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SOME SUM FORMULAS AND NEW IDENTITIES OF BI-PERIODIC JACOBSTHAL AND BI-PERIODIC JACOBSTHAL LUCAS SEQUENCE

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Abstract. In this article, it is considered that the new generalizations of the Jacobsthal sequence and Jacobsthal-Lucas sequence. These sequences can arise in the study of continued fractions of quadratic irrationals. Some well-known sequences are special cases of this generalization. Jacobsthal and Jacobsthal-Lucas sequence is a special case with $a = b = 1$. Some new identities and properties of these generalized sequences are investigated with the aid of its Binet formula, recurrence relation etc. We study specially some different sum formulas for these sequences. Then, by these formulas we get properties for Jacobsthal sequence, Jacobsthal-Lucas sequence.

1. Introduction

There are so many articles about the special integer sequences and their generalized sequences in the literaure. Bi-periodic sequence is a type of generalization of the special integer sequences. Bi-periodic sequence are important since the recurrence relation changes with respect to index n. Yayenie found interesting properties of the bi-periodic Fibonacci sequence in [2, 3]. Bilgici [4] introduced into literature the bi-periodic Lucas sequence and gave some properties of bi-periodic Lucas sequence and relationship between bi-periodic Fibonacci sequence. The authors investigated the identities on the bi-periodic Pell, Pell Lucas sequences [7, 8]. Choo studied some identities of the generalized bi-periodic Fibonacci sequences in [11].

In the recent years, many studies have been seen about the Jacobsthal and Jacobsthal-Lucas sequences. Jacobsthal numbers have applications in such areas as tiling, graph matching, alternating sign matrices, etc. ([12]-[14]). The Jacobsthal sequence $\{j_n\}_{n=0}^{\infty}$ is defined as $j_n = j_{n-1} + 2j_{n-2}$ with initial conditions $j_0 = 0,\ j_1 = 1$. The Jacobsthal-Lucas sequence $\{c_n\}_{n=0}^{\infty}$ is defined by the same recurrence relation $c_n = c_{n-1} + 2c_{n-2}$ with initial conditions $c_0 = 2,\ c_1 = 1$ in[1]. In this paper, there will be a much broader study on the properties of bi-periodic Jacobsthal sequences. The bi-periodic Jacobsthal sequence $\{\hat{j}_n\}_{n=0}^{\infty}$ is defined as

$$\hat{j}_0 = 0,\ \hat{j}_1 = 1,\ \hat{j}_n = \begin{cases} a\hat{j}_{n-1} + 2\hat{j}_{n-2}, & \text{if } n \text{ is even} \\ b\hat{j}_{n-1} + 2\hat{j}_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2, \tag{1}$$

Similarly the recurrence relation is defined for bi-periodic Jacobsthal sequence as follows $\hat{j}_{n+2} = a^{1-\xi(n)} b^{\xi(n)} \hat{j}_{n+1} + 2\hat{j}_n$.

Uygun and Owusu defined a new generalization for the Jacobsthal Lucas sequence $\{\hat{c}_n\}_{n=0}^{\infty}$ recursively as

$$\hat{c}_0 = 2,\ \hat{c}_1 = a,\ \ \hat{c}_n = \begin{cases} b\hat{c}_{n-1} + 2\hat{c}_{n-2}, & \text{if } n \text{ is even} \\ a\hat{c}_{n-1} + 2\hat{c}_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2. \tag{2}$$

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Similarly the recurrence relation is defined for bi-periodic Jacobsthal Lucas sequence as follows $\hat{c}_{n+2} = b^{1-\xi(n)} a^{\xi(n)} \hat{c}_{n+1} + 2\hat{c}_n$ [6]. From the above definitions we have the nonlinear quadratic equation

$$x^2 - abx - 2ab = 0$$

with roots α and β defined by

$$\alpha = \frac{ab + \sqrt{a^2b^2 + 8ab}}{2}, \quad \beta = \frac{ab - \sqrt{a^2b^2 + 8ab}}{2}$$

[5, 6]. Uygun developed the bi-periodic Jacobsthal Lucas sequence in [9, 10].

Proposition 1. *The bi-periodic Jacobsthal sequence and bi-periodic Jacobsthal Lucas sequence hold the following relations*

- *The extended Binet formula*

$$\hat{j}_m = \left(\frac{a^{1-\xi(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}} \right) \frac{\alpha^m - \beta^m}{\alpha - \beta} \quad n \geq 3. \tag{3}$$

$$\hat{c}_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\alpha^n + \beta^n \right), \quad n \geq 4. \tag{4}$$

where $\lfloor a \rfloor$ *is the floor function of* a *and* $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ *is the parity function in* [5].

- *The generating function*

$$J(x) = \frac{x(1 + ax - 2x^2)}{1 - (ab + 4)x^2 + 4x^4}$$

$$C(x) = \frac{2 + ax - (ab + 4)x^2 + 2ax^3}{1 - (ab + 4)x^2 + 4x^4}$$

- $$\begin{aligned} \hat{j}_{n+4} &= (ab + 4)\, \hat{j}_{n+2} - 4\hat{j}_n \quad n \geq 5. \\ \hat{c}_{n+4} &= (ab + 4)\hat{c}_{n+2} - 4\hat{c}_n \end{aligned} \tag{5}$$

- $$\begin{aligned} (\alpha + 2)(\beta + 2) &= 4 \quad n \geq 6. \\ \alpha + \beta &= ab, \qquad \alpha\beta = -2ab \\ \beta + 2 &= \frac{\beta^2}{ab}, \qquad \alpha + 2 = \frac{\alpha^2}{ab} \\ -(\alpha + 2)\beta &= 2\alpha, \qquad -(\beta + 2)\alpha = 2\beta. \end{aligned} \tag{6}$$

Now in this paper, there will be a much broader study on the generalized Jacobsthal and Jacobsthal Lucas sequence in order to obtain some new crucial identities and properties of the bi-periodic Jacobsthal, bi-periodic Jacobsthal Lucas sequences.

2. New Identities of Bi-periodic Jacobsthal and Bi-periodic Jacobsthal Lucas Sequence

Theorem 1. *Let n is positive integer, the following equalities are satisfied:*

$$\begin{aligned} \hat{j}_{n+6} &= (ab + 6)\, a^{1-\xi(n)} b^{\xi(n)} \hat{j}_{n+3} + 8\hat{j}_n, \\ \hat{c}_{n+6} &= (ab + 6)\, a^{1-\xi(n)} b^{\xi(n)} \hat{c}_{n+3} + 8\hat{c}_n. \end{aligned}$$

Proof. The theorem can be proved by using of the Binet's formula or the recurrence relation as follows. By (1) and (5) we have

$$\begin{aligned}
\hat{c}_{n+6} &= (ab+4)\left[a^{1-\xi(n)}b^{\xi(n)}\hat{c}_{n+3}+2\hat{c}_{n+2}\right]-4\hat{c}_{n+2} \\
&= (ab+4)\,a^{1-\xi(n)}b^{\xi(n)}\hat{c}_{n+3}+(2ab+4)\,\hat{c}_{n+2} \\
&= (ab+6)\,a^{1-\xi(n)}b^{\xi(n)}\hat{c}_{n+3}+(2ab+4)\,\hat{c}_{n+2}-2a^{1-\xi(n)}b^{\xi(n)}\hat{c}_{n+3} \\
&= (ab+6)\,a^{1-\xi(n)}b^{\xi(n)}\hat{c}_{n+3}+(2ab+4)\,\hat{c}_{n+2} \\
&\quad -2a^{1-\xi(n)}b^{\xi(n)}\left[a^{\xi(n)}b^{1-\xi(n)}\hat{c}_{n+2}+2\hat{c}_{n+1}\right] \\
&= (ab+6)\,a^{1-\xi(n)}b^{\xi(n)}\hat{c}_{n+3}+4(\hat{c}_{n+2}-a^{1-\xi(n)}b^{\xi(n)}\hat{c}_{n+1}) \\
&\quad (ab+6)\,a^{1-\xi(n)}b^{\xi(n)}\hat{c}_{n+3}+8\hat{c}_{n}
\end{aligned}$$

The other formula is proved similarly. When $a=b=1$ the above result reduces to a known identity of Jacobsthal numbers and Jacobsthal-Lucas numbers

$$\begin{aligned}
j_{n+6} &= 7j_{n+3}+8j_{n}, \\
c_{n+6} &= 7c_{n+3}+8c_{n},
\end{aligned}$$

□

Theorem 2. *If $ab\neq 1$, then the sum formulas for the bi-periodic Jacobsthal, bi-periodic Jacobsthal-Lucas sequences are given as*

$$\begin{aligned}
\sum_{k=0}^{n-1}\hat{j}_{k} &= \frac{4(\hat{j}_{n-1}+\hat{j}_{n-2})-(\hat{j}_{n+1}+\hat{j}_{n})+a-1}{1-ab}, \\
\sum_{k=0}^{n-1}\hat{c}_{k} &= \frac{4(\hat{c}_{n-2}+\hat{c}_{n-1})-(\hat{c}_{n}+\hat{c}_{n+1})-ab-2+3a}{1-ab}.
\end{aligned}$$

Proof. For proof, we use Binet formula. If n is even, then

$$\begin{aligned}
\sum_{k=0}^{n-1}\hat{j}_{k} &= \sum_{k=0}^{\frac{n-2}{2}}\hat{j}_{2k+1}+\sum_{k=0}^{\frac{n-2}{2}}\hat{j}_{2k} \\
&= \frac{1}{\alpha-\beta}\left[\sum_{k=0}^{\frac{n-2}{2}}\frac{\alpha^{2k+1}-\beta^{2k+1}}{(ab)^{k}}+a\sum_{k=0}^{\frac{n-2}{2}}\frac{\alpha^{2k}-\beta^{2k}}{(ab)^{k}}\right] \\
&= \frac{1}{\alpha-\beta}\left[\begin{array}{c}\frac{(\alpha^{n+1}-\alpha(ab)^{\frac{n}{2}})(\beta^{2}-ab)-(\beta^{n+1}-\beta(ab)^{\frac{n}{2}})(\alpha^{2}-ab)}{(ab)^{\frac{n}{2}-1}(\alpha^{2}-ab)(\beta^{2}-ab)} \\ +a\frac{(\alpha^{n}-(ab)^{\frac{n}{2}})(\beta^{2}-ab)-(\beta^{n}-(ab)^{\frac{n}{2}})(\alpha^{2}-ab)}{(ab)^{\frac{n}{2}-1}(\alpha^{2}-ab)(\beta^{2}-ab)}\end{array}\right] \\
&= \frac{1}{\alpha-\beta}\left[\begin{array}{c}\frac{4a^{2}b^{2}(\alpha^{n-1}-\beta^{n-1})-ab(\alpha^{n+1}-\beta^{n+1})+(ab)^{\frac{n}{2}+1}(\alpha-\beta)-2(ab)^{\frac{n}{2}+1}(\alpha-\beta)}{(ab)^{\frac{n}{2}-1}(a^{2}b^{2}(1-ab))} \\ +a\frac{4a^{2}b^{2}(\alpha^{n-2}-\beta^{n-2})-ab(\alpha^{n}-\beta^{n})+(ab)^{\frac{n}{2}}(\alpha^{2}-\beta^{2})}{(ab)^{\frac{n}{2}-1}(a^{2}b^{2}(1-ab))}\end{array}\right] \\
&= \frac{4\hat{j}_{n-1}-\hat{j}_{n+1}-1+4\hat{j}_{n-2}-\hat{j}_{n}+a}{1-ab} \\
&= \frac{4(\hat{j}_{n-1}+\hat{j}_{n-2})-(\hat{j}_{n+1}+\hat{j}_{n})+a-1}{1-ab}
\end{aligned}$$

If n is odd, then

$$\begin{aligned}
\sum_{k=0}^{n-1} \hat{\jmath}_k &= \sum_{k=0}^{\frac{n-3}{2}} \hat{\jmath}_{2k+1} + \sum_{k=0}^{\frac{n-1}{2}} \hat{\jmath}_{2k} \\
&= \frac{1}{\alpha-\beta}\left[\sum_{k=0}^{\frac{n-3}{2}} \frac{\alpha^{2k+1}-\beta^{2k+1}}{(ab)^k} + a\sum_{k=0}^{\frac{n-1}{2}} \frac{\alpha^{2k}-\beta^{2k}}{(ab)^k}\right] \\
&= \frac{1}{\alpha-\beta}\left[\begin{array}{c} \frac{(\alpha^n-\alpha(ab)^{\frac{n-1}{2}})(\beta^2-ab)-(\beta^n-\beta(ab)^{\frac{n-1}{2}})(\alpha^2-ab)}{(ab)^{\frac{n-3}{2}}(\alpha^2-ab)(\beta^2-ab)} \\ +a\frac{(\alpha^{n+1}-(ab)^{\frac{n+1}{2}})(\beta^2-ab)-(\beta^{n+1}-(ab)^{\frac{n+1}{2}})(\alpha^2-ab)}{(ab)^{\frac{n-1}{2}}(\alpha^2-ab)(\beta^2-ab)} \end{array}\right] \\
&= \frac{1}{\alpha-\beta}\left[\begin{array}{c} \frac{4a^2b^2(\alpha^{n-2}-\beta^{n-2})-ab(\alpha^n-\beta^n)-(ab)^{\frac{n+1}{2}}(\alpha-\beta)}{(ab)^{\frac{n+1}{2}}(1-ab)} \\ +a\frac{4a^2b^2(\alpha^{n-1}-\beta^{n-1})-ab(\alpha^{n+1}-\beta^{n+1})+(ab)^{\frac{n+1}{2}}(\alpha^2-\beta^2)}{(ab)^{\frac{n+3}{2}}(1-ab)} \end{array}\right] \\
&= \frac{4(\hat{\jmath}_{n-1}+\hat{\jmath}_{n-2})-(\hat{\jmath}_{n+1}+\hat{\jmath}_n)+a-1}{1-ab}
\end{aligned}$$

The sum formula for the bi-periodic Jacobsthal-Lucas numbers is proved similarly. □

Theorem 3. *The sum of the product of bi-periodic Jacobsthal numbers and bi-periodic Jacobsthal-Lucas numbers is*

$$\sum_{k=0}^{n-1} \frac{\hat{\jmath}_k \hat{c}_k}{2^k} = \frac{a\hat{\jmath}_{2n-1}}{2^{n-1}} + \frac{1}{b}.$$

Proof. By (3), (4), we get

$$\begin{aligned}
\sum_{k=0}^{n-1} \frac{\hat{\jmath}_k \hat{c}_k}{2^k} &= \frac{a}{\alpha-\beta}\sum_{k=0}^{n-1}\left[\left(\frac{\alpha^2}{2ab}\right)^k - \left(\frac{\beta^2}{2ab}\right)^k\right] \\
&= \frac{a}{\alpha-\beta}\left(\frac{\left(\frac{\alpha^2}{2ab}\right)^n - 1}{\left(\frac{\alpha^2}{2ab}\right)-1} - \frac{\left(\frac{\beta^2}{2ab}\right)^n - 1}{\left(\frac{\beta^2}{2ab}\right)-1}\right) \\
&= \frac{a}{\alpha-\beta}\left(\frac{\left(\frac{\alpha+2}{2}\right)^n - 1}{\frac{\alpha}{2}} - \frac{\left(\frac{\beta+2}{2}\right)^n - 1}{\frac{\beta}{2}}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{a}{\alpha-\beta}\left(\frac{-\frac{\alpha}{2}\left(\frac{\alpha+2}{2}\right)^{n-1} - \frac{\beta}{2} + \frac{\beta}{2}\left(\frac{\beta+2}{2}\right)^{n-1} + \frac{\alpha}{2}}{-\frac{ab}{2}}\right) \\
&= \frac{a}{\alpha-\beta}\left(\frac{\alpha^{2n-1}-\beta^{2n-1}}{(2ab)^{n-1}}\right) + \frac{1}{b} \\
&= \frac{a\hat{\jmath}_{2n-1}}{2^{n-1}} + \frac{1}{b}
\end{aligned}$$

The other formula is proved similarly. When $a = b = 1$, the above result reduces to Jacobsthal numbers and Jacobsthal-Lucas numbers

$$\sum_{k=0}^{n-1} \frac{j_k c_k}{2^k} = \frac{j_{2n-1}}{2^{n-1}} + 1.$$

□

Theorem 4. *The sum formulas for the squares of the first n terms of the bi-periodic Jacobsthal and bi-periodic Jacobsthal-Lucas sequences are given as*

$$\sum_{i=1}^{n} \left(\frac{2b}{a}\right)^{\xi(i+1)} \left(\frac{\hat{j}_i}{2^{\lfloor \frac{i}{2} \rfloor}}\right)^2 = \frac{1}{a}\frac{\hat{j}_n \hat{j}_{n+1}}{2^{n-1}},$$

$$\sum_{i=1}^{n} \left(\frac{2b}{a}\right)^{\xi(i)} \left(\frac{\hat{c}_i}{2^{\lfloor \frac{i}{2} \rfloor}}\right)^2 = \frac{1}{a}\frac{\hat{c}_n \hat{c}_{n+1}}{2^{n}} - 2.$$

Proof. By using Binet form of the bi-periodic Jacobsthal sequence we have

$$\left(\frac{2b}{a}\right)^{\xi(i+1)} \left(\frac{\hat{j}_i}{2^{\lfloor \frac{i}{2} \rfloor}}\right)^2 = \frac{2ab}{(\alpha-\beta)^2}\left[\left(\frac{\alpha^2}{2ab}\right)^i + \left(\frac{\beta^2}{2ab}\right)^i - 2(-1)^i\right].$$

Using the properties $ab(\alpha+2) = \alpha^2$ and $ab(\beta+2) = \beta^2$ and geometric sum formula, it is obtained that

$$\begin{aligned}
&\sum_{i=1}^{n} \left(\frac{2b}{a}\right)^{\xi(i+1)} \left(\frac{\hat{j}_i}{2^{\lfloor \frac{i}{2} \rfloor}}\right)^2 \\
&= \frac{2ab}{(\alpha-\beta)^2}\left[\sum_{i=1}^{n}\left(\frac{\alpha^2}{2ab}\right)^i + \sum_{i=1}^{n}\left(\frac{\beta^2}{2ab}\right)^i - 2\sum_{i=1}^{n}(-1)^i\right] \\
&= \frac{2ab}{(\alpha-\beta)^2}\left[\frac{\left(\frac{\alpha^2}{2ab}\right)^{n+1} - \left(\frac{\alpha^2}{2ab}\right)}{\left(\frac{\alpha^2}{2ab}\right) - 1} + \frac{\left(\frac{\beta^2}{2ab}\right)^{n+1} - \left(\frac{\beta^2}{2ab}\right)}{\left(\frac{\beta^2}{2ab}\right) - 1} - (-1)^n + 1\right] \\
&= \frac{2ab}{(\alpha-\beta)^2}\left[\frac{\left(\frac{\alpha+2}{2}\right)^{n+1} - \left(\frac{\alpha+2}{2}\right)}{\frac{\alpha}{2}} + \frac{\left(\frac{\beta+2}{2}\right)^{n+1} - \left(\frac{\beta+2}{2}\right)}{\frac{\beta}{2}} - (-1)^n + 1\right] \\
&= \frac{2ab}{(\alpha-\beta)^2}\left[\frac{\frac{-2\alpha}{4}\left(\frac{\alpha+2}{2}\right)^n + \frac{\alpha}{2} - \frac{2\beta}{4}\left(\frac{\beta+2}{2}\right)^n + \frac{\beta}{2}}{\frac{-ab}{2}} - (-1)^n + 1\right] \\
&= \frac{2ab}{(\alpha-\beta)^2}\left[\frac{\alpha\left(\frac{\alpha+2}{2}\right)^n - \alpha + \beta\left(\frac{\beta+2}{2}\right)^n - \beta}{ab} - (-1)^n + 1\right] \\
&= \frac{2ab}{(\alpha-\beta)^2}\left[\frac{\frac{\alpha^{2n+1}}{(2ab)^n} + \frac{\beta^{2n+1}}{(2ab)^n}}{ab} - 1 - (-1)^n + 1\right] \\
&= \frac{1}{(\alpha-\beta)^2}\left[\frac{\alpha^{2n+1} + \beta^{2n+1}}{2^{n-1}(ab)^n} - 2ab(-1)^n\right].
\end{aligned}$$

If we take care of that

$$\begin{aligned}\frac{\hat{\jmath}_n\hat{\jmath}_{n+1}}{a2^{n-1}} &= \frac{1}{a2^{n-1}}\left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor\frac{n}{2}\rfloor}}\right)\frac{\alpha^n-\beta^n}{\alpha-\beta}\left(\frac{a^{1-\xi(n+1)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}\right)\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\\ &= \frac{\alpha^{2n+1}+\beta^{2n+1}-(-2ab)^n ab}{(\alpha-\beta)^2\,2^{n-1}\,(ab)^n}\\ &= \frac{\alpha^{2n+1}+\beta^{2n+1}}{(\alpha-\beta)^2\,2^{n-1}\,(ab)^n}-2(-1)^n ab\end{aligned}$$

Hence the proof is completed. The other formula is also proved by similar way. When $a=b=1$ the above result reduces to Jacobsthal numbers and Jacobsthal Lucas numbers:

$$\begin{aligned}\sum_{i=1}^{n}2^{\xi(i+1)}\left(\frac{j_i}{2^{\lfloor\frac{i}{2}\rfloor}}\right)^2 &= \frac{j_nj_{n+1}}{2^{n-1}},\\ \sum_{i=1}^{n}2^{\xi(i)}\left(\frac{c_i}{2^{\lfloor\frac{i}{2}\rfloor}}\right)^2 &= \frac{c_nc_{n+1}}{2^n}-2.\end{aligned}$$

□

Theorem 5. *The following property holds for the bi-periodic Jacobsthal and Jacobsthal Lucas sequences*

$$\begin{aligned}&\sum_{i=0}^{n}\left(\frac{2b}{a}\right)^{\xi(i+1)}\left(\frac{\hat{\jmath}_i}{2^{\lfloor\frac{i}{2}\rfloor}\left|r\right|^i}\right)^2\\ =\;&\frac{2ab}{(\alpha-\beta)^2}\left[\frac{4\hat{c}_{2n}-2\left|r\right|\hat{c}_{2n+2}-4\left(2\left|r\right|\right)^n+\left(2\left|r\right|\right)^{n+1}(ab+4)}{\left(2\left|r\right|\right)^n\left(4-2\left|r\right|(ab+4)+4\left|r\right|^2\right)}-2\frac{\left|r\right|^{2n+2}+(-1)^n}{\left|r\right|^{2n}\left(\left|r\right|^2+1\right)}\right],\end{aligned}$$

$$\begin{aligned}&\sum_{i=0}^{n}\left(\frac{b}{2a}\right)^{\xi(i)}\left(\frac{\hat{c}_i}{2^{\lfloor\frac{i}{2}\rfloor}\left|r\right|^i}\right)^2\\ =\;&\frac{4\hat{c}_{2n}-2\left|r\right|^2\hat{c}_{2n+2}+\left(2\left|r\right|^2\right)^{n+2}-\left(2\left|r\right|^2\right)^{n+1}(ab+4)}{\left(2\left|r\right|^2\right)^n\left(4-2\left|r\right|^2(ab+4)+4\left|r\right|^4\right)}+2\frac{\left|r\right|^{2n+2}+(-1)^n}{\left|r\right|^{2n}\left(\left|r\right|^2+1\right)}.\end{aligned}$$

Proof. By using Binet forms of the bi-periodic Jacobsthal sequence, we have

$$\left(\frac{2b}{a}\right)^{\xi(i+1)}\left(\frac{\hat{\jmath}_i}{2^{\lfloor\frac{i}{2}\rfloor}\left|r\right|^i}\right)^2=\frac{2ab}{(\alpha-\beta)^2}\left[\left(\frac{\alpha^2}{2ab\left|r\right|^2}\right)^i+\left(\frac{\beta^2}{2ab\left|r\right|^2}\right)^i-2\left(-\frac{1}{\left|r\right|^2}\right)^i\right].$$

Using (6) and the sum of geometric series, it is obtained that

$$\begin{aligned}&\sum_{i=1}^{n}\left(\frac{2b}{a}\right)^{\xi(i+1)}\left(\frac{\hat{\jmath}_i}{2^{\lfloor\frac{i}{2}\rfloor}\left|r\right|^i}\right)^2\\ =\;&\frac{2ab}{(\alpha-\beta)^2}\left[\sum_{i=0}^{n}\left(\frac{\alpha^2}{2ab\left|r\right|^2}\right)^i+\sum_{i=0}^{n}\left(\frac{\beta^2}{2ab\left|r\right|^2}\right)^i-2\sum_{i=0}^{n}\left(-\frac{1}{\left|r\right|^2}\right)^i\right]\\ =\;&\frac{2ab}{(\alpha-\beta)^2}\left[\frac{\left(\frac{\alpha^2}{2ab|r|^2}\right)^{n+1}-1}{\left(\frac{\alpha^2}{2ab|r|^2}\right)-1}+\frac{\left(\frac{\beta^2}{2ab|r|^2}\right)^{n+1}-1}{\left(\frac{\beta^2}{2ab|r|^2}\right)-1}-2\frac{\left|r\right|^{2n+2}-(-1)^{n+1}}{\left|r\right|^{2n}\left(\left|r\right|^2+1\right)}\right]\end{aligned}$$

Then

$$\sum_{i=0}^{n}\left(\frac{2b}{a}\right)^{\xi(i+1)}\left(\frac{\hat{j}_i}{2^{\lfloor\frac{i}{2}\rfloor}|r|^i}\right)^2$$

$$= \frac{2ab}{(\alpha-\beta)^2}\left[\begin{array}{c}\frac{\alpha^{2n+2}-\left(2ab|r|^2\right)^{n+1}}{\left(2ab|r|^2\right)^n(\alpha^2-2ab|r|^2)}+\frac{\beta^{2n+2}-\left(2ab|r|^2\right)^{n+1}}{\left(2ab|r|^2\right)^n(\beta^2-2ab|r|^2)}\\ -2\frac{|r|^{2n+2}-(-1)^{n+1}}{|r|^{2n}(|r|^2+1)}\end{array}\right]$$

$$= \frac{2ab}{(\alpha-\beta)^2}\left[\begin{array}{c}\frac{\begin{array}{c}4(ab)^2(\alpha^{2n}+\beta^{2n})-2ab|r|^2(\alpha^{2n+2}+\beta^{2n+2})\\ -\left(2ab|r|^2\right)^{n+1}(\alpha^2+\beta^2)+2\left(2|r|^2ab\right)^{n+2}\end{array}}{\left(2|r|^2ab\right)^n(\alpha^2-2ab|r|^2)(\beta^2-2ab|r|^2)}\\ -2\frac{|r|^{2n+2}-(-1)^{n+1}}{|r|^{2n}(|r|^2+1)}\end{array}\right]$$

$$= \frac{2ab}{(\alpha-\beta)^2}\left[\frac{4\hat{c}_{2n}-2|r|^2\hat{c}_{2n+2}+2\left(2|r|^2\right)^{n+2}-\left(2|r|^2\right)^{n+1}(ab+4)}{\left(2|r|^2\right)^n\left(4-2|r|^2(ab+4)+4|r|^4\right)}-2\frac{|r|^{2n+2}-(-1)^{n+1}}{|r|^{2n}(|r|^2+1)}\right].$$

The other formula is proved similarly. When $a=b=1$ the above result reduces to Jacobsthal numbers and Jacobsthal Lucas numbers:

$$\sum_{i=0}^{n}\left(\frac{j_i}{2^{\lfloor\frac{i}{2}\rfloor}|r|^i}\right)^2$$

$$= \frac{2}{9}\left[\frac{4c_{2n}-2|r|^2c_{2n+2}+2\left(2|r|^2\right)^{n+2}-5\left(2|r|^2\right)^{n+1}}{\left(2|r|^2\right)^n\left(4-10|r|^2+4|r|^4\right)}-2\frac{|r|^{2n+2}-(-1)^{n+1}}{|r|^{2n}(|r|^2+1)}\right],$$

$$\sum_{i=0}^{n}\left(\frac{c_i}{2^{\lfloor\frac{i}{2}\rfloor}|r|^i}\right)^2$$

$$= \frac{4c_{2n}-2|r|^2c_{2n+2}-4(2|r|)^n+5\left(2|r|^2\right)^{n+1}}{\left(2|r|^2\right)^n\left(4-10|r|^2+4|r|^4\right)}+2\frac{|r|^{2n+2}-(-1)^{n+1}}{|r|^{2n}(|r|^2+1)}.$$

□

Theorem 6. *Let m is any positive integer, then the following property is obtained*

$$\frac{a^{1-\xi(m)}}{8}(ab+2)+\sum_{k=0}^{m-2}\frac{a^{\xi(k)-\xi(m)}(ab+6)^{-k-1}}{(ab)^{\lfloor\frac{k+1}{2}\rfloor}}\hat{j}_{3k}=\frac{\hat{j}_{3m}}{(ab+6)^{m-1}(ab)^{\lfloor\frac{m-1}{2}\rfloor}}$$

$$\frac{3a^{\xi(m)}}{8}ab(ab+6)+\sum_{k=0}^{m-2}\frac{a^{\xi(m)-\xi(k)}(ab+6)^{-k-1}}{(ab)^{\lfloor\frac{k}{2}\rfloor}}\hat{c}_{3k}=\frac{\hat{c}_{3m}}{8(ab)^{\lfloor\frac{m-2}{2}\rfloor}(ab+6)^{m-1}}$$

Proof. Since $\alpha^2-ab\alpha-2ab=0$, multiplying by $\frac{\alpha^2}{ab}=\alpha+2$ and using $\alpha\beta=-2ab$, we get

$$\alpha^3+\beta^3=a^2b^2(ab+6)$$

$$\left(\alpha^3-(ab+6)(ab)^2\right)\left(\beta^3-(ab+6)(ab)^2\right)=-8a^3b^3$$

$$\sum_{k=0}^{m-2} \frac{a^{\xi(k)-\xi(m)} (ab+6)^{-k-1}}{(ab)^{\lfloor \frac{k+1}{2} \rfloor}} \hat{\jmath}_{3k} = \frac{a^{1-\xi(m)}}{\alpha-\beta} \sum_{k=0}^{m-2} \frac{(ab+6)^{-k-1}}{(ab)^{\lfloor \frac{k+1}{2} \rfloor + \lfloor \frac{3k}{2} \rfloor}} \left(\alpha^{3k}-\beta^{3k}\right)$$

$$= \frac{a^{1-\xi(m)}}{(\alpha-\beta)(ab+6)} \sum_{k=0}^{m-2} \frac{\alpha^{3k}-\beta^{3k}}{(ab+6)^{k}(ab)^{2k}}$$

$$= \frac{a^{1-\xi(m)} \left[\frac{\alpha^{3(m-1)}-(ab+6)^{m-1}(ab)^{2m-2}}{\alpha^{3}-(ab+6)(ab)^{2}} - \frac{\beta^{3(m-1)}-(ab+6)^{m-1}(ab)^{2m-2}}{\beta^{3}-(ab+6)(ab)^{2}} \right]}{(\alpha-\beta)(ab+6)^{m-1}(ab)^{2m-4}}$$

$$= \frac{a^{1-\xi(m)}}{8(ab+6)^{m-1}(ab)^{2m-1}} \left[\frac{\alpha^{3m-3}\left((ab+6)(ab)^{2}-\beta^{3}\right) - \beta^{3m-3}\left((ab+6)(ab)^{2}-\alpha^{3}\right)}{\alpha-\beta} \right]$$

$$- \frac{a^{1-\xi(m)}}{8(ab+6)^{m-1}(ab)^{2m-1}} \left[\frac{\left(\alpha^{3}-\beta^{3}\right)(ab+6)^{m-1}(ab)^{2m-2}}{\alpha-\beta} \right]$$

$$= \frac{a^{1-\xi(m)}}{8(ab+6)^{m-1}(ab)^{2m-1}} \left[\frac{\alpha^{3m}-\beta^{3m}}{\alpha-\beta} \right] - \frac{a^{1-\xi(m)}}{8}(ab+2)$$

Then

$$\sum_{k=0}^{m-2} \frac{a^{\xi(k)-\xi(m)} (ab+6)^{-k-1}}{(ab)^{\lfloor \frac{k+1}{2} \rfloor}} \hat{\jmath}_{3k} + \frac{a^{1-\xi(m)}}{8}(ab+2)$$

$$= \frac{a^{1-\xi(m)}}{8(ab+6)^{m-1}(ab)^{2m-1}} \left[\frac{\alpha^{3m}-\beta^{3m}}{\alpha-\beta} \right]$$

$$= \frac{a^{1-\xi(m)}}{8(ab+6)^{m-1}(ab)^{2m-1}} \frac{a^{1-\xi(3m)}(ab)^{\lfloor \frac{3m}{2} \rfloor}}{a^{1-\xi(3m)}(ab)^{\lfloor \frac{3m}{2} \rfloor}} \left[\frac{\alpha^{3m}-\beta^{3m}}{\alpha-\beta} \right]$$

$$= \frac{(ab)^{\lfloor \frac{3m}{2} \rfloor -2m+1} a^{1-\xi(3m)}}{8(ab+6)^{m-1}(ab)^{\lfloor \frac{3m}{2} \rfloor}} \left[\frac{\alpha^{3m}-\beta^{3m}}{\alpha-\beta} \right] = \frac{\hat{\jmath}_{3m}}{8(ab)^{\lfloor \frac{m-1}{2} \rfloor}(ab+6)^{m-1}}$$

The other formula is proved similarly. When $a = b = 1$ the above result reduces to Jacobsthal and Jacobsthal-Lucas numbers,

$$\frac{7}{8} + \sum_{k=0}^{m-2} \frac{j_{3k}}{7^{k+1}} = \frac{j_{3m}}{8.7^{m-1}},$$

$$\frac{21}{8} + \sum_{k=0}^{m-2} \frac{c_{3k}}{7^{k+1}} = \frac{c_{3m}}{8.7^{m-1}}.$$

□

Theorem 7. *For any positive integer n, we have*

$$\sum_{k=0}^{n} a^{\xi(mk)} (ab)^{\lfloor \frac{mk}{2} \rfloor} \hat{\jmath}_{mk}$$

$$= \frac{a^{\xi(mn+m)}(ab)^{\lfloor \frac{mn+m}{2} \rfloor} \hat{\jmath}_{mn+m} - (-2)^{m} a^{\xi(mn)}(ab)^{\lfloor \frac{mn+2m}{2} \rfloor} \hat{\jmath}_{mn} - a^{\xi(m)}(ab)^{\lfloor \frac{m}{2} \rfloor} \hat{\jmath}_{m}}{\alpha^{m}+\beta^{m}-(-2ab)^{m}-1},$$

$$\sum_{k=0}^{n} a^{\xi(mk)} (ab)^{\left\lfloor \frac{mk+1}{2} \right\rfloor} \hat{c}_{mk}$$
$$= \frac{\begin{array}{c} -a^{-\xi(mn+m)} (ab)^{\left\lfloor \frac{mn+m+1}{2} \right\rfloor} \hat{c}_{mn+m} + (-2ab)^{m} a^{-\xi(mn)} (ab)^{\left\lfloor \frac{mn+1}{2} \right\rfloor} \hat{c}_{mn} \\ -a^{-\xi(m)} (ab)^{\left\lfloor \frac{m}{2} \right\rfloor} \hat{c}_{m} + 2 \end{array}}{\alpha^{m} + \beta^{m} - (-2ab)^{m} - 1}$$

Proof. Using the Binet's formula, it is obtained that

$$\begin{aligned} \sum_{k=0}^{n} a^{\xi(mk)} (ab)^{\left\lfloor \frac{mk}{2} \right\rfloor} \hat{\jmath}_{mk} &= a \sum_{k=0}^{n} a^{\xi(mk)-1} (ab)^{\left\lfloor \frac{mk}{2} \right\rfloor} \hat{\jmath}_{mk} \\ a \sum_{k=0}^{n} \frac{\alpha^{mk} - \beta^{mk}}{\alpha - \beta} &= \frac{a}{\alpha - \beta} \left(\frac{\alpha^{mn+m} - 1}{\alpha^{m} - 1} - \frac{\beta^{mn+m} - 1}{\beta^{m} - 1} \right) \\ &= \frac{a \left(\frac{\alpha^{mn+m} - \beta^{mn+m}}{\alpha - \beta} - (\alpha\beta)^{m} \frac{\alpha^{mn} - \beta^{mn}}{\alpha - \beta} - \frac{\alpha^{m} - \beta^{m}}{\alpha - \beta} \right)}{\alpha^{m} + \beta^{m} - (\alpha\beta)^{m} - 1} \\ &= \frac{\begin{array}{c} a^{\xi(mn+m)} (ab)^{\left\lfloor \frac{mn+m}{2} \right\rfloor} \hat{\jmath}_{mn+m} \\ - (\alpha\beta)^{m} a^{\xi(mn)} (ab)^{\left\lfloor \frac{mn}{2} \right\rfloor} \hat{\jmath}_{mn} - a^{\xi(m)} (ab)^{\left\lfloor \frac{m}{2} \right\rfloor} \hat{\jmath}_{m} \end{array}}{\alpha^{m} + \beta^{m} - (\alpha\beta)^{m} - 1}. \end{aligned}$$

Since $\alpha\beta = -2ab$, we get the desired result

$$\sum_{k=0}^{n} a^{\xi(mk)} (ab)^{\left\lfloor \frac{mk}{2} \right\rfloor} \hat{\jmath}_{mk}$$
$$= \frac{a^{(mn+m)} (ab)^{\left\lfloor \frac{mn+m}{2} \right\rfloor} \hat{\jmath}_{mn+m} - (-2)^{m} a^{\xi(mn)} (ab)^{\left\lfloor \frac{mn+2m}{2} \right\rfloor} \hat{\jmath}_{mn} - a^{\xi(m)} (ab)^{\left\lfloor \frac{m}{2} \right\rfloor} \hat{\jmath}_{m}}{\alpha^{m} + \beta^{m} - (-2)^{m} (ab)^{m} - 1}.$$

The other formula is proved similarly. If $a = b = 1$, the above result reduces to Jacobsthal and Jacobsthal-Lucas numbers,

$$\begin{aligned} \sum_{k=0}^{n} \jmath_{mk} &= \frac{\jmath_{mn+m} - (-2)^{m} \jmath_{mn} - \jmath_{m}}{2^{m} + (-1)^{m} - (-2)^{m} - 1}, \\ \sum_{k=0}^{n} c_{mk} &= \frac{-c_{mn+m} + (-2)^{m} c_{mn} - c_{m} + 2}{2^{m} + (-1)^{m} - (-2)^{m} - 1}. \end{aligned}$$

□

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