TJMM
15 (2023), No. 1-2, 07-15

A NEW TYPE RANDOM ITERATION SCHEME FOR RANDOM COMMON FIXED POINT OF THREE OPERATORS

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Abstract. In this paper, we introduce a random iteration scheme for three asymptotically nonexpansive random operators defined on a uniformly convex separable Banach space and prove its convergence to a common fixed point of three random operators.

1. Introduction

The study of random fixed points is a crucial subject in this paper. Špaček [13] first proved the random fixed point theorems for random contraction mappings on separable complete metric spaces in 1955. In his works on the subject that was first initiated by Špaček, Hans [4, 5] furthered the study of random fixed points in his publications from 1957 and 1961. The random Ishikawa scheme was put forth in [3] to generate iterative methods for finding fixed points of random operators defined on linear spaces. Recent research has been devoted to the examination of fixed points of random operators, wherein various scholars have explored the theoretical underpinnings and practical implications of this topic (see [1, 3, 6, 7, 11]). After, Beg and Abbas in 2006, the work conducted by Plubtieng et al. [8] concerned the weak and strong convergence theorems set up for a modified Noor iterative scheme with errors that three-step asymptotically nonexpansive mappings in the context of Banach spaces.

2. Preliminaries

Suppose that Θ is a nonempty subset of a Banach space $\mathfrak{X}$. Denote by $(\mho, \Sigma)$ a measurable space with Σ a sigma-algebra of subsets of $\mho$. For a given mapping $f : \mho \to \mathfrak{X}$, we say that measurable mapping if $f^{-1}(U) \in \Sigma$ for each open subset U of $\mathfrak{X}$. We also say that a measurable mapping $f : \mho \to \mathfrak{X}$ is the random fixed point of the random map $E : \mho \times \Theta \to \mathfrak{X}$ if $E(\ell, p(\ell)) = p(\ell)$, for each $\ell \in \mho$. Denote by $RF(E)$ and $E^m(\ell, u_0)$ the set of all random fixed points of a random map E and the m-th iterate $E(\ell, E(\ell, E(, \ldots, E(\ell, u_0))))$ of E, respectively. Moreover, the letter I denotes the random mapping $I : \mho \times \Theta \to \Theta$ defined by $I(\ell, u_0) = u_0$ and $E^0 = I$.

Definition 1. *Suppose that Θ is a nonempty subset of a separable Banach space $\mathfrak{X}$ and $E : \mho \times \Theta \to \Theta$ is a random map. The map E is said to be*

(a) a nonexpansive random operator if arbitrary $u_0, \upsilon_0 \in \Theta$, one has

$$\|E(\ell, u_0) - E(\ell, \upsilon_0)\| \leq \|u_0 - \upsilon_0\| \tag{1}$$

for each $\ell \in \mho$;

2010 *Mathematics Subject Classification.* 47H09, 47H10.

Key words and phrases. Common random fixed point, asymptotically nonexpansive random operators, uniformly convex Banach spaces.

(b) an asymptotically nonexpansive random operator if there exists a sequence of measurable mappings $r_m : \mho \to [0,\infty)$ with $\lim_{m\to\infty} \mu_m(\ell) = 0$, for each $\ell \in \mho$, such that for arbitrary $u_0, \upsilon_0 \in \Theta$

$$\|E^m(\ell, u_0) - E^m(\ell, \upsilon_0)\| \leq (1 + \mu_m(\ell)) \|u_0 - \upsilon_0\| \tag{2}$$

for each $\ell \in \mho$;

(c) a uniformly L-Lipschitzian random operator if arbitrary $u_0, \upsilon_0 \in \Theta$, one has

$$\|E^m(\ell, u_0) - E^m(\ell, \upsilon_0)\| \leq L\|u_0 - \upsilon_0\| \tag{3}$$

where $m = 1, 2, \ldots$, and L is a positive constant;

(d) a semi-compact random operator if for a sequence of measurable mappings $\{\xi_m\}$ from $\mho$ to Θ, with $\lim_{m\to\infty} \|\xi_m(\ell) - E(\ell, \xi_m(\ell))\| = 0$,

for every $\ell \in \mho$, one has a subsequence $\{\xi_{m_k}\}$ of $\{\xi_m\}$ and a measurable mapping $f : \mho \to \Theta$ such that $\{\xi_{m_k}\}$ converges to f as $k \to \infty$;

(e) completely continuous random operator if for every bounded sequence of measurable mappings $\{\xi_m\}$ from $\mho$ to Θ, there exists a subsequence say $\{\xi_{m_k}\}$ of $\{\xi_m\}$ such that the sequence $\{E(\ell, \xi_{m_k}(\ell))\}$ converges to some element of the range of E for every $\ell \in \mho$.

Definition 2. *([2]) Suppose that $E : \mho \times \Theta \to \Theta$ is a random operator, where Θ is a nonempty convex subset of a separable Banach space $\mathfrak{X}$. Suppose that $f_0 : \mho \to \Theta$ is a measurable mapping from $\mho$ to Θ. Moreover, suppose that sequences of functions $\{\varrho_m\}$, $\{\varsigma_m\}$, and $\{\xi_m\}$ are defined as follows:*

$$\begin{aligned} \varrho_m(\omega) &= \alpha''_m E^m(\omega, \xi_m(\ell)) + \beta''_m \xi_m(\ell), \\ \varsigma_m(\omega) &= \alpha'_m E^m(\omega, \varrho_m(\ell)) + \beta'_m \xi_m(\ell), \\ \xi_{m+1}(\omega) &= \alpha_m E^m(\omega, \varsigma_m(\ell)) + \beta_m \xi_m(\ell) \end{aligned} \tag{4}$$

for each $\ell \in \mho$, $m = 0, 1, 2, \ldots$, where $\{\alpha_m\}, \{\alpha'_m\}, \{\alpha''_m\}, \{\beta_m\}, \{\beta'_m\}$, and $\{\beta''_m\}$ are sequences of real numbers in $[0,1]$. Obviously $\{\varrho_m\}$, $\{\varsigma_m\}$, and $\{\xi_m\}$ are sequences of measurable functions from $\mho$ to Θ.

Beg and Abbas [2] obtained some fixed point results for weakly contractive and asymptotically nonexpansive random operators on arbitrary Banach spaces using the above iterative process.

After, Plubtieng et al. [9] introduced random the following iterative processes with errors for three asymptotically nonexpansive random operators and studied the necessary conditions for the convergence of these processes. Their results extended and improved the recent ones announced by Beg and Abbas [2].

Definition 3. *Let $E_1, E_2, E_3 : \mho \times \Theta \to \Theta$ be three random operators, where Θ is a nonempty convex subset of a separable Banach space $\mathfrak{X}$. Let $\xi_0 : \mho \to \Theta$ be a measurable mapping from $\mho$ to Θ, let $\{f_m\}, \{g_m\}, \{h_m\}$ be bounded sequences of measurable functions from $\mho$ to Θ. Define sequences of functions $\{\varrho_m\}, \{\varsigma_m\}$, and $\{\xi_m\}$, as given below:*

$$\begin{aligned} \varrho_m(\ell) &= \gamma_m E_3^m \xi_m(\ell) + \gamma'_m \xi_m(\ell) + \gamma''_m h_m(\ell), \\ \varsigma_m(\ell) &= \beta_m E_2^m(\ell, \varrho_m(\ell)) + \beta'_m \xi_m(\ell) + \beta''_m g_m(\ell), \\ \xi_{m+1}(\ell) &= \alpha_m E_1^m(\ell, \varsigma_m(\ell)) + \alpha'_m \xi_m(\ell) + \alpha''_m f_m(\ell), \end{aligned} \tag{5}$$

for each $\ell \in \mho$, $m = 0, 1, 2, \ldots$, where $\{\alpha_m\}, \{\alpha'_m\}, \{\alpha''_m\}, \{\beta_m\}, \{\beta'_m\}, \{\beta''_m\}$, $\{\gamma_m\}$, $\{\gamma'_m\}$, and $\{\gamma''_m\}$ are sequences of real numbers in $[0,1]$ with $\alpha_m + \alpha'_m + \alpha''_m = \beta_m + \beta'_m + \beta''_m = \gamma_m + \gamma'_m + \gamma''_m = 1$. Taking $E_1 = E_2 = E_3 \equiv E$, and $\alpha''_m = \beta''_m = \gamma''_m \equiv 0$ at the above iteration, then (5) reduces to (4).

Remark 1. *If we take $E : \Theta \to \Theta$ is an operator, where Θ is a nonempty convex subset of a normed space $\mathfrak{X}$, then the iteration (4) reduces to Mann iteration (6). Also, taking $E_1, E_2, E_3 : \Theta \to \Theta$ are three operators, where Θ is a nonempty convex subset of a normed space $\mathfrak{X}$, then the iteration (5) reduces to Mann iteration with errors as follows:*

$$\begin{aligned} \varrho_m &= \alpha''_m E^m \xi_m + \beta''_m \xi_m, \\ \varsigma_m &= \alpha'_m E^m \varrho_m + \beta'_m \xi_m, \\ \xi_{m+1} &= \alpha_m E^m \varsigma_m + \beta_m \xi_m, \end{aligned} \tag{6}$$

and

$$\begin{aligned} \varrho_m &= \gamma_m E_3^m \xi_m + \gamma'_m \xi_m + \gamma''_m h_m, \\ \varsigma_m &= \beta_m E_2^m \varrho_m + \beta'_m \xi_m + \beta''_m g_m, \\ \xi_{m+1} &= \alpha_m E_1^m \varsigma_m + \alpha'_m \xi_m + \alpha''_m f_m. \end{aligned} \tag{7}$$

Based on the above studies, we define the following iteration, which is more effective and useful than the above iterations. And, we obtain some convergence results of this iteration for three asymptotically nonexpansive random operators. Our iteration process is as follows:

Definition 4. *Let $E_1, E_2, E_3 : \mho \times \Theta \to \Theta$ be three random operators, where Θ is a nonempty convex subset of a separable Banach space $\mathfrak{X}$. Let $\xi_0 : \mho \to \Theta$ be a measurable mapping from $\mho$ to Θ, let $\{f_m\}, \{g_m\}, \{h_m\}$ be bounded sequences of measurable functions from $\mho$ to Θ. Define sequences of functions $\{\varrho_m\}, \{\varsigma_m\}$, and $\{\xi_m\}$, as given below:*

$$\begin{aligned} \varrho_m(\ell) &= \gamma_m \xi_m(\ell) + \gamma'_m E_3^m(\xi_m, \ell) + \gamma''_m h_m(\ell), \\ \varsigma_m(\ell) &= \beta_m E_2^m(\ell, \varrho_m(\ell)) + \beta'_m g_m(\ell), \\ \xi_{m+1}(\ell) &= \alpha_m E_1^m(\ell, \varsigma_m(\ell)) + \alpha'_m f_m(\ell), \end{aligned} \tag{8}$$

for each $\ell \in \mho$, $m = 0, 1, 2, \ldots$, where $\{\alpha_m\}, \{\beta_m\}, \{\gamma_m\}, \{\alpha'_m\}, \{\beta'_m\}, \{\gamma'_m\}$, and $\{\gamma''_m\}$ are sequences of real numbers in $[0, 1]$ with $\alpha_m + \alpha'_m = \beta_m + \beta'_m = \gamma_m + \gamma'_m + \gamma''_m = 1$.

Remark 2. *If we take $E : \Theta \to \Theta$ is an operator, where Θ is a nonempty convex subset of a normed space $\mathfrak{X}$, then the iteration (4) reduces to the following iteration:*

$$\begin{aligned} \varrho_m &= \gamma_m \xi_m + \gamma'_m E_3^m \xi_m + \gamma''_m h_m, \\ \varsigma_m &= \beta_m E_2^m \varrho_m + \beta'_m g_m, \\ \xi_{m+1} &= \alpha_m E_1^m \varsigma_m + \alpha'_m f_m, \end{aligned} \tag{9}$$

In the sequel, we will need the following lemma.

Lemma 1. *([12]) Let $\{\xi_m\}, \{\varrho_m\}$ and $\{\delta_m\}$ be sequences of nonnegative real numbers such that*

$$\xi_{m+1} \le (1 + \delta_m)\, \xi_m + \varrho_m.$$

If $\sum \delta_m < \infty$ and $\sum \varrho_m < \infty$, then

(i) $\lim_{m\to\infty} \xi_m$ exists,

(ii) $\lim_{m\to\infty} \xi_m = 0$ whenever $\liminf_{m\to\infty} \xi_m = 0$.

Lemma 2. *([10]) Let $\mathfrak{X}$ be a uniformly convex Banach space with $\xi_m, \varrho_m \in \mathfrak{X}$, real numbers $r \ge 0, \alpha, \beta \in (0, 1)$, and let $\{\alpha_m\}$ be a real sequence of numbers which satisfies*

(i) $0 < \alpha \le \alpha_m \le \beta < 1$, for all $m \ge m_0$ and for some $m_0 \in \mathbb{N}$;

(ii) $\limsup_{m\to\infty} \|\xi_m\| \le r$ and $\limsup_{m\to\infty} \|\varrho_m\| \le r$;

(iii) $\lim_{m\to\infty} \|\alpha_m \xi_m + (1 - \alpha_m)\, \varrho_m\| = r$.

Then $\lim_{m\to\infty} \|\xi_m - \varrho_m\| = 0$.

3. Main results

In this section, we will give the following two lemmas to prove our main results.

Lemma 3. *Let $\mathfrak{X}$ be a uniformly convex separable Banach space, let Θ be a nonempty closed and convex subset of $\mathfrak{X}$. Let E_1, E_2, E_3 be asymptotically nonexpansive random operators from $\mho \times \Theta$ to Θ with a sequence of measurable mappings $\mu_{i_m}(\ell) : \mho \to [0, \infty)$ satisfying $\sum_{m=1}^{\infty} \mu_{i_m}(\ell) < \infty$, for each $\ell \in \mho$ and for all $i = 1, 2, 3$, and $F = \bigcap_{i=1}^{3} RF(E_i) \neq \varnothing$. Let $\{\xi_m(\ell)\}$ be the sequence as defined by (8) with $\sum_{m=1}^{\infty} \alpha'_m < \infty, \sum_{m=1}^{\infty} \beta'_m < \infty$, and $\sum_{m=1}^{\infty} \gamma'_m < \infty$. Then $\lim_{m\to\infty} \|\xi_m(\ell) - p(\ell)\|$ exists for all $p(\ell) \in F$ and for each $\ell \in \mho$.*

Proof. Assume that $p : \mho \to \Theta$ is the random common fixed point of the random operators E_1, E_2 and E_3. Since $\{f_m\}, \{g_m\}$, and $\{h_m\}$ are bounded sequences of measurable functions from $\mho$ to Θ, there exists a finite number $M(\ell)$ as follows:

$$M(\ell) = \sup_{m\geq 1} \{\|f_m(\ell) - p(\ell)\|, \|g(\ell) - p(\ell)\|, \|h_m(\ell) - p(\ell)\|\}. \tag{10}$$

For all $m \geq 1$, we write $\mu_m(\ell) = \max\{\mu_{i_m}(\ell) \mid i = 1, 2, 3\}$. Then, we obtain $\mu_m(\ell) \geq 0, \lim_{m\to 0} \mu_{i_m}(\ell) = 0$, and

$$\begin{aligned} \|\xi_{m+1}(\ell) - p(\ell)\| &= \|\alpha_m E_1^m(\ell, \varsigma_m(\ell)) + \alpha'_m f_m(\ell) - p(\ell)\| \\ &\leq \alpha_m \|E_1^m(\ell, \varsigma_m(\ell)) - p(\ell)\| + \alpha'_m \|f_m(\ell) - p(\ell)\| \\ &\leq \alpha_m (1 + \mu_m(\ell)) \|\varsigma_m(\ell) - p(\ell) + l_m\| \\ &+ \alpha'_m \|f_m(\ell) - p(\ell)\| \\ &\leq \alpha_m (1 + \mu_m(\ell) M(\ell)) \|\varsigma_m(\ell) - p(\ell)\| \\ &+ \alpha_m l_m + \alpha'_m \|f_m(\ell) - p(\ell)\|. \end{aligned} \tag{11}$$

Similarly, we obtain

$$\begin{aligned} \|\varsigma_m(\ell) - p(\ell)\| \leq\ & \beta_m (1 + \mu_m(\ell) M(\ell)) \|\varrho_m(\ell) - p(\ell)\| \\ & + \beta_m l_m + \beta'_m \|g_m(\ell) - p(\ell)\|, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \|\varrho_m(\ell) - p(\ell)\| \leq\ & \gamma_m (1 + \gamma_m(\ell) + \mu_m(\ell) M(\ell)) \|\xi_m(\ell) - p(\ell)\| \\ & + \gamma'_m l_m + \gamma''_m \|h_m(\ell) - p(\ell)\|. \end{aligned} \tag{13}$$

If we combine (13) in (12), we have

$$\begin{aligned} & \|\varsigma_m(\ell) - p(\ell)\| \\ & \leq \gamma_m \beta_m (1 + \mu_m(\ell) M(\ell)) (1 + \gamma_m + \mu_m(\ell) M(\ell)) \|\xi_m(\ell) - p(\ell)\| \\ & \quad + \beta_m (1 + \mu_m(\ell) M(\ell)) \gamma'_m l_m + \beta_m (1 + \mu_m(\ell) M(\ell)) \gamma''_m \|h_m(\ell) - p(\ell)\| \\ & \quad + \beta_m l_m + \beta'_m \|g_m(\ell) - p(\ell)\| \\ & = \gamma_m (1 - \alpha_m - \gamma_m) (1 + \mu_m(\ell) M(\ell)) (1 + \gamma_m + \mu_m(\ell) M(\ell)) \|\xi_m(\ell) - p(\ell)\| \\ & \quad + \beta_m l_m [1 + \gamma'_m (1 + \mu_m(\ell) M(\ell))] + m_m(\ell) \\ & \quad \leq (1 + \mu_m(\ell) M(\ell)) (1 + \gamma_m + \mu_m(\ell) M(\ell)) \|\xi_m(\ell) - p(\ell)\| \\ & \quad + \beta_m l_m [1 + \gamma'_m (1 + \mu_m(\ell) M(\ell))] + m_m(\ell) \end{aligned} \tag{14}$$

where

$$m_m(\ell) = \beta_m (1 + \mu_m(\ell) M(\ell)) \gamma''_m \|h_m(\ell) - p(\ell)\| + \beta'_m \|g_m(\ell) - p(\ell)\|. \tag{15}$$

From the conditions in the hypothesis of the theorem, we get that $\sum_{m=1}^{\infty} m_m(\ell) < \infty$. Substituting (14) in (11), we obtain

$$\begin{aligned}
\|\xi_{m+1}(\ell)-p(\ell)\| \leq & \alpha_m\left(\mu_m(\ell)M(\ell)\right)^2\left(1+\gamma_m+\mu_m(\ell)M(\ell)\right)\|\xi_m(\ell)-p(\ell)\| \\
& +\alpha_m\left(1+\mu_m(\ell)M(\ell)\right)\beta_m l_m[1+\gamma_m'(1+\mu_m(\ell)M(\ell))] \\
& +\alpha_m\left(1+\mu_m(\ell)M(\ell)\right)m_m(\ell)+\alpha_m l_m+\alpha_m'\|f_m(\ell)-p(\ell)\| \\
\leq & \left(1+\mu_m(\ell)\right)\left(1+\gamma_m+\mu_m(\ell)M(\ell)\right)\|\xi_m(\ell)-p(\ell)\| \\
& +b_m(\ell)+\alpha_m l_m
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
b_m(\ell) = & \alpha_m\left(1+\mu_m(\ell)M(\ell)\right)\beta_m l_m[1+\gamma_m'(1+\mu_m(\ell)M(\ell))] \\
& +\alpha_m\left(1+\mu_m(\ell)M(\ell)\right)m_m(\ell)+\alpha_m'\|f_m(\ell)-p(\ell)\|.
\end{aligned} \tag{17}$$

This implies that $\sum_{m=1}^{\infty}\mu_m(\ell) < \infty$ and $\sum_{m=1}^{\infty} b_m(\ell) < \infty$. From Lemma 1, we obtain that $\lim_{m\to\infty}\|\xi_{m+1}(\ell)-p(\ell)\|$ exists for all $\ell \in \mho$. This completes the proof. □

Lemma 4. *Let $\mathfrak{X}$ be a uniformly convex separable Banach space, and let Θ be a nonempty closed and convex subset of $\mathfrak{X}$. Let E_1, E_2, E_3 be asymptotically nonexpansive random operators from $\mho$ to Θ with sequence of measurable mappings $\mu_m(\ell) : \mho \to [0,\infty)$ satisfying $\sum_{m=1}^{\infty}\mu_m(\ell) < \infty$, for each $\ell \in \mho$ and for all $i = 1,2,3$, and $F = \bigcap_{i=1}^{3} RF(E_i) \neq \varnothing$. Let $\{\xi_m(\ell)\}$ be the sequence defined as in (8) with the following restrictions:*
(1) $0 < \alpha \leq \alpha_m, \beta_m, \gamma_m \leq 1-\alpha$, for some $\alpha \in (0,1)$, for all $m \geq m_0, \exists m_0 \in \mathbb{N}$,
(2) $\sum_{m=1}^{\infty}\alpha' < \infty, \sum_{m=1}^{\infty}\beta_m' < \infty$, and $\sum_{m=1}^{\infty}\gamma_m'' < \infty$.
Then

$$\begin{aligned}
\lim_{m\to\infty}\|E_1^m(\ell,\varsigma_m(\ell))-\xi_m(\ell)\| &= \lim_{m\to\infty}\|E_2^m(\ell,\varrho_m(\ell))-\xi_m(\ell)\| \\
&= \lim_{m\to\infty}\|E_3^m(\ell,\xi_m(\ell))-\xi_m(\ell)\| = 0,
\end{aligned} \tag{18}$$

for all $\ell \in \mho$.

Proof. Let $p(\ell) \in F$. From Lemma 3, we know that $\lim_{m\to\infty}\|\xi_{m+1}(\ell)-p(\ell)\|$ exists, for all $\ell \in \mho$. Let's say $\lim_{m\to\infty}\|\xi_m(\ell)-p(\ell)\| = r$ for some $r \geq 0$. For each $m \geq 1$, let $\mu_m(\ell) = \max\{\mu_{i_m}(\ell) \mid i = 1,2,3\}$. If we take lim sup of both sides of the inequality (14), we obtain that

$$\limsup_{m\to\infty}\|\varsigma_m(\ell)-p(\ell)\| \leq \limsup_{m\to\infty}\|\xi_m(\ell)-p(\ell)\| = r. \tag{19}$$

From the definition of the operator E_1 and (19), we have

$$\limsup_{m\to\infty}\|E_1^m(\ell,\varsigma_m(\ell))-p(\ell)\| \leq \limsup_{m\to\infty}\left(1+\mu_m(\ell)\right)\|\varsigma_m(\ell)-p(\ell)\| \leq r. \tag{20}$$

Now, consider the following inequality

$$\begin{aligned}
&\limsup_{m\to\infty}\|E_1^m(\ell,\varsigma_m(\ell))-p(\ell)+\alpha_m'\left(f_m(\ell)-\xi_m(\ell)\right)\| \\
&\leq \limsup_{m\to\infty}\|E_1^m(\ell,\varsigma_m(\ell))-p(\ell)\|+\|\alpha_m'\left(f_m(\ell)-\xi_m(\ell)\right)\|.
\end{aligned} \tag{21}$$

It follows from (20) and (21) that

$$\limsup_{m\to\infty}\|E_1^m(\ell,\varsigma_m(\ell))-p(\ell)+\alpha_m'\left(f_m(\ell)-\xi_m(\ell)\right)\| \leq r. \tag{22}$$

Using the triangle inequality, we obtain

$$\limsup_{m\to\infty}\|\xi_m(\ell)-p(\ell)+\alpha_m'\left(f_m(\ell)-\xi_m(\ell)\right)\| \leq r. \tag{23}$$

From (8) and the definition of the operator E_1, we also obtain that

$$\begin{aligned} r &= \lim_{m\to\infty} \|\xi_{m+1}(\ell) - p(\ell)\| \\ &= \lim_{m\to\infty} \|\alpha_m E_1^m(\ell, \varsigma_m(\ell)) + \alpha'_m f_m(\ell) - p(\ell)\| \\ &= \lim_{m\to\infty} \left\| \begin{array}{c} \alpha_m E_1^m(\ell, \varsigma_m(\ell)) + \alpha'_m f_m(\ell) - (1-\alpha_m)\, p(\ell) - \alpha_m p(\ell) \\ +\alpha_m \alpha'_m f_m(\ell) - \alpha_m \alpha'_m \xi_m(\ell) - \alpha_m \alpha'_m f_m(\ell) + \alpha_m \alpha'_m \xi_m(\ell) \end{array} \right\| \\ &= \lim_{m\to\infty} \|\alpha_m \left(E_1^m(\ell, \varsigma_m(\ell)) - p(\ell) + \alpha'_m (f_m(\ell) - \xi_m(\ell))\right) \\ &\quad + (1-\alpha_m)\left(\xi_m(\ell) - p(\ell) + \alpha'_m (f_m(\ell) - \xi_m(\ell))\right)\|. \end{aligned} \tag{24}$$

Using (22), (23) and Lemma 3, we have that

$$\lim_{m\to\infty} \|E_1^m(\ell, \varsigma_m(\ell)) - \xi_m(\ell)\| = 0. \tag{25}$$

Now, we will prove that $\lim_{m\to\infty} \|E_2^m(\ell, \varrho_m(\ell)) - \xi_m(\ell)\| = 0$. For all $m \geq 1$,

$$\begin{aligned} \|\xi_m(\ell) - p(\ell)\| &\leq \|E_1^m(\ell, \varsigma_m(\ell)) - \xi_m(\ell)\| + \|E_1^m(\ell, \varsigma_m(\ell)) - p(\ell)\| \\ &\leq \|E_1^m(\ell, \varsigma_m(\ell)) - \xi_m(\ell)\| + (1+\mu_m(\ell))\, \|\varsigma_m(\ell) - \xi_m(\ell)\|. \end{aligned} \tag{26}$$

Since $\lim_{m\to\infty} \mu_m(\ell) = \lim_{m\to\infty} \|E_1^m(\ell, \varsigma_m(\ell)) - \xi_m(\ell)\| = 0$, it follows from (19) and (26) that

$$\begin{aligned} r &= \lim_{m\to\infty} \|\xi_m(\ell) - p(\ell)\| \leq \liminf_{m\to\infty} \|\varsigma_m(\ell) - \xi_m(\ell)\| \\ &\leq \limsup_{m\to\infty} \|\varsigma_m(\ell) - \xi_m(\ell)\| \leq r. \end{aligned} \tag{27}$$

Thus, $\lim_{m\to\infty} \|\varsigma_m(\ell) - p(\ell)\| = r$.
Taking (8), we get that

$$\begin{aligned} \|\varrho_m(\ell) - p(\ell)\| &\leq (1+\mu_m(\ell))\, \|\xi_m(\ell) - p(\ell)\| \\ &\quad + \gamma''_m \|h(\ell) - p(\ell)\|. \end{aligned}$$

Using boundedness of $\{h_m(\ell)\}$ and $\lim_{m\to\infty} \mu_m(\ell) = 0 = \lim_{m\to\infty} \gamma''_m$, we have

$$\limsup_{m\to\infty} \|\varrho_m(\ell) - p(\ell)\| \leq \limsup_{m\to\infty} \|\xi_m(\ell) - p(\ell)\| \leq r$$

and

$$\begin{aligned} &\limsup_{m\to\infty} \|E_2^m(\ell, \varrho_m(\ell)) - p(\ell)\| \\ \leq\ &\limsup_{m\to\infty} (1+\mu_m(\ell))\, \|(\ell, \varrho_m(\ell)) - p(\ell)\| \leq r. \end{aligned}$$

Next, we consider

$$\begin{aligned} &\|E_2^m(\ell, \varrho_m(\ell)) - p(\ell) + \beta'_m (g(\ell) - \xi_m(\ell))\| \\ &\leq \|E_2^m(\ell, \varrho_m(\ell)) - p(\ell)\| + \beta'_m \|(g_m(\ell) - \xi_m(\ell))\|. \end{aligned} \tag{28}$$

Taking $\limsup$ in both sides at the above inequality, we get that

$$\limsup_{m\to\infty} \|E_2^m(\ell, \varrho_m(\ell)) - p(\ell) + \beta'_m (g_m(\ell) - \xi_m(\ell))\| \leq r.$$

From again the triangle inequality, we have

$$\limsup_{m\to\infty} \|\xi_m(\ell) - p(\ell) + \beta'_m (g_m(\ell) - \xi_m(\ell))\| \leq r.$$

Since $\lim_{m\to\infty}\|\varsigma_m(\ell)-p(\ell)\|=r$, we obtain

$$r=\lim_{m\to\infty}\|\xi_m(\ell)-p(\ell)\|=\lim_{m\to\infty}\|\beta_m E_2^m(\ell,\varrho_m(\ell))+\beta'_m g_m(\ell)-p(\ell)\| \tag{29}$$
$$=\lim_{m\to\infty}\left\|\begin{array}{c}\beta_m(E_2^m(\ell,\varrho_m(\ell))-p(\ell)+\beta'_m(g_m(\ell)-\xi_m(\ell)))\\+(1-\beta_m)(\xi_m(\ell)-p(\ell)+\beta'_m(g_m(\ell)-\xi_m(\ell)))\end{array}\right\|.$$

By Lemma 1, we get that

$$\lim_{m\to\infty}\|E_2^m(\ell,\varrho_m(\ell))-\xi_m(\ell)\|=0.$$

By using similarly argument as above, we have

$$\lim_{m\to\infty}\|E_3^m(\ell,\xi_m(\ell))-\xi_m(\ell)\|=0$$

for all $\ell\in\mho$. This completes the proof. $\square$

Theorem 1. *Let Θ be a nonempty closed and convex subset of a uniformly convex separable Banach space $\mathfrak{X}$. Assume that one of $E_1,E_2,E_3:\mho\times\Theta\to\Theta$ is either completely continuous or semi-compact asymptotically nonexpansive random operators with a sequence of measurable mappings $\mu_{i_m}(\ell):\mho\to[0,\infty)$ satisfying $\sum_{m=1}^{\infty}\mu_{i_m}(\ell)<\infty$, for all $\ell\in\mho$ and for all $i=1,2,3$ and $F=\bigcap_{i=1}^{3}RF(E_i)\neq\varnothing$. Moreover, assume that f_0 is a measurable mapping from $\mho$ to Θ. Define the sequence of functions $\{\xi_m\},\{\varsigma_m\}$, and $\{\varrho_m\}$ by (8) with $\{\alpha_m\},\{\alpha'_m\},\{\beta_m\},\{\beta'_m\},\{\gamma_m\},\{\gamma'_m\}$, and $\{\gamma''_m\}$ satisfying*

(1) $0<\alpha\le\alpha_m,\alpha'_m\le 1-\alpha$, for some $\alpha\in(0,1)$, for all $m\ge m_0,\exists m_0\in\mathbb{N}$,

(2) $\sum_{m=1}^{\infty}\alpha'_m<\infty,\sum_{m=1}^{\infty}\beta'_m<\infty$, and $\sum_{m=1}^{\infty}\gamma''_m<\infty$.

Then sequences $\{\xi_m\},\{\varsigma_m\}$, and $\{\varrho_m\}$ converge to a common random fixed point of F.

Proof. Let $p:\mho\to\Theta$ be the common random fixed point in F. From Lemma 4, we have

$$\lim_{m\to\infty}\|E_1^m(\ell,\xi_m(\ell))-\xi_m(\ell)\|=\lim_{m\to\infty}\|E_2^m(\ell,\varsigma_m(\ell))-\xi_m(\ell)\| \tag{30}$$
$$=\lim_{m\to\infty}\|E_3^m(\ell,\xi_m(\ell))-\xi_m(\ell)\|=0$$

for all $\ell\in\mho$. This implies that $\|\xi_{m+1}(\ell)-\xi_m(\ell)\|\le\alpha_m\|E_1^m(\ell,\varsigma_m(\ell))-\xi_m(\ell)\|+\alpha'_m\|f_m(\ell)-\xi_m(\ell)\|\to 0$, as $m\to\infty$, for all $\ell\in\mho$. Using the triangle inequality, we get that

$$\begin{aligned}&\|E_1^m(\ell,\xi_{m+1}(\ell))-\xi_{m+1}(\ell)\|\le\|E_1^m(\ell,\xi_{m+1}(\ell))-E_1^m(\ell,\xi_m(\ell))\|\\&+\|E_1^m\xi_m(\ell)-\xi_m(\ell)\|+\|\xi_m(\ell)-\xi_{m+1}(\ell)\|\\&\le(1+\mu_m(\ell))\|\xi_{m+1}(\ell)-\xi_m(\ell)\|+\|E_1^m(\ell,\xi_m(\ell))-\xi_m(\ell)\|\\&+\|\xi_m(\ell)-\xi_{m+1}(\ell)\|\longrightarrow 0,\quad\text{as } m\longrightarrow\infty,\end{aligned} \tag{31}$$

for all $\ell\in\mho$. Using (31), we have

$$\begin{aligned}&\|E_1(\ell,\xi_{m+1}(\ell))-\xi_{m+1}(\ell)\|\\&\le\left\|E_1(\ell,\xi_{m+1}(\ell))-E_1^{m+1}(\ell,\xi_m(\ell))\right\|+\left\|E_1^{m+1}(\ell,\xi_{m+1}(\ell))-\xi_{m+1}(\ell)\right\|\\&\le(1+\mu_m(\ell))\|\xi_{m+1}(\ell)-E_1^m\xi_{m+1}(\ell)\|\\&\quad+\left\|E_1^{m+1}(\ell,\xi_{m+1}(\ell))-\xi_{m+1}(\ell)\right\|\longrightarrow 0,\quad\text{as } m\longrightarrow\infty,\end{aligned} \tag{32}$$

for each $\ell\in\mho$. Then, we have $\lim_{m\to\infty}\|E_1(\ell,\xi_m(\ell))-\xi_m(\ell)\|=0$ for all $\ell\in\mho$. Similarly, we can show that

$$\lim_{m\to\infty}\|E_2(\ell,\xi_m(\ell))-\xi_m(\ell)\|=\lim_{m\to\infty}\|E_3(\ell,\xi_m(\ell))-\xi_m(\ell)\|=0. \tag{33}$$

Suppose that E_1 is a semicompact continuous random operator and

$\lim_{m\to\infty} \|E_1(\ell, \xi_m(\ell)) - \xi_m(\ell)\| = 0$ for all $\ell \in \mho$, then there exist a subsequence $\{\xi_{m_k}\}$ of $\{\xi_m\}$ and a measurable mapping $\xi_0 : \mho \to \Theta$ such that ξ_{m_k} converges to ξ_0. The mapping $\xi_0 : \mho \to \Theta$, being a limit of measurable mappings $\{\xi_{m_k}\}$, is measurable. Now,

$$\lim_{k\to\infty} \|\xi_{m_k}(\ell) - E_1(\ell, \xi_{m_k}(\ell))\| = \|\xi_0(\ell) - E_1(\ell, \xi_0(\ell))\| = 0 \tag{34}$$

for all $\ell \in \mho$. Thus, $\xi_0(\ell)$ is a random fixed point of E_1. Since the limit $\lim_{m\to\infty} \|\xi_m(\ell) - \xi_0(\ell)\|$ exists, we write $\lim_{m\to\infty} \xi_m(\ell) = \xi_0(\ell)$ for all $\ell \in \mho$. That is, using similarly method, we can show that $\xi_0(\ell)$ is also a random fixed point of E_2 and E_3. Note that

$$\|\varsigma_m(\ell) - \xi_m(\ell)\| \leq \beta_m \|E_2^m(\ell, \varrho_m(\ell)) - \xi_m(\ell)\| + \beta'_m \| g_m(\ell) - \xi_m(\ell)\| \to 0, \tag{35}$$

and

$$\|\varrho_m(\ell) - \xi_m(\ell)\| \leq \gamma'_m \|E_3^m(\ell, \xi_m(\ell)) - \xi_m(\ell)\| + \gamma''_m \|h_m(\ell) - \xi_m(\ell)\| \to 0, \tag{36}$$

as $m \to \infty$, for all $\ell \in \mho$. Then, $\lim_{m\to\infty} \varsigma_m(\ell) = \xi_0(\ell)$ and $\lim_{m\to\infty} \varrho_m(\ell) = \xi_0(\ell)$ for all $\ell \in \mho$. Thus $\{\xi_m\}, \{\varsigma_m\}$, and $\{\varrho_m\}$ converge to a common random fixed point in F.

Now, we suppose that one of $E_1, E_2, E_3 : \mho \times \Theta \to \Theta$ is completely continuous random operator, say E_1, then there exists a subsequence $\{E_1(\ell, \xi_{m_k}(\ell))\}$ of $\{E_1(\ell, \xi_m(\ell))\}$ such that $E_1(\ell, \xi_{m_k}(\ell)) \to \xi_0(\ell)$ as $k \to \infty$ which $\xi_0 : \mho \to \Theta$ is a measurable mapping for all $\ell \in \mho$. From Lemma 4, we know that $\lim_{k\to\infty} \|\xi_{m_k}(\ell) - E_1(\ell, \xi_{m_k}(\ell))\| = 0$. Using the continuity of E_1, we get that $\lim_{k\to\infty} \xi_{m_k}(\ell) = \xi_0(\ell)$ for all $\ell \in \mho$. This implies that $E_1(\ell, \xi_0(\ell)) = \xi_0(\ell)$ for all $\ell \in \mho$. Thus, $\xi_0(\ell)$ is a random fixed point of E_1. From Lemma 3, we know that $\lim_{m\to\infty} \|\xi_m(\ell) - \xi_0(\ell)\|$ exists and $\lim_{k\to\infty} \|\xi_{m_k}(\ell) - \xi_0(\ell)\| = 0$ for all $\ell \in \mho$. Therefore, $\lim_{m\to\infty} \|\xi_m(\ell) - \xi_0(\ell)\| = 0$, that is $\lim_{m\to\infty} \xi_m(\ell) = \xi_0(\ell)$ for all $\ell \in \mho$. Using the same inequalities 35 and 36, we have $\lim_{m\to\infty} \varsigma_m(\ell) = \xi_0(\ell)$ and $\lim_{m\to\infty} \varrho_m(\ell) = \xi_0(\ell)$ for all $\ell \in \mho$. This completes the proof. □

If $E_1 = E_2 = E_3 := E$ and $\alpha'_m = \beta'_m = \gamma''_m \equiv 0$ at 8, then we have the following iteration process:

$$\begin{aligned} \varrho_m(\ell) &= \gamma_m \xi_m(\ell) + \gamma'_m E_3^m(\xi_m, \ell), \\ \varsigma_m(\ell) &= \beta_m E_2^m(\ell, \varrho_m(\ell)), \\ \xi_{m+1}(\ell) &= \alpha_m E_1^m(\ell, \varsigma_m(\ell)). \end{aligned} \tag{37}$$

Then, we obtain the following result from Theorem 1.

Corollary 1. *Let Θ be a nonempty closed bounded and convex subset of a uniformly convex separable Banach space $\mathfrak{X}$. Let $E : \mho \times \Theta \to \Theta$ be either completely continuous or semi-compact asymptotically nonexpansive random operators with a sequence of measurable mappings $\mu_{i_m}(\ell) : \mho \to [0, \infty)$ satisfying $\sum_{m=1}^{\infty} \mu_{i_m}(\ell) < \infty$, for all $\ell \in \mho$ and $RF(E) \neq \varnothing$. Moreover, assume that f_0 is a measurable mapping from $\mho$ to Θ. Define the sequence of functions $\{\xi_m\}, \{\varsigma_m\}$, and $\{\varrho_m\}$ by (8) with $\{\alpha_m\}, \{\beta_m\}, \{\gamma_m\}$, and $\{\gamma'_m\}$ satisfying $0 < \alpha \leq \alpha_m \leq 1 - \alpha$, for some $\alpha \in (0, 1)$, for all $m \geq m_0, \exists m_0 \in \mathbb{N}$. Then sequences $\{\xi_m\}, \{\varsigma_m\}$, and $\{\varrho_m\}$ converge to a random fixed point of F.*

4. Conclusions

In the presented paper, we introduce a new type iterative algorithm faster than the other iterative algorithms in literature and use it for asymptotically nonexpansive random operators. Moreover, we prove some strong convergence theorems for such mappings under the appropriate conditions in uniformly convex Banach spaces.

5. Conflicts of Interest

The authors declare that they have no conflicts of interest.

6. Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

7. Acknowledgement

The first author would like to thank TUBITAK (Scientific and Technological Research Council of Turkey) for financial support with the TUBITAK 2211 Domestic Graduate Scholarship Program during his graduate studies.

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