	$-1.7045E-17$ $2.0001E-06$ $2.0002E-06$	$-3.7509E-17$ $4.0001E-06$ $4.0002E-06$	$-3.6116E-17$ $6.0001E-06$ $6.0002E-06$	$-6.9415E-17$ $8.0001E-06$ $8.0002E-06$	$-6.1079E-17$ $1.00001E-05$ $1.00002E-05$

Table 1. Numerical results for $p = 2$ and various a and ε.

Table 2 details the values of $S_{\ell_{\frac{3}{2}}}\left(\frac{\varepsilon}{2}\right)$, for $N = 32$ and various ε, and various quadrature (Newton-Cotes) rules (including absolute differences).

Next is a graphical representation of the absolute error for various integer p,

Int Type		$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
Ostrowski		-0.031583	-0.001000	$-3.1623E-05$	$-1.0000E-06$	$-3.1623E-08$
Trapezoidal	$N = 32$	$-2.71442E-05$ (0.031556)	$-8.5859E-09$ $(1.0000E-03)$	$-2.7151E-12$ $(3.1623E-05)$	$-8.5859E-16$ $(1.0000E-06)$	$-9.0388E-19$ $(3.1623E-08)$
Simpson's	$N = 32$	$-2.7098E-05$ (0.031556)	$-8.5713E-09$ $(1.0000E-03)$	$-2.7105E-12$ $(3.1623E-05)$	$-8.5713E-16$ $(1.0000E-06)$	$-9.0314E-19$ $(3.1623E-08)$
Mid Point	$N = 32$	$-2.7076E-05$ (0.031556)	$-8.5641E-09$ $(1.0000E-03)$	$-2.7082E-12$ $(3.1623E-05)$	$-8.5642E-16$ $(1.0000E-06)$	$-2.7082E-19$ $(3.1623E-08)$
Upper Bound		0.047414	0.001500	$4.7434E-05$	$1.5000E-06$	$(4.7434E-08)$

TABLE 2. Numerical results comparing various quadrature (Newton-Cotes) rules to the Ostrowski approximation for $a = 0$, various ε, and $p = \frac{3}{2}$.

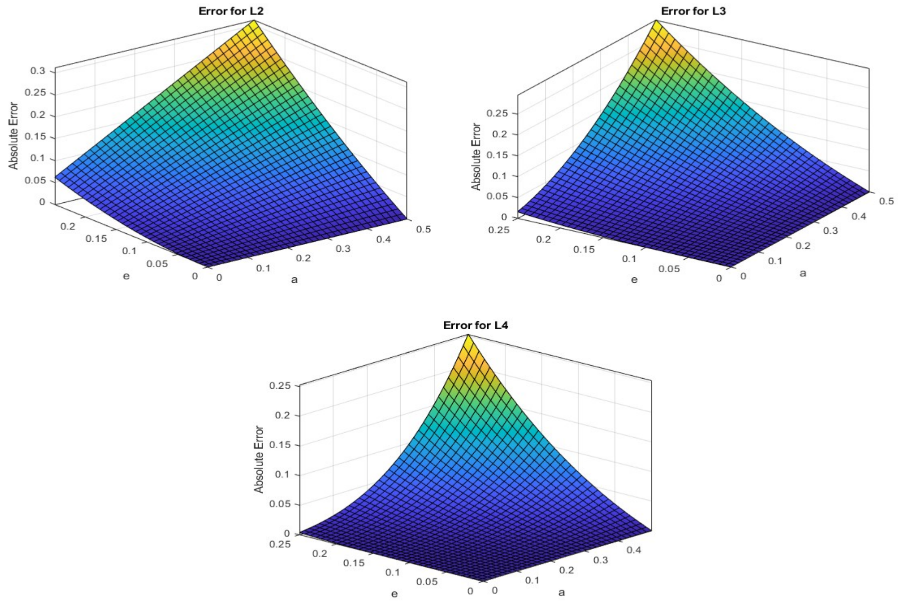

FIGURE 1. Graphical representation of the Absolute Error for $p = 2, 3$ and 4 and various x and ε.

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