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ON A REDUCTION METHOD USING MAX-PLUS ALGEBRA FOR A INITIAL VALUE PROBLEM IN CLASSIC ALGEBRA AND THE SOLUTION OF THE PROBLEM

ZELIHA AYDOĞMUŞ AND AHMET İPEK

Abstract. In this paper, we first will develop a reduction method in max-plus algebra for the initial value problem given by

$$\begin{cases} x(t+n) = \max\{a_{n-1}(t) + x(t+n-1), \cdots, a_1(t) + x(t+1), \\ \qquad a_0(t) + x(t), f(t)\} \\ x(t_0) = c_1, x(t_0+1) = c_2, \cdots, x(t_0+n-1) = c_n. \end{cases}$$

and then we obtain the solutions to the this equation.

1. Introduction

Max-plus algebra $\mathbb{R}_{max} = \mathbb{R} \cup \{-\infty\}$ is an analogue of linear algebra and has many analogies for linear algebra (see [1], [3] and [7]). Max-plus algebra $\mathbb{R}_{max} = \mathbb{R} \cup \{-\infty\}$ is equipped with the operations: addition $a \oplus b = \max(a, b)$ and multiplication $a \otimes b = a + b$. In max-plus algebra, $\varepsilon = -\infty$ is a neutral element for $\oplus$ and $e = 0$ is a neutral element for $\otimes$. The fact that $(\mathbb{R}_{max}, \oplus, \otimes, \varepsilon, e)$ algebraic structure is known as idempotent semiring.

If $A = (a_{ij}), B = (b_{ij})$ are matrices with elements from $\mathbb{R}_{max}$ of compatible sizes, then the max-plus addition, product and scalar multiples of matrices are defined by

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij},$$

$$(A \otimes B)_{ij} = \sum_k^{\oplus} a_{ik} \otimes b_{kj} = \max_k (a_{ik} + b_{kj})$$

and

$$(\lambda \otimes A)_{ij} = \lambda \otimes a_{ij}$$

for $\lambda \in \mathbb{R}_{max}$ respectively.

Max-plus algebra is widely used in system theory and optimal control [8], scheduling of energy flows [6], speech recognition [5] and control of an electroplating [9]. For a large survey on max-plus algebra theory, we refer to books [1], [2] and [4].

In this paper, we first consider the initial value problem given by

$$\left.\begin{array}{c} x(t+n) = \max\{a_{n-1}(t) + x(t+n-1), \cdots, a_1(t) + x(t+1), \\ a_0(t) + x(t), f(t)\} \\ x(t_0) = c_1, x(t_0+1) = c_2, \cdots, x(t_0+n-1) = c_n. \end{array}\right\} \quad (1)$$

in classic algebra. Eq. (1) is an nth order difference equation for $x(t)$, where $a_0(t), a_1(t), \ldots, a_n(t)$ and $f(t)$ are assumed known. Then, before seeking solutions to the difference equations in (1) in Section 3, we will first develop techniques for reducing these equations to fundamental form in Section 2.

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2. Method of reduction

A method of reduction, particularly useful for difference equations defined by system (1), is the following:
Step 1. Rewrite (1) in max-plus algebra:

$$\begin{cases} x(t+n) = a_{n-1}(t) \otimes x(t+n-1) \oplus \cdots \oplus a_1(t) \otimes x(t+1) \\ \oplus a_0(t) \otimes x(t) \oplus f(t) \\ x(t_0) = c_1, x(t_0+1) = c_2, \cdots, x(t_0+n-1) = c_n. \end{cases} \tag{2}$$

Step 2. Define n new variables, $x_1(t), x_2(t), \ldots, x_n(t)$ by the equations:

$$\begin{aligned} x_1(\mathrm{t}) &= x(t) \\ x_2(\mathrm{t}) &= x_1(\mathrm{t}+1) \\ x_3(\mathrm{t}) &= x_2(\mathrm{t}+1) \\ &\vdots \\ x_{n-1}(\mathrm{t}) &= x_{\mathrm{n}-2}(\mathrm{t}+1) \\ x_n(\mathrm{t}) &= x_{n-1}(\mathrm{t}+1). \end{aligned} \tag{3}$$

Step 3. It is immediate from system (3) that we also have the following relationships between $x_1, x_2, \ldots, x_n$ and the unknown $x(t)$:

$$\begin{aligned} x_1(\mathrm{t}) &= x(t) \\ x_2(\mathrm{t}) &= x(t+1) \\ x_3(\mathrm{t}) &= x(t+2) \\ &\vdots \\ x_{n-1}(\mathrm{t}) &= x(t+n-2) \\ x_n(\mathrm{t}) &= x(t+n-1) \end{aligned} \tag{4}$$

and

$$x_n(\mathrm{t}+1) = x(t+n). \tag{5}$$

Step 4. Rewrite $x(t+n)$ in (2) in terms of the new variables $x_1, x_2, \ldots, x_n$. Substituting (4) and (5) into the equation (2), we obtain

$$\begin{aligned} x_n(\mathrm{t}+1) = & a_{n-1}(t) \otimes x_n(\mathrm{t}) \oplus \cdots \oplus a_1(t) \otimes x_2(\mathrm{t}) \\ & \oplus a_0(t) \otimes x_1(\mathrm{t}) \oplus f(t). \end{aligned} \tag{6}$$

Step 5. Form a system for $x_1, x_2, \ldots, x_n$. Using (4), (5) and (6), we obtain the system:

$$\begin{aligned} x_1(\mathrm{t}+1) &= x_2(\mathrm{t}) \\ x_2(\mathrm{t}+1) &= x_3(\mathrm{t}) \\ x_3(\mathrm{t}+1) &= x_2(\mathrm{t}) \\ &\vdots \\ x_{\mathrm{n}-2}(\mathrm{t}+1) &= x_{n-1}(\mathrm{t}) \\ x_{n-1}(\mathrm{t}+1) &= x_n(\mathrm{t}) \\ x_n(\mathrm{t}+1) &= a_0(t) \otimes x_1(\mathrm{t}) \oplus a_1(t) \otimes x_2(\mathrm{t}) \\ &\quad \oplus \cdots \oplus a_{n-1}(t) \otimes x_n(\mathrm{t}) \oplus f(t). \end{aligned} \tag{7}$$

Step 6. Put (7) into matrix form. Define

$$X(t)=\begin{bmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\\ \vdots\\ x_{n-1}(t)\\ x_n(t)\end{bmatrix}, X(t+1)=\begin{bmatrix} x_1(t+1)\\ x_2(t+1)\\ x_3(t+1)\\ \vdots\\ x_{n-1}(t+1)\\ x_n(t+1)\end{bmatrix}, F(t)=\begin{bmatrix} \varepsilon\\ \varepsilon\\ \varepsilon\\ \vdots\\ \varepsilon\\ f(t)\end{bmatrix}$$

and

$$\mathrm{A}(t)=\begin{bmatrix} \varepsilon & 0 & \varepsilon & \cdots & \varepsilon & \varepsilon\\ \varepsilon & \varepsilon & 0 & \cdots & \varepsilon & \varepsilon\\ \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & 0\\ a_0(t) & a_1(t) & a_2(t) & \cdots & a_{\mathrm{n}-2}(t) & a_{\mathrm{n}-1}(t)\end{bmatrix}.$$

Then (7) can be written as:

$$X(t+1)=\mathrm{A}(t)\otimes X(t)\oplus F(t). \tag{8}$$

Step 7. Rewrite the initial conditions in matrix form:

$$X\left(t_0\right)=\begin{bmatrix} x_1\left(t_0\right)\\ x_2\left(t_0\right)\\ x_3\left(t_0\right)\\ \vdots\\ x_{n-1}\left(t_0\right)\\ x_n\left(t_0\right)\end{bmatrix}=\begin{bmatrix} x\left(t_0\right)\\ x\left(t_0+1\right)\\ x\left(t_0+2\right)\\ \vdots\\ x\left(t_0+n-2\right)\\ x\left(t_0+n-1\right)\end{bmatrix}=\begin{bmatrix} c_1\\ c_2\\ c_3\\ \vdots\\ c_{\mathrm{n}-1}\\ c_{\mathrm{n}}\end{bmatrix}.$$

Thus, if we define

$$C=\begin{bmatrix} c_1\\ c_2\\ c_3\\ \vdots\\ c_{\mathrm{n}-1}\\ c_{\mathrm{n}}\end{bmatrix}$$

the initial conditions can be put into matrix form

$$X\left(t_0\right)=C. \tag{9}$$

Consequently, Eqs. (8) and (9) together represent the fundamental form for (1):

$$\begin{aligned} X(t+1)&=\mathrm{A}(t)\otimes X(t)\oplus F(t)\\ X\left(t_0\right)&=C.\end{aligned} \tag{10}$$

3. Solutions of matrix difference equations in max-plus algebra

In this section, firstly, if one has a homogeneous initial value problem ($F(t)=0$) when one reduces the initial value problem given by 1 to fundamental form, then we seek the solution to the initial value problem in the fundamental form

$$\begin{aligned} X(t+1)&=\mathrm{A}(t)\otimes X(t), t\geq t_0\\ X\left(t_0\right)&=C,\end{aligned} \tag{11}$$

where it is assumed that $\mathrm{A}(t) \neq 0$ and $\mathrm{A}(t)$ are real-valued matrix function defined for $t \geq t_0 \geq 0$.
One may obtain the solution of 11 by a simple iteration:

$$\begin{aligned} X\left(t_0+1\right) &= A\left(t_0\right) \otimes X\left(t_0\right) = A\left(t_0\right) \otimes C \\ X\left(t_0+2\right) &= A\left(t_0+1\right) \otimes X\left(t_0+1\right) = A\left(t_0+1\right) \otimes A\left(t_0\right) \otimes C \\ X\left(t_0+3\right) &= A\left(t_0+2\right) \otimes X\left(t_0+2\right) = A\left(t_0+2\right) \otimes A\left(t_0+1\right) \otimes A\left(t_0\right) \otimes C. \end{aligned}$$

Therefore, inductively, we obtain the solution of 11 that

$$\begin{aligned} X(t) &= X\left(t_0+t-t_0\right) \\ &= A(t-1) \otimes A(t-2) \otimes \cdots \otimes A\left(t_0\right) \otimes X_0 \\ &= A(t-1) \otimes A(t-2) \otimes \cdots \otimes A\left(t_0\right) \otimes C \\ &= \left[\bigotimes_{i=t_0}^{t-1} A(i)\right] \otimes C. \end{aligned}$$

In this section, finally, if one has a nonhomogeneous initial value problem $(F(t) \neq 0)$ when one reduces the initial value problem given by 1 to fundamental form, then we seek the solution to the initial value problem in the fundamental form

$$\begin{aligned} X(t+1) &= \mathrm{A}(t) \otimes X(t) \oplus F(t), t \geq t_0 \\ X\left(t_0\right) &= C, \end{aligned} \tag{12}$$

where it is assumed that $\mathrm{A}(t)$ and $F(t)$ are real-valued matrix functions defined for $t \geq t_0 \geq 0$ and $\mathrm{A}(t) \neq 0$.
The solution of the nonhomogeneous 12 may be found as follows. One may obtain the solution of 12 by a simple iteration:

$$\begin{aligned} X\left(t_0+1\right) &= A\left(t_0\right) \otimes X_0 \oplus F\left(t_0\right) \\ &= A\left(t_0\right) \otimes C \oplus F\left(t_0\right) \\ X\left(t_0+2\right) &= A\left(t_0+1\right) \otimes X\left(t_0+1\right) \oplus F\left(t_0+1\right) \\ &= A\left(t_0+1\right) \otimes \left[A\left(t_0\right) \otimes C \oplus F\left(t_0\right)\right] \oplus F\left(t_0+1\right) \\ &= A\left(t_0+1\right) \otimes A\left(t_0\right) \otimes C \oplus A\left(t_0+1\right) \otimes F\left(t_0\right) \oplus F\left(t_0+1\right). \end{aligned}$$

We now use mathematical induction to show that, for all $t \in \mathbb{Z}^+$,

$$X(t) = \left[\bigotimes_{i=t_0}^{t-1} A(i)\right] \otimes C \oplus \bigoplus_{r=t_0}^{t-1} \left[\bigotimes_{i=r+1}^{t-1} A(i)\right] \otimes F(r) \tag{13}$$

To establish this, assume that formula 13 holds for $t = k$. Notice that we have adopted the notation $\underset{i=k+1}{\overset{k}{\otimes}} A(i) = 1$ and $\underset{i=k+1}{\overset{k}{\oplus}} A(i) = 0$. Then from 12,

$$X(t+1) = \mathrm{A}(t) \otimes X(t) \oplus F(t),$$

which by formula 13 yields

$$
\begin{aligned}
X(t+1) &= A(k)\otimes X(k)\oplus F(k)\\
&= A\left(k\right)\otimes\Bigg\{\left[\bigotimes_{i=t_0}^{k-1}A(i)\right]\otimes C\\
&\quad\oplus\bigoplus_{r=t_0}^{k-1}\left[\bigotimes_{i=r+1}^{k-1}A(i)\right]\otimes F\left(r\right)\Bigg\}\oplus F\left(k\right)\\
&=\left\{A(k)\otimes\left[\bigotimes_{i=t_0}^{k-1}A(i)\right]\otimes C\right\}\\
&\quad\oplus\left\{\bigoplus_{r=t_0}^{k-1}\left[A(k)\otimes\bigotimes_{i=r+1}^{k-1}A(i)\right]\otimes F(r)\right\}\oplus F(k)\\
&=\left\{\left[\bigotimes_{i=t_0}^{k}A(i)\right]\otimes C\right\}\oplus\left\{\bigoplus_{r=t_0}^{k-1}\left(\bigotimes_{i=r+1}^{k}A(i)\right)\otimes F(r)\right\}\\
&\quad\oplus\left\{\left(\bigotimes_{i=k+1}^{k}A(i)\right)\otimes F(k)\right\}\\
&=\left\{\left[\bigotimes_{i=t_0}^{k}A(i)\right]\otimes C\right\}\oplus\left\{\bigoplus_{r=t_0}^{k}\left(\bigotimes_{i=r+1}^{k}A(i)\right)\otimes F(r)\right\}
\end{aligned}
$$

Hence formula 13 holds for all $t\in\mathbb{Z}^+$.

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Karamanoğlu Mehmetbey University
Kamil Özdağ Science Faculty
Department of Mathematics
Yunus Emre Campus, 70100, Karaman, Turkey
E-mail address: z_dilekci@hotmail.com, ahmetipek@kmu.edu.tr (or dr.ahmetipek@gmail.com)