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CONVERGENCE RESULTS FOR SEQUENTIAL HENSTOCK STIELTJES INTEGRAL IN REAL VALUED SPACE

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Abstract. In this paper, we prove the convergence theorems for the Sequential Henstock Stieltjes integral of the real valued functions and give an example to show its applicability.

1. Introduction

In the early twentieth century, Mathematician sought to refine the rigorous foundation of integration theory. Researchers like Denjoy, Perron, Henstock, Kursweil and Lebesgues have made great achievements towards this end, with the recent one being the Henstock integral, developed independently by R. Henstock and J. Kurzweil in 1955 and 1957 respectively (see [1]-[11]). It is a kind of non-absolute integral which includes the Riemann, Improper Riemann, Newton and Lebesgue integral. Paxton [11] examined a theory for a specific definition of Henstock integral that was defined and called the Sequential Henstock integral. The relevance of developing the theory of Sequential Henstock is to expand the overall theory of integrals into more abstract mathematical settings involving the use of generalised sequences. It is well known that the Henstock integral is equivalent to the Denjoy integral, Perron integral and Denjoy-Perron integral. The equivalence of the Henstock integral and Sequential Henstock integral has been discussed in Paxton [11]. The authors [7] have studied dominated and bounded convergence results of Sequential Henstock Stieltjes Integral in real valued space. The aim of this paper is to prove a convergence theorem of Sequential Henstock Stieltjes Integral in real valued space and give an example to show its applicability.

Throughout this paper, we use $\mathbb{R}$ and $\mathbb{N}$ as set of real and natural numbers, $\{\delta_n(x)\}_{n=1}^{\infty}$ as sequence of gauge functions of $x \in [a,b]$ and P_n as sequence of partitions of subintervals of a compact interval $[a,b]$ for n = 1,2,3,...

Firstly, we recall the following definitions due to Paxton [11].

Definition 1. *A gauge on $[a,b]$ is a positive real-valued function $\delta : [a,b] \to \mathbb{R}^+$. This gauge is δ-fine if $[u_{i-1}, u_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$.*

Definition 2. *A sequence of tagged partition P_n of [a,b] is a finite collection of ordered pairs $P_n = \{(u_{(i-1)_n} u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ where $[u_{i-1}, u_i] \in [a,b]$, $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ and $a = u_0 < u_{i_1} < ... < u_{m_n} = b$.*

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Definition 3. *(Henstock integral). A function $f : [a,b] \to \mathbb{R}$ is Henstock integrable if there exists a number $\alpha \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a positive gauge function $\delta(x) > 0$ such that*

$$|\sum_{i=1}^{n} f(t_i)(u_i - u_{i-1}) - \alpha| < \varepsilon,$$

whenever $P = \{(u_{(i-1)}, u_i), t_i\}_{i=1}^{n}$ is a $\delta_n(x) - fine$ tagged partition on $[a,b]$. We say that α is Henstock integral of f on $[a,b]$ i.e $\alpha = (H)\int_a^b f$. We use $H[a,b]$ to denote the set of all Henstock integrable functions on $[a,b]$.

Definition 4. *(Sequential Henstock integral). A function $f : [a,b] \to \mathbb{R}$ is Sequential Henstock integrable if there exists a number $\alpha \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a sequence of positive gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that*

$$|\sum_{i=1}^{m_n} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \alpha| < \varepsilon,$$

whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ is a $\delta_n(x) - fine$ tagged partition on $[a,b]$. We say that α is Sequential Henstock integral of f on $[a,b]$ i.e $\alpha = (SH)\int_a^b f$. We use $SH[a,b]$ to denote the set of all Sequential Henstock integrable functions on $[a,b]$.

Remark 1. *If $\delta_n = \delta$ where $n = 1$ in Definition 4, we have our definition for the Henstock integral..*

Definition 5. *(Henstock Stieltjes Integral). Let $g : [a,b] \to \mathbb{R}$ be an increasing function. A real valued function $f : [a,b] \to \mathbb{R}$ is Henstock Stieltjes integrable with respect to g on $[a,b]$ if there exists a number $\alpha \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a positive gauge functions $\delta(x) > 0$ such that*

$$|\sum_{i=1}^{n} f(t_i)[g(u_i) - g(u_{i-1})] - \alpha| < \varepsilon,$$

whenever $P = \{(u_{i-1}, u_i), t_i\}$ is a $\delta - fine$ partition on $[a,b]$. We say that α is Henstock Stieltjes integral of f on $[a,b]$ i.e $\alpha = (HS)\int_a^b fdg$. We use $HS[a,b]$ to denote the set of all Henstock Stieltjes integrable functions on $[a,b]$.

We define newly the following:

Definition 6. *(Sequential Henstock Stieltjes Integral). Let $g : [a,b] \to \mathbb{R}$ be an increasing function. A function $f : [a,b] \to \mathbb{R}$ is Sequential Henstock Stieltjes integrable with respect to g on $[a,b]$ if there exists a number $\alpha \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a sequence of positive gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that*

$$|\sum_{i=1}^{m_n} f(t_{i_n})[g(u_{i_n} - g(u_{(i-1)_n}))] - \alpha| < \varepsilon,$$

whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ is a $\delta_n(x) - fine$ tagged partition on $[a,b]$. We say that α is Sequential Henstock Stieltjes integral of f on $[a,b]$ i.e $\alpha = (SHS)\int_a^b fdg$. We use $SHS[a,b]$ to denote the set of all Sequential Henstock Stieltjes integrable functions on $[a,b]$.

Definition 7. [11] *Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of function for $n \in \mathbb{N}$ and a function $g : [a, b] \to \mathbb{R}$. Then f_n is uniformly integrable with respect to g on $[a, b]$ if*
(i) The integral $\int_a^b f_n dg$ exists for each $n \in \mathbb{N}$.
(ii) for $\varepsilon > 0$ there exists a sequence of gauges $\delta_\mu(x) = sup_{n\in\mathbb{N}}\{\delta_n(x)\}_{n=1}^{\infty}$ on $[a, b]$ such that the inequality

$$|\int_a^b f_n dg - U(f_n, dg, P_n)| < \varepsilon,$$

holds for each $\delta_n(x)$-fine partition P_n of $[a, b]$ and for every $n \in \mathbb{N}$

Next, we state the theorems on the basic properties of the Sequential Henstock Stieltjes integral for real valued function.

Theorem 1. *Let $f_1, f_2 : [a, b] \to \mathbb{R}$ be Sequential Henstock Steiltjes integrable with respect to an increasing function $g : [a, b] \to \mathbb{R}$ and $c \in \mathbb{R}$ be a constant. Then*

(i) If $f_1 \in SH(g, [a, b]) \geq 0$ and $f_2 \in SH(g, [a, b]) \geq 0$, then $\int_{[a,b]} f_1 dg \geq 0$ and $\int_{[a,b]} f_2 dg \geq 0$;
(ii) If $cf_1 \in SH(g, [a, b])$ then $\int_{[a,b]} cf_1 dg = c\int_{[a,b]} f_1 dg$;
(iii) If $(f_1 + f_2) \in SH(g, [a, b])$, then $\int_{[a,b]}(f_1 + f_2)dg = \int_{[a,b]} f_1 dg + \int_{[a,b]} f_2 dg$;
(iv) If $(f_1, g)(x) \leq (f_2, g)(x)$ for all $x \in [a, b]$, then $\int_{[a,b]} f_1 dg \leq \int_{[a,b]} f_2 dg$;
(v) If $|f_1, g| \in SH(g, [a, b])$, then $|\int_{[a,b]} f_1 dg| = \int_{[a,b]}|f_1 dg|$;
(vi) If $|(f_1, g(x))| \leq k$ for all $x \in [a, b]$, $k \in \mathbb{R}$, then $|\int_{[a,b]} f_1 dg| = k(b - a)$.

Proof. The proof of (i)-(vi) results follows easily from the Definition 6 and so the details are omitted. □

Lemma 1. *Let $f, g : [a, b] \to \mathbb{R}$ be such that the integral $\int_a^b f dg$ exists. Then, for any $\varepsilon > 0$, there exists a sequence of gauges $\{\delta_n(x)\}_{n=1}^{\infty}$ on $[a, b]$ such that for all $\delta_n(x) - fine$ tagged partitions $P_n = \{([u_{(i-1)_n}, u_{i_n}], t_{i_n})\}$ where $[u_{(i-1)_n}, u_{i_n}] \in [a, b]$ and $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ we have*

$$|U(f, dg, P_n) - \int_a^b f dg| < \varepsilon.$$

If $\{([u_{(i-1)_n}, u_{i_n}], t_{i_n})\} : i = 1, 2, ..., n$ is an arbitrary system satisfying

$$a \leq u_0 \leq t_{i_n} \leq u_{1_n} \leq u_{2_n} \leq ... \leq u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n} \leq b, \tag{1}$$

with

$$[u_{(i-1)_n}, u_{i_n}] \subset (t_{i_n} - \delta_n(t_{i_n}), t_{i_n} + \delta_n(t_{i_n})),$$

then

$$|\sum_{i=1}^{n}(f(t_{i_n})(g(u_{i_n}) - g(u_{(i-1)_n})) - \int_a^b f dg| < \varepsilon.$$

Proof. Assume the system $\{([u_{(i-1)_n}, u_{i_n})], t_{i_n} : i \in 1, 2, ..., n\}$ satisfies the (1). We set $u_{(i-1)_n} = a$ and $u_{i_n} = b$. Now, let $\beta_n = 0$ and $i \in \{0, 1, ..., n\}$ be given. Assume that $u_{i_n} > u_{(i-1)_n}$, then if the sequence of gauges δ_n, δ_0 are such that $\delta_0 < \delta_n$ on $[a, b]$, then every $\delta_0 - fine$ partition of $[a, b]$ is also $\delta_n - fine$, so there are sequence of gauges $\{\delta_{i_\mu}(x)\}_{n=1}^{\infty}$, $\mu \in \mathbb{R}$. So for every $\delta_{i_\mu} - fine$ partitions on $[u_{(i-1)_n}, u_{i_n}]$ and $\delta_{i_\mu} - fine$ partitions $P_{i_n} = \{([u_{(i-1)_n}, u_{i_n}], t_{i_n})\}$ where $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ of $[u_{(i-1)_n}, u_{i_n}]$ such that

$\delta_{i_n}(x) \leq \delta_n(x)$ for $x \in [u_{(i-1)_n}, u_{i_n}]$ and

$$|U(f,g,P_n) - \int_a^b f dg| < \frac{\beta_n}{n+1}, \forall n \in N. \tag{2}$$

Now, we form $\delta_n(x) - fine$ tagged partitions $Q_{i_n} = \{([u_{(i-1)_n}, u_{i_n})], t_{i_n}\}$ of the interval $[a,b]$,such that

$$U(Q,g,P_n) = \sum_{i=1}^{n} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] + \sum_{i=0}^{n} U(f,g,P_n).$$

If $u_{i_n} > u_{(i-1)_n}$ and we set $U(f,g,P_n) = 0$, then

$$|\sum_{i=1}^{n}(f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] + \sum_{i=0}^{n} U(f,g,P_n) - \sum_{i=0}^{n} U(f,g,P_n) - \int_a^b f dg|$$

$$= \quad |U(Q,g,P_n) - \int_a^b f dg| < \varepsilon.$$

This together with (2) yields

$$|\sum_{i=1}^{n}(f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] - \int_a^b f dg|$$

$$\leq \quad |U(Q,g,P_n) - \int_a^b f dg| + |\sum_{i=0}^{n} U(f,g,P_n) - \int_a^b f dg|$$

$$< \quad \varepsilon + \beta_n.$$

Since $\beta_n > 0$ was arbitrary, (3) follows. This completes the proof.
The Saks' Lemma plays a very important role in the proof of some of the theorems on convergence.

□

2. Main results

Theorem 2. Let $f_n : [a,b] \to \mathbb{R}$ be integrable with respect to a function $g : [a,b] \to \mathbb{R}$ and suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a,b]$. Then, there exists both integrals $\int_a^b f dg$ and $\lim_{n\to\infty} \int_a^b f_n(x)$ and

$$\int_a^b f dg = \lim_{n\to\infty} \int_a^b f_n(x) dg. \tag{3}$$

Moreover,

$$\lim_{n\to\infty} f_n(x)[\sup_{t_{i_n} \in [a,b]} |\int_a^t f_n dg - \int_a^t f dg|] = 0,$$

holds whenever g is bounded on $[a,b]$.

Proof. Let $\varepsilon > 0$, then for every $\delta_n(x) - fine$ tagged partitions $P_n = \{([u_{(i-1)_n}, u_{i_n}], t_{i_n})\}$ on $[a, b]$,

$$U(f, dg, P_n) = lim_{n\to\infty} U(f_n, dg, P_n).$$

Let $n_0 \in \mathbb{N}$ such that

$$|U(f_m, dg, P_n) - U(f_n, dg, P_n)| < \varepsilon,$$

holds for all $m, n \geq n_0$; where

$$U(f_m, dg, P_n) = |\sum_{i=1}^{n}(f_m(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n}))]|.$$

By definition of uniform integrability, we have

$$\begin{aligned}
|\int_a^b f_m dg - \int_a^b f_n dg| &= |\int_a^b f_m dg - \sum_{i=1}^{n} f_m(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] \\
&\quad + \sum_{i=1}^{n} f_m(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] \\
&\quad - \sum_{i=1}^{n} f_n(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] \\
&\quad + \sum_{i=1}^{n} f_n(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] - \int_a^b f_n dg| \\
&\leq |\int_a^b f_m dg - \sum_{i=1}^{n} f_m(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})]| \\
&\quad + |\sum_{i=1}^{n} f_m(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] \\
&\quad - \sum_{i=1}^{n} f_n(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})]| \\
&\quad + |\sum_{i=1}^{n} f_n(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] - \int_a^b f_n dg| \\
&< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,
\end{aligned}$$

for all $m, n \geq n_0$. In fact, $\int_a^b f_n dg$ is a Cauchy Sequence and thus it has a finite limit, say

$$\lim_{n\to\infty} \int_a^b f_n dg = \alpha \in \mathbb{R}.$$

Now, let $P_n = \{([u_{(i-1)_n}, u_{i_n}], t_{i_n})\}$ be an arbitrary $\delta_n(x)$- fine partitions of $[a, b]$. Choose an $n_i \in \mathbb{N}$ for $i = 1, ..., r \in \mathbb{N}$ such that

$$|U(f_{n_i}, dg, P_n) - U(f, dg, P_n)| < \varepsilon$$

and

$$|\int_a^b f_{n_i} dg - \alpha| < \varepsilon.$$

Then

$$\begin{aligned}
|U(f,dg,P_n)-\alpha| &= |\sum_{i=1}^{n} f(t_{i_n})[g(u_{i_n})-g(u_{(i-1)_n})] - \sum_{i=1}^{n} f_{n_i}(t_{i_n})[g(u_{i_n})-g(u_{(i-1)_n})] \\
&\quad + \sum_{i=1}^{n} f_{n_i}(t_{i_n})[g(u_{i_n})-g(u_{(i-1)_n})] - \int_a^b f_{n_i}dg + \int_a^b f_{n_i}dg - \alpha| \\
&\leq |\sum_{i=1}^{n} f(t_{i_n})[g(u_{i_n})-g(u_{(i-1)_n})] - \sum_{i=1}^{n} f_{n_i}(t_{i_n})[g(u_{i_n})-g(u_{(i-1)_n})]| \\
&\quad + |\sum_{i=1}^{n} f_{n_i}(t_{i_n})[g(u_{i_n})-g(u_{(i-1)_n})] - \int_a^b f_{n_i}dg| + |\int_a^b f_n dg - \alpha| \\
&< 3\varepsilon.
\end{aligned}$$

It follows that

$$\int_a^b f dg = \alpha.$$

Let $h_n(x) = f_n(x) - f(x)$, for $n \in \mathbb{N}$ and $x \in [a,b]$. Assume g is bounded on $[a,b]$ and suppose $\varepsilon > 0$, there exists a $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for all $\delta_n(x) - fine$ tagged partitions $P_n = \{([u_{(i-1)_n}, u_{i_n}], t_{i_n})\}$ on $[a,b]$ we have

$$|\int_a^b h_n dg - U(h_n, dg, P_n)| < \varepsilon.$$

Since $h_n(x) \to 0$ for $x \in [a,b]$ and g is bounded, there exists an $n_0 \in \mathbb{N}$ such that $|h_n(t_{i_n})| \|g\| < \dfrac{\varepsilon}{2n}$ for all $n \geq n_0$ and $i \in \{1,2,...n\}$.
Let $x \in [a,b]$ be arbitrary and $n \in \mathbb{N} \cap [n_0, \infty]$ be given and let $i \in 1,...,n$ be such that $x \in [u_{(i-1)_n}, u_{i_n}]$. Then

$$\begin{aligned}
|\int_a^b h_n dg| &= |\int_a^x h_n dg - \sum_{i=1}^{m_n} h_n(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})) + \sum_{i=1}^{m_n} h_n(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})) \\
&\quad - \int_a^x h_n dg + \sum_{i=1}^{m_n} h_n(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})) - \sum_{i=1}^{m_n} h_n(t_{i_n})[g(x) - g(u_{(i-1)_n})) \\
&\quad + \sum_{i=1}^{m_n} h_n(t_{i_n})[g(x) - g(u_{(i-1)_n})) - \int_x^b h_n dg| \\
&\leq |\int_a^x h_n dg - \sum_{i=1}^{m_n} h_n(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n}))| \\
&\quad + |\int_x^b h_n dg - \sum_{i=1}^{m_n \in \mathbb{N}} h_n(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n}))| \\
&\quad + |\sum_{i=1}^{m_n} |h_n||g(u_{i_n}) - g(u_{(i-1)_n}))| - \sum_{i=1}^{m_n} |h_n(t_{i_n})||g(x) - g(u_{(i-1)_n})| \\
&\leq |\int_a^x h_n dg - \sum_{i=1}^{m_n} h_n(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n}))|
\end{aligned}$$

$$+|\int_x^b h_n dg - \sum_{i=1}^{m_n} h_n(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n}))|$$

$$\begin{aligned}
&+|\sum_{i=1}^{m_n}|h_n(t_{i_n})||g(u_{i_n}) - g(u_{(i-1)_n})) - |h_n(t_{i_n})|g(x) - g(u_{(i-1)_n})| \\
= \ & |\int_a^x h_n dg - h_n(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n}))|, +|\int_x^b h_n dg - h_n(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n}))| \\
\leq \ & 2\varepsilon.
\end{aligned}$$

To summarise, we have shown that

$$|\int_a^b h_n dg| \leq 2\varepsilon,$$

for all $n \geq n_0$ and $x \in [a, b]$. □

The following example shows that the boundedness of the integral and is essential to ensure the uniform convergence of the definite integrals of the Sequential Henstock in (3).

Example 1. *Let* $f_n(x) = \dfrac{1}{n+1}$, *for* $x \in [a, b]$ *and let* $g : [a, b] \to \mathbb{R}$ *be arbitrary with* $g(a) = 0$, *then* $\{f_n\}$ *tends pointwise on* $[a, b]$ *to zero function. Furthermore, as* $U(f_n, dg, P_n) = (\dfrac{1}{n+1}), g(b)$ *for each* $n \in N$ *and sequence partitions* P_n *of* $[a, b]$ *, we see that* $\int_a^b f_n dg = (\dfrac{1}{n+1})g(b)$, *for each* $n \in \mathbb{N}$ *and the sequence* $\{f_n\}$ *is Sequential Henstock Stieltjes uniformly integrable with respect to* g. *Suppose* g *is unbounded on the other hand, then for any* $\varepsilon > 0$ *and for each* $n \in \mathbb{N}$, *there is a* $x \in [a, b]$ *such that*

$$f_n(x) = (\frac{1}{n+1})g(x) > \varepsilon.$$

That is f_n *does not converge uniformly to the zero function.*

3. Discussion and Conclusion

Convergence theorem connotes that the integrability of a sequence of functions is preserved by taking limits. In other words, suppose a sequence of integrable functions$(f_n)_{n=1}^{\infty}$ has a limit function f, then one can conclude that f is also integrable as well as the equality

$$\int f = lim_{n\to\infty} \int f_n.$$

The analysis of the dynamics of integrability of a sequence of function often produce results that involve looking for sequence of functions $(f_n)_{n=1}^{\infty}$ that is approximants for f or construct an integrable function, approximants and examine the existence of the limit using the $\varepsilon - \delta$ definition. In this paper, we studied the convergence theorem of Sequential Henstock Stieltjes integral in real space by introducing and applying new concepts like Sequential Sak's lemma to prove the convergence properties. and proving the theorems on the Sequential Henstock Stieltjes integral. The results obtained show that the integrability of the Sequential Henstock Stieltjes holds for convergence theorem for sequence of function. equivalence between these family of Henstock integrals. To this

end, an example to show the applicability of the result is also given.
Up until this research work, however, the convergence theory of Sequential Henstock Stieltjes integral did not include definitions and theorems based on sequence and it is in our viewpoint that the Sequential Henstock Steiltjes integral can be used to renew the interest of integration theorists and researchers on convergence theory of Henstock integral. In line with this, the results of this research can now be extended to studies in more abstract spaces and applications arising from this as well as to the conclusion of Sequential Henstock Stieltjes integral in introductory Calculus can be assessed for possible pedagogical benefits. In conclusion, can convergence result of Sequential Henstock Stieltjes integral hold for classes of functions, such as step functions, measurable functions, absolutely integrable functions? It is of the view of the authors that these problems could be considered for further research.

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