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# ANALYTIC PARAMETRIZATION OF THE ALGEBRAIC POINTS OF GIVEN DEGREE ON THE CURVE OF AFFINE EQUATION $y^2 = 157(x^2-2)(x^2+x)(x^2+1)$

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Abstract. We give an explicit parametrization of the set of algebraic points of given degree on $\mathbb{Q}$ over the affine equation curve : $y^2 = 157(x^2-2)(x^2+x)(x^2+1)$.
This note treat aspecial case of the curves described by Anna ARNTH-JENSEN and Victor FLYNN in [1], where the generators of the Mordell-Weil group explained.

## 1. Introduction

Let $\mathcal{C}$ be a projective algebraic curve of definite over $\mathbb{Q}$. For any field of numbers $\mathbb{K}$, $\mathcal{C}(\mathbb{K})$ is the set of points on $\mathcal{C}$ with coordinates in $\mathbb{K}$ and $\bigcup_{[\mathbb{K}:\mathbb{Q}]\leqslant l} \mathcal{C}(\mathbb{K})$ the set points on $\mathcal{C}$ a coordinates in $\mathbb{K}$ of degree at most $l$ on $\mathbb{Q}$. The degree of a point $R$ is the degree of its defining field on $\mathbb{Q}$, that is $\deg(R) = [\mathbb{Q}(R) : \mathbb{Q}]$. We denote by $\mathcal{J}$ the Jacobian of $\mathcal{C}$ and by $j(P)$ the class $[P - P_\infty]$ of $P - P_\infty$, that is $j$ is the Jacobian fold:

$$\begin{array}{cccc} j & : & \mathcal{C} & \longrightarrow & \mathcal{J}(\mathbb{Q}), \\ & & P & \longmapsto & [P - P_\infty] \end{array}$$

where $\mathcal{J}(\mathbb{Q})$ represents the Mordell-Weil group of rational points of the Jacobian of $\mathcal{C}$ (see [7]); this group is finite (see [1, page 10], Lemma 2.).
Our curve $\mathcal{C}$ of affine equation $y^2 = 157(x^2-2)(x^2+x)(x^2+1)$ is a special case of the curve family

$$\mathcal{C} : y^2 = q(x^2-2)(x^2+x)(x^2+1) \quad \text{with} \quad q \ \equiv \ 13[24]$$

is a prime, sttudy in [1, page 4], where the mordell-Weill group was explained.
In this note we define the set :

$$\bigcup_{[\mathbb{K}:\mathbb{Q}]\leqslant l} \mathcal{C}(\mathbb{K}) \qquad \text{with} \qquad l \geq 5$$

## 2. Main results

Our main result is as follows:

**Theorem 1.** *The set of algebraic points of degree at most $l \geq 5$ over $\mathbb{Q}$ on the curve $\mathcal{C}$ of affine equation $y^2 = 157(x^2-2)(x^2+x)(x^2+1)$ is given by*

$$\bigcup_{[\mathbb{K}:\mathbb{Q}] \ \leqslant \ l} \mathcal{C}(\mathbb{K}) = \mathcal{M}_1 \bigcup \mathcal{M}_2 \bigcup \mathcal{M}_3 \bigcup \mathcal{M}_4,$$

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*with:*

$$\mathcal{M}_1=\left\{\left(x,\ -\frac{\sum_{i=0}^{\frac{l}{2}} b_i x^i + a x^{\frac{5}{2}}}{\sum_{j=0}^{\frac{l-7}{2}} c_j x^{\frac{2j+1}{2}}}\right) \middle| \begin{array}{c} (b_0 \bigwedge c_0)\neq 0,\ b_{\frac{l}{2}}\neq 0 \textit{ if } l \textit{ is even },\\ c_{\frac{l-7}{2}}\neq 0 \textit{ if } l \textit{ is odd and } x \textit{ is a solution}\\ \textit{of the equation :}\end{array} \right. \left. \left(\sum_{i=0}^{\frac{l}{2}} b_i x^i + a x^{\frac{5}{2}}\right)^2 = 157\left(\sum_{j=0}^{\frac{l-7}{2}} c_j x^{\frac{2j+1}{2}}\right)^2 (x^2-2)(x^2+x)(x^2+1)\right\}$$

$$\mathcal{M}_2=\left\{\left(x,\ -\frac{a\left(x^{\frac{5}{2}}-\rho^{\frac{5}{2}}\right)+\sum_{i=2}^{\frac{l+2}{2}} b_i\left(x^i-\rho^i\right)}{\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j+1}{2}}}\right) \middle| \begin{array}{c} b_{\frac{l+2}{2}}\neq 0 \textit{ if } l \textit{ is even, } c_{\frac{l-5}{2}}\neq 0\\ \textit{if } l \textit{ is odd and } x \textit{ is a solution}\\ \textit{of the equation :}\end{array} \right. \left. \left(a\left(\frac{x^{\frac{5}{2}}-\rho^{\frac{5}{2}}}{x}\right)+\sum_{i=2}^{\frac{l+2}{2}} b_i\left(\frac{x^i-\rho^i}{x}\right)\right)^2 = 157\left(\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j-1}{2}}\right)^2 (x^2-2)(x^2+x)(x^2+1) \quad \textit{with } \rho=-1 \right\}$$

$$\mathcal{M}_3=\left\{\left(x,\ -\frac{a\left(x^{\frac{5}{2}}+\eta\right)+\sum_{i=1}^{\frac{l+2}{2}} b_i\left(x^i+\varsigma^i\right)}{\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j+1}{2}}}\right) \middle| \begin{array}{c} b_{\frac{l+4}{2}}\neq 0 \textit{ if } l \textit{ is even, } c_{\frac{l-3}{2}}\neq 0\\ \textit{if } l \textit{ is odd and } x \textit{ is a solution}\\ \textit{of the equation :}\end{array} \right. \left. \left(a\left(\frac{x^{\frac{5}{2}}+\eta}{x}\right)+\sum_{i=1}^{\frac{l+2}{2}} b_i\left(\frac{x^i+\varsigma^i}{x}\right)\right)^2 = 157\left(\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j-1}{2}}\right)^2 (x^2-2)(x^2+x)(x^2+1) \quad \textit{with } \eta=-\frac{1}{2}\left((\sqrt{2})^{\frac{5}{2}}+(-\sqrt{2})^{\frac{5}{2}}\right) \textit{ and } \varsigma^i=-\frac{1}{2}\left((\sqrt{2})^i+(-\sqrt{2})^i\right)\right\}$$

$$\mathcal{M}_4=\left\{\left(x,\ -\frac{a\left(x^{\frac{5}{2}}+\mu\right)+\sum_{i=1}^{\frac{l+2}{2}} b_i\left(x^i+\nu^i\right)}{\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j+1}{2}}}\right) \middle| \begin{array}{c} b_{\frac{l+2}{2}}\neq 0 \textit{ if } l \textit{ is even, } c_{\frac{l-5}{2}}\neq 0\\ \textit{if } l \textit{ is odd and } x \textit{ is a solution}\\ \textit{of the equation :}\end{array} \right. \left. \left(a\left(\frac{x^{\frac{5}{2}}+\mu}{x}\right)+\sum_{i=1}^{\frac{l+2}{2}} b_i\left(\frac{x^i+\nu^i}{x}\right)\right)^2 = 157\left(\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j-1}{2}}\right)^2 (x^2-2)(x^2+x)(x^2+1) \quad \textit{with } \mu=\frac{1}{3}\left((\imath)^{\frac{5}{2}}+(-\imath)^{\frac{5}{2}}\right) \textit{ and } \nu^i=\frac{1}{3}\left((\imath)^i+(-\imath)^i\right)\right\}$$

## 3. Auxiliary results

The text of the following environments should be in italics. For a divisor $\mathfrak{D}$ on $\mathcal{C}$, let $\mathcal{L}(\mathfrak{D})$ denote the $\mathbb{Q}$-vector space of rational functions $f$ of definite over $\mathbb{Q}$ such that $f=0$ or $div(f)\geq -\mathfrak{D}$ ; $\mathfrak{l}(\mathfrak{D})$ denotes the $\mathbb{Q}$-dimension of $\mathcal{L}(\mathfrak{D})$ (see [4]).

**Lemma 1.** *We have :* $\mathcal{J}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

*Proof.* For the proof see [1, page 8]. □

The projective form of the equation of the curve :

$$\mathcal{C}: \ Z^4Y^2 = 157(X^2 - 2Z^2)(X^2 + XZ)(X^2 + Z^2),$$

we note $P_0, P_1, P_2, P_3, P_4, P_5$ and $P_\infty$ the points of $\mathcal{C}$, defined by: $P_0 = [0:0:1]$, $P_1 = [-1:0:1]$, $P_2 = [\sqrt{2}:0:1]$, $P_3 = [-\sqrt{2}:0:1]$, $P_4 = [\imath:0:1]$, $P_5 = [-\imath:0:1]$ and $P_\infty = [1:0:0]$ with $\imath^2 = -1$.

**Corollary 1.** *For the curve* $\mathcal{C}: y^2 = 157(x^2-2)(x^2+x)(x^2+1)$, *we have :*
$(i)$**:** $div(x) = 2P_0 - 2P_\infty$,
$(vii)$**:** $div(y) = P_0 + P_1 + P_2 + P_3 + P_4 + P_5 - 6P_\infty$.

*Proof.* We define by $x$, $y$ the affine coordinates of the curve $\mathcal{C}$ in the following way : $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$.

$(i)$ $div(x) = div\left(\dfrac{X}{Z}\right) = (X=0)\cdot\mathcal{C} - (Z=0)\cdot\mathcal{C}$.

- **:** For $X = 0$, imply that: $Y^2 = 0$ or $Z^4 = 0$.
  We thus obtain the points : $P_0 = [0:0:1]$ and $P_\infty = [0:1:0]$ with an order multiplicity of 2 and 4 respectively. Hence
  $$(X=0)\cdot\mathcal{C} = 2P_0 + 4P_\infty \quad (1)$$
- **:** The same for $Z = 0$, this implies : $X^6 = 0$.
  We therefore obtain the point $P_\infty = [0:1:0]$ with an equal order of multiplicity 6. Hence
  $$(Z=0)\cdot\mathcal{C} = 6P_\infty. \quad (2)$$
  From relations (1) and (2), we can infer that : $div(x) = 2P_0 - 2P_\infty$.

$(ii)$ $div(y) = div\left(\dfrac{Y}{Z}\right) = (Y=0)\cdot\mathcal{C} - (Z=0)\cdot\mathcal{C}$.

- **:** For $Y = 0$, equivalent to :
  $$X(X+Z)(X-\sqrt{2}Z)(X+\sqrt{2}Z)(X-\imath Z)(X+\imath Z) = 0.$$
  Thus, we obtain the points : $P_0 = [0:0:1]$, $P_1 = [-1:0:1]$, $P_2 = [\sqrt{2}:0:1]$, $P_3 = [-\sqrt{2}:0:1]$, $P_4 = [\imath:0:1]$ and $P_5 = [-\imath:0:1]$ with a multiplicative order equal to 1 for each point. Hence
  $$(Y=0)\cdot\mathcal{C} = P_0 + P_1 + P_2 + P_3 + P_4 + P_5. \quad (3)$$
- **:** For $Z = 0$,his is equivalent to obtaining the relation (0.2).
  Thus the relations (2) and (4), we deduce that :
  $$div(y) = P_0 + P_1 + P_2 + P_3 + P_4 + P_5 - 6P_\infty.$$

□

**Corollary 2.** *The following results are the consequences of Lemma 2.*
$a)$**:** $j(P_0) + j(P_1) + j(P_2) + j(P_3) + j(P_4) + j(P_5) = 0$,
$b)$**:** $2j(P_0) = 0$

**Lemma 2.** *We have :*

$$\mathcal{J}(\mathbb{Q}) = \langle [P_0 + P_1 - 2P_\infty], \ [P_2 + P_3 - 2P_\infty] \rangle$$

*Proof.* see [1, page 10], Lemma 3.3. □

**Lemma 3.** *A* $\mathbb{Q}$*-base of* $\mathcal{L}(dP_\infty)$ *is given by :*

$$\mathcal{B}_m = \left\{x^i, \ 0 \le i \le \frac{m}{2}\right\} \bigcup \left\{yx^{\frac{2j+1}{2}}, \ 0 \le j \le \frac{m-7}{2}\right\} \bigcup \left\{x^{\frac{5}{2}}\right\}$$

*Proof.* It is easy to show that $\mathcal{B}_d$ is a free family, it then remains to show that $\#\mathcal{B}_d = \dim \mathcal{L}(dP_\infty)$. We know that the genus of $\mathcal{C}$ is $g = 2$ (see [1, page 174]). Since the curve has genus 2, according to the Riemann-Roch theorem, we have $\dim \mathcal{L}(mP_\infty) = d - g + 1 = m - 1$ since $m \geq 2g - 1 = 3$. Two cases are possible:

**First case :** suppose that $m$ is even, then $m = 2h$, we obtain : $i \leq \frac{m}{2} \Leftrightarrow i \leq \frac{2h}{2} = h$ the same $j \leq \frac{m-7}{2} \Leftrightarrow j \leq \frac{2h-7}{2} \Leftrightarrow j \leq h - \frac{7}{2} \Longrightarrow j < h - \frac{6}{2} = h - 3 \Longrightarrow j \leqslant h - 4$. It follows that :

$$\mathcal{B}_m = \left\{x^{\frac{5}{2}}\right\} \bigcup \{1,\ x,\ \ldots, x^h\} \bigcup \left\{yx^{\frac{1}{2}},\ \ldots, yx^{\frac{2h-7}{2}}\right\}.$$

So we have :
$\#\mathcal{B}_m = 1 + h + 1 + h - \dfrac{7}{2} - \dfrac{1}{2} + 1 = 2h - 1 = m - 1 = \dim \mathcal{L}(mP_\infty).$

**Second case :** suppose that $m$ is odd, then $m = 2h + 1$, we get: $i \leq \frac{m}{2} \Leftrightarrow i \leq \frac{2h+1}{2} \Leftrightarrow i \leq h + \frac{1}{2} \Longrightarrow i < h + 1 \Longrightarrow i \leqslant h$ the same $j \leq \frac{m-7}{2} \Leftrightarrow j \leq \frac{2h-6}{2} = h - 3$. Thus we have :

$$\mathcal{B}_m = \left\{x^{\frac{5}{2}}\right\} \bigcup \{1,\ x,\ \ldots, x^h\} \bigcup \left\{yx^{\frac{1}{2}},\ \ldots, yx^{h-\frac{5}{2}}\right\}.$$

It follows that :
$\#\mathcal{B}_m = 1 + h + 1 + h - \dfrac{5}{2} - \dfrac{1}{2} + 1 = 2h = m - 1 = \dim \mathcal{L}(mP_\infty).$

□

## 4. Proof of the Theorem

The proof of **Theorem 1** being the theorem is given as follows:

*Proof.* Let $R \in \mathcal{C}(\mathbb{Q})$ of degree $[\mathbb{Q}(R) : \mathbb{Q}] = l$ with $l \geq 5$. Consider $R_1, \ldots\ldots, R_l$ the Galois conjugates of $R$ and let $t = [R_1 + \ldots + R_l - lP_\infty] \in \mathcal{J}(\mathbb{Q})$ where $\mathcal{J}(\mathbb{Q}) = \{\alpha j(P_0) + \alpha j(P_1) + \beta j(P_2) + \beta j(P_3),\ \text{with}\ \ \alpha,\ \beta \in \{0\ ,\ 1\}\}$ from **Lemma 2**, so $t = \alpha j(P_0) + \alpha j(P_1) + \beta j(P_2) + \beta j(P_3),$ with $\alpha,\ \beta \in \{0,\ 1\}$, which gives the following formula :

$$[R_1 + \ldots\ldots + R_l - lP_\infty] = [\alpha P_0 + \alpha P_1 + \beta P_2 + \beta P_3 - (2\alpha + 2\beta)\, P_\infty],$$

according to the Abel-Jacobi theorem ([4, page 156]), there exists a rational function of definite on $\mathbb{Q}$ such that :

$$div(f) = R_1 + \ldots + R_l - \alpha P_0 + \alpha P_1 + \beta P_2 + \beta P_3 - (2\alpha + 2\beta + l)\, P_\infty \qquad (\star)$$

We study the following cases :

**1st Case :** $\alpha = \beta = 0$.

The formula $(\star)$ becomes : $div(f) = R_1 + \ldots + R_l - lP_\infty$, so $f \in \mathcal{L}(lP_\infty)$. By the **Lemma 3**, we have : $f = ax^{\frac{5}{2}} + \sum\limits_{i=0}^{\frac{l}{2}} b_i x^i + \sum\limits_{j=0}^{\frac{l-7}{2}} c_j y x^{\frac{2j+1}{2}}$ with $a_0$, $b_0$ and $c_0$ not simultaneously zero which we denote by $b_0 \bigwedge c_0 \neq 0$ ( otherwise of the $R_i$'s should be at $P_4$, which would be absurd), $a_{\frac{l}{2}} \neq 0$ ( otherwise one of the $R_i$'s should be at $P_\infty$, which would be absurd ) and $b_{\frac{l-7}{2}} \neq 0$ ( otherwise one of the $R_i$'s should be at $P_\infty$, which would be absurd ). At points $R_i$, we have:

$ax^{\frac{5}{2}}+\sum_{i=0}^{\frac{l}{2}}b_ix^i+\sum_{j=0}^{\frac{l-7}{2}}c_jyx^{\frac{2j+1}{2}}=0$, implicating thus, $y=-\dfrac{\sum_{i=0}^{\frac{l}{2}}b_ix^i+ax^{\frac{5}{2}}}{\sum_{j=0}^{\frac{l-7}{2}}c_jx^{\frac{2j+1}{2}}}$. By replacing the expression for $y$ in $y^2=157(x^2-2)(x^2+x)(x^2+1)$, we obtain the equation :

$$\left(\sum_{i=0}^{\frac{l}{2}}b_ix^i+ax^{\frac{5}{2}}\right)^2=157\left(\sum_{j=0}^{\frac{l-7}{2}}c_jx^{\frac{2j+1}{2}}\right)^2(x^2-2)(x^2+x)(x^2+1) \qquad (4)$$

The expression (4) is an equation of degree at mos $l$. Indeed, whether $l$ is even or odd, the first member of the equation (4) is of degree $2\times\left(\dfrac{l}{2}\right)=l$ and the second member is of degree $2\times\left(\dfrac{l-5}{2}\right)+5=l$.

This gives a degree point family $l$:

$$\mathcal{M}_1=\left\{\left(x,\ -\frac{\sum_{i=0}^{\frac{l}{2}}b_ix^i+ax^{\frac{5}{2}}}{\sum_{j=0}^{\frac{l-7}{2}}c_jx^{\frac{2j+1}{2}}}\right)\ \middle|\ \begin{array}{c}(b_0\bigwedge c_0)\neq 0,\ b_{\frac{l}{2}}\neq 0 \text{ if } l \text{ is even },\\ c_{\frac{l-7}{2}}\neq 0 \text{ if } l \text{ is odd and } x \text{ is a solution}\\ \text{of the equation :}\end{array}\right.$$
$$\left.\left(\sum_{i=0}^{\frac{l}{2}}b_ix^i+ax^{\frac{5}{2}}\right)^2=157\left(\sum_{j=0}^{\frac{l-7}{2}}c_jx^{\frac{2j+1}{2}}\right)^2(x^2-2)(x^2+x)(x^2+1)\right\}$$

**2nd Case :** $\alpha=1$ **and** $\beta=0$**.**

The formula $(\star)$ becomes : $div(f)=R_1+\ldots+R_l+P_0+P_1-(l+2)P_\infty$, then $f\in\mathcal{L}((l+2)P_\infty)$, according to **Lemma 3**, we have :

$f=ax^{\frac{5}{2}}+\sum_{i=0}^{\frac{l+4}{2}}b_ix^i+\sum_{j=0}^{\frac{l-5}{2}}c_iyx^{\frac{2j+1}{2}}$ and since $ord_{P_0}f=ord_{P_1}f=1$, thus implied that $b_0=0$ and $b_1=-\left(a\,(-1)^{\frac{5}{2}}\right)-\left(\sum_{i=2}^{\frac{l+2}{2}}b_i\,(-1)^i\right)$. The expression of $b_1$ can be put in the form $b_1=a\left(\rho^{\frac{5}{2}}\right)+\sum_{i=2}^{\frac{l+2}{2}}b_i\left(\rho^i\right)$ with $\rho=-1$, implying that $f=a\left(x^{\frac{5}{2}}-\rho^{\frac{5}{2}}\right)+\sum_{i=2}^{\frac{l+2}{2}}b_i\left(x^i-\rho^i\right)+\sum_{j=0}^{\frac{l-5}{2}}c_iyx^{\frac{2j+1}{2}}$ with $b_{\frac{l+2}{2}}\neq 0$ if $l$ is even (otherwise one of the $R_i$'s should be at $P_\infty$, which would be absurd) and $c_{\frac{l-5}{2}}\neq 0$ if $l$ is odd (otherwise one of the $R_i$'s should be at $P_\infty$, which would be absurd). At point $R_i$, we have:

$$a\left(x^{\frac{5}{2}}-\rho^{\frac{5}{2}}\right)+\sum_{i=2}^{\frac{l+2}{2}}b_i\left(x^i-\rho^i\right)+\sum_{j=0}^{\frac{l-5}{2}}c_iyx^{\frac{2j+1}{2}}=0,$$

implying so, $y = -\dfrac{a\left(x^{\frac{5}{2}} - \rho^{\frac{5}{2}}\right) + \sum_{i=2}^{\frac{l+2}{2}} b_i\left(x^i - \rho^i\right)}{\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j+1}{2}}}$. By replacing the expression for $y$ in $y^2 = 157(x^2-2)(x^2+x)(x^2+1)$, we obtain the following equation :

$$\left(a\left(x^{\frac{5}{2}} - \rho^{\frac{5}{2}}\right) + \sum_{i=2}^{\frac{l+2}{2}} b_i\left(x^i - \rho^i\right)\right)^2 = 157\left(\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j+1}{2}}\right)^2 (x^2-2)(x^2+x)(x^2+1)$$

This equation can be noted as follows :

$$\left(a\left(\frac{x^{\frac{5}{2}} - \rho^{\frac{5}{2}}}{x}\right) + \sum_{i=2}^{\frac{l+2}{2}} b_i\left(\frac{x^i - \rho^i}{x}\right)\right)^2 = 157\left(\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j-1}{2}}\right)^2 (x^2-2)(x^2+x)(x^2+1) \quad (5)$$

The expression (5) is an equation of degree at mos $l$. Indeed, whether $l$, the first member of equation (5) is of degree $2\left(\frac{l+2}{2} - 1\right) = l$ and the second member is of degree $2\left(\frac{2 \times (\frac{l-5}{2}) - 1}{2}\right) + 6 = l$.

This gives a degree point family $l$:

$$\mathcal{M}_2 = \left\{ \left( x,\ -\frac{a\left(x^{\frac{5}{2}} - \rho^{\frac{5}{2}}\right) + \sum_{i=2}^{\frac{l+2}{2}} b_i\left(x^i - \rho^i\right)}{\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j+1}{2}}} \right) \middle| \begin{array}{c} b_{\frac{l+2}{2}} \neq 0 \text{ if } l \text{ is even, } c_{\frac{l-5}{2}} \neq 0 \\ \text{if } l \text{ is odd and } x \text{ is a solution} \\ \text{of the equation :} \end{array} \right.$$
$$\left. \left(a\left(\frac{x^{\frac{5}{2}} - \rho^{\frac{5}{2}}}{x}\right) + \sum_{i=2}^{\frac{l+2}{2}} b_i\left(\frac{x^i - \rho^i}{x}\right)\right)^2 = 157\left(\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j-1}{2}}\right)^2 (x^2-2)(x^2+x)(x^2+1) \text{ with } \rho = -1 \right\}$$

**3rd Case :** $\alpha = 0$ **and** $\beta = 1$.

The formula $(\star)$ becomes : $div(f) = R_1 + \ldots + R_l + P_2 + P_3 - (l+2)P_\infty$, then $f \in \mathcal{L}((l+2)P_\infty)$, according to **Lemma 3**, we have:

$f = ax^{\frac{5}{2}} + \sum_{i=0}^{\frac{l+4}{2}} b_i x^i + \sum_{j=0}^{\frac{l-3}{2}} c_i y x^{\frac{2j+1}{2}}$ and since $ord_{P_2} f = ord_{P_3} f = 1$, thus implied that $b_0 = -\frac{1}{2}a\left(\left(\sqrt{2}\right)^{\frac{5}{2}} + \left(-\sqrt{2}\right)^{\frac{5}{2}}\right) - \frac{1}{2}\sum_{i=1}^{\frac{l+2}{2}} b_i\left(\left(\sqrt{2}\right)^i + \left(-\sqrt{2}\right)^i\right)$ this writing cam be put the form $b_0 = a\eta + \sum_{i=1}^{\frac{l+2}{2}} b_i \varsigma^i$; with $\eta = -\frac{1}{2}\left((\sqrt{2})^{\frac{5}{2}} + (-\sqrt{2})^{\frac{5}{2}}\right)$ and $\varsigma^i = -\frac{1}{2}\left((\sqrt{2})^i + (-\sqrt{2})^i\right)$, implying that

$f = a\left(x^{\frac{5}{2}}+\eta\right)+\sum_{i=1}^{\frac{l+2}{2}} b_i\left(x^i+\varsigma^i\right)+\sum_{j=0}^{\frac{l-5}{2}} c_i y x^{\frac{2j+1}{2}}$ with $b_{\frac{l+2}{2}} \neq 0$ if $l$ is even (otherwise one of the $R_i$'s should be at $P_\infty$, which would be absurd) and $c_{\frac{l-5}{2}} \neq 0$ if $l$ is odd (otherwise one of the $R_i$'s should be at $P_\infty$, which would be absurd). At point $R_i$, we have :

$$a\left(x^{\frac{5}{2}}+\eta\right)+\sum_{i=1}^{\frac{l+2}{2}} b_i\left(x^i+\varsigma^i\right)+\sum_{j=0}^{\frac{l-5}{2}} c_i y x^{\frac{2j+1}{2}}=0,$$

implying so, $y=-\dfrac{a\left(x^{\frac{5}{2}}+\eta\right)+\sum_{i=1}^{\frac{l+2}{2}} b_i\left(x^i+\varsigma^i\right)}{\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j+1}{2}}}$. By replacing the expression for $y$ in $y^2=157(x^2-2)(x^2+x)(x^2+1)$, we obtain the following equation :

$$\left(a\left(x^{\frac{5}{2}}+\eta\right)+\sum_{i=1}^{\frac{l+2}{2}} b_i\left(x^i+\varsigma^i\right)\right)^2=157\left(\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j+1}{2}}\right)^2(x^2-2)(x^2+x)(x^2+1)$$

This equation can be noted as follows :

$$\left(a\left(\frac{x^{\frac{5}{2}}+\eta}{x}\right)+\sum_{i=1}^{\frac{l+2}{2}} b_i\left(\frac{x^i+\varsigma^i}{x}\right)\right)^2=157\left(\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j-1}{2}}\right)^2(x^2-2)(x^2+x)(x^2+1) \quad (6)$$

The expression (6) is an equation of degree at mos $l$. Indeed, whether $l$, the first member of equation (6) is of degree $2\left(\dfrac{l+2}{2}-1\right)=l$ and the second member is of degree $2\left(\dfrac{2\times(\frac{l-5}{2})-1}{2}\right)+6=l$.

This gives a degree point family $l$:

$$\mathcal{M}_3=\left\{\begin{array}{c|c}\left(x,\ -\dfrac{a\left(x^{\frac{5}{2}}+\eta\right)+\sum_{i=1}^{\frac{l+2}{2}} b_i\left(x^i+\varsigma^i\right)}{\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j+1}{2}}}\right) & \begin{array}{c} b_{\frac{l+4}{2}}\neq 0 \text{ if } l \text{ is even, } c_{\frac{l-3}{2}}\neq 0 \\ \text{if } l \text{ is odd and } x \text{ is a solution} \\ \text{of the equation :}\end{array} \\ \multicolumn{2}{c}{\left(a\left(\frac{x^{\frac{5}{2}}+\eta}{x}\right)+\sum_{i=1}^{\frac{l+2}{2}} b_i\left(\frac{x^i+\varsigma^i}{x}\right)\right)^2=157\left(\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j-1}{2}}\right)^2(x^2-2)(x^2+x)(x^2+1)} \\ \multicolumn{2}{c}{\text{with } \eta=-\frac{1}{2}\left((\sqrt{2})^{\frac{5}{2}}+(-\sqrt{2})^{\frac{5}{2}}\right) \text{ and } \varsigma^i=-\frac{1}{2}\left((\sqrt{2})^{i}+(-\sqrt{2})^{i}\right)}\end{array}\right\}$$

**4th Case : :** $\alpha=1$ **and** $\beta=1$**.**

The formula $(\star)$ becomes : $div(f)=R_1+\ldots+R_l+P_0+P_1+P_2+P_3-(l+4)P_\infty$, from Corollary 1 we have hence : $div(f)=R_1+\ldots+R_l+P_4+P_5-(l+2)P_\infty$, then $f\in\mathcal{L}((l+2)P_\infty)$, according to **Lemma 3**, we have:

$f = ax^{\frac{5}{2}} + \sum_{i=0}^{\frac{l+2}{2}} b_i x^i + \sum_{j=0}^{\frac{l-5}{2}} c_i y x^{\frac{2j+1}{2}}$ and since $ord_{P_4} f = ord_{P_5} f = 1$, thus implied that $b_0 = -\frac{1}{2}\left(a\left((\imath)^{\frac{5}{2}} + (-\imath)^{\frac{5}{2}}\right)\right) + \frac{1}{2}\left(\sum_{i=1}^{\frac{l+2}{2}} b_i \left((\imath)^i + (-\imath)^i\right)\right)$ this writing cam be put the form $b_0 = a\left(x^{\frac{5}{2}} + \mu\right) + \sum_{i=1}^{\frac{l+2}{2}} b_i \left(x^i + \nu^i\right)$ with $\mu = -\frac{1}{2}\left((\imath)^{\frac{5}{2}} + (-\imath)^{\frac{5}{2}}\right)$ and $\nu^i = -\frac{1}{2}\left((\imath)^i + (-\imath)^i\right)$, implying that

$f = a\left(x^{\frac{5}{2}} + \mu\right) + \sum_{i=1}^{\frac{l+2}{2}} b_i \left(x^i + \nu^i\right) + \sum_{j=0}^{\frac{l-5}{2}} c_i y x^{\frac{2j+1}{2}}$ with $b_{\frac{l+2}{2}} \neq 0$ if $l$ is even (otherwise one of the $R_i$'s should be at $P_\infty$, which would be absurd) and $c_{\frac{l-5}{2}} \neq 0$ if $l$ is odd (otherwise one of the $R_i$'s should be at $P_\infty$, which would be absurd). At point $R_i$, we have : $a\left(x^{\frac{5}{2}} + \mu\right) + \sum_{i=1}^{\frac{l+2}{2}} b_i \left(x^i + \nu^i\right) + \sum_{j=0}^{\frac{l-5}{2}} c_i y x^{\frac{2j+1}{2}} = 0$, implying so, $y = -\frac{a\left(x^{\frac{5}{2}} + \mu\right) + \sum_{i=1}^{\frac{l+2}{2}} b_i \left(x^i + \nu^i\right)}{\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j+1}{2}}}$. By replacing the expression for $y$ in $y^2 = 157(x^2-2)(x^2+x)(x^2+1)$, we obtain the following equation :

$$\left(a\left(x^{\frac{5}{2}} + \mu\right) + \sum_{i=1}^{\frac{l+2}{2}} b_i \left(x^i + \nu^i\right)\right)^2 = 157\left(\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j+1}{2}}\right)^2 (x^2-2)(x^2+x)(x^2+1)$$

This equation can be noted as follows :

$$\left(a\left(\frac{x^{\frac{5}{2}} + \mu}{x}\right) + \sum_{i=1}^{\frac{l+2}{2}} b_i \left(\frac{x^i + \nu^i}{x}\right)\right)^2 = 157\left(\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j-1}{2}}\right)^2 (x^2-2)(x^2+x)(x^2+1) \qquad (7)$$

The expression (7) is an equation of degree at mos $l$. Indeed, whether $l$, the first member of equation (7) is of degree $2\left(\frac{l+2}{2} - 1\right) = l$ and the second member is of degree $2\left(\frac{2 \times (\frac{l-5}{2}) - 1}{2}\right) + 6 = l$.

This gives a degree point family $l$:

$$\mathcal{M}_4 = \left\{ \begin{array}{c} \left( x,\ -\dfrac{a\left(x^{\frac{5}{2}}+\mu\right)+\displaystyle\sum_{i=1}^{\frac{l+2}{2}} b_i\left(x^i+\nu^i\right)}{\displaystyle\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j+1}{2}}} \right) \;\middle|\; \begin{array}{c} b_{\frac{l+2}{2}} \neq 0 \text{ if } l \text{ is even, } c_{\frac{l-5}{2}} \neq 0 \\ \text{if } l \text{ is odd and } x \text{ is a solution} \\ \text{of the equation :} \end{array} \\ \left(a\left(\dfrac{x^{\frac{5}{2}}+\mu}{x}\right)+\displaystyle\sum_{i=1}^{\frac{l+2}{2}} b_i\left(\dfrac{x^i+\nu^i}{x}\right)\right)^2 = 157\left(\displaystyle\sum_{j=0}^{\frac{l-5}{2}} c_i x^{\frac{2j-1}{2}}\right)^2 (x^2-2)(x^2+x)(x^2+1) \\ \text{with } \mu = \dfrac{1}{3}\left((\imath)^{\frac{5}{2}}+(-\imath)^{\frac{5}{2}}\right) \quad \text{and} \quad \nu^i = \dfrac{1}{3}\left((\imath)^i+(-\imath)^i\right) \end{array} \right\}$$

□

**Conclusion :** The set of algebraic points of degree at most $l \geq 5$ on $\mathbb{Q}$ on the curve $\mathcal{C}$ is given by:

$$\bigcup_{[\mathbb{K}:\mathbb{Q}] \leqslant l} \mathcal{C}(\mathbb{K}) = \mathcal{M}_1 \bigcup \mathcal{M}_2 \bigcup \mathcal{M}_3 \bigcup \mathcal{M}_4$$

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