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IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING THE CAPUTO-HADAMARD FRACTIONAL DERIVATIVE IN A BANACH SPACE

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Abstract. In this paper we establish existence results for a class of initial value problems for impulsive fractional differential equations involving the Caputo-Hadamard fractional derivative of order $1 < r \leq 2$.

1. Introduction

This paper deals with the existence of solutions to the initial value problem (IVP) for fractional order differential equations

$$^{CH}D^r y(t) = f(t, y(t)), \tag{1}$$

for a.e. $t \in J = [a, T]$, $a > 0$, $t \neq t_k,\ k = 1, ..., m, 1 < r \leq 2,$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, ..., m, \tag{2}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, ..., m, \tag{3}$$

$$y(a) = y_1, y'(a) = y_2 \tag{4}$$

where $^{CH}D^r$ is the Caputo-Hadamard fractional derivative, $f : J \times E \to E$ is a function, I_k and overline $I_k : E \to E$, $k = 1, ..., m$ are functions, $a = t_0 < t_1 < ... < t_m < t_{m+1} = T$ $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $\Delta y'|_{t=t_k} = y'(t_k^+) - y'(t_k^-), y(t_k^+) = \underset{\varepsilon \to 0^+}{lim} y(t_k + \varepsilon)$ and $y(t_k^-) = \underset{\varepsilon \to 0^-}{lim} y(t_k + \varepsilon)$ represent the right and left limits of y at $t = t_k$, $k = 1, ..., m$ and E is a Banach space. Differential equations of fractional order have recently proved valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. There has been a significant development in theory of fractional calculus and fractional ordinary and partial differential equations in recent years; see e.g. The monographs of Hilfer [26], Kilbas *et al.* [29], Podlubny [35], Momani *et al.* [33], and the papers by Agarwal *et al.* [2] and Benchohra *et al.* [12]. Applied problems require the definitions of fractional derivatives allowing the utilization of physically interpretable initial data, that contain $y(0)$, $y'(0)$, and so on. Caputo's fractional derivative satisfies these demands. For more details concerning geometric and physical interpretation of fractional derivatives of Riemann-Liouville type and Caputo type, see [35]. However, the literature on Hadamard-type fractional differential equations has not undergone as much development; see [4, 38]. The fractional derivative that Hadamard

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[22] introduced in 1892, differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains a logarithmic function of arbitrary exponent. Detailed descriptions of the Hadamard fractional derivative and integral can be found in [17, 18, 19]. Recently, Hadamard fractional calculus is getting attention which is an important part of theory of fractional calculus ; see [29] and [4, 17, 18, 19, 28, 30, 38]. A Caputo-type modification of the Hadamard fractional derivative which is called the Caputo-Hadamard fractional derivative was given in [27], and its fundamental theorems were proved in [20, 1].
The web site http://people.tuke.sk/igor.podlubny/, authored by Igor Podlubny contains more information on fractional calculus and its applications, and hence it is very useful for those that are interested in this field. The impulsive differential equations (for $r \in \mathbb{N}$) have become important in recent years as mathematical models of phenomena in both physical and social sciences. There has been a significant development in impulsive theory, especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Bainov and Simeonov [7], Benchohra et *al* [12], Lakshmikantham et *al.* [31], Samoilenko and Perestyuk [37], and the references therein. In [16], Benchohra and Slimani have initiated the study of fractional differential equations with impulses. To the best knowledge of the authors, no papers exist in the literature devoted to differential inclusions with Caputo-Hadamard fractional derivatives and impulses. Thus, the results of the present paper initiate this study.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that are used in the remainder of this paper.

Let $J := [a, T]$, let $C([a, T], E)$ be the Banach space of continuous functions $y : [a, T] \to E$ with the norm,

$$\|y\| = \sup\{||y(t)||_E : a \leq t \leq T\},$$

and we denote by $L^1([a, T], E)$ the Banach space of functions $y : [a, T] \to E$ that are Bohner integrable with norm

$$\|y\|_{L^1} = \int_a^T ||y(t)||_E dt.$$

$AC([a, T], E)$ ix the space of functions $y : [a, T] \to E$, which are absolutely continuous. Let $AC^1([a, T], E)$ the space of functions $y : [a, T] \to E$, that are absolutely continuous and whose first derivative, y', is absolutely continuous.
Let

$$V(t) = \{v(t) : v \in V\}, t \in J,$$
$$V(J) = \{v(t) : v \in V,\ t \in J\}.$$

Definition 1. *(*[29]*). The Hadamard fractional integral of order r for a function $h : [1, +\infty) \to \mathbb{R}$ is defined as*

$$I^r h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds,\ r > 0,$$

provided the integral exists.

Definition 2. *(*[29]*). For a function h given on the interval $[1, +\infty)$, the r Hadamard fractional-order derivative of his defined by*

$$({}^H D^r h)(t) \quad = \quad \frac{1}{\Gamma(n-r)} \left(t \frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-r-1} \frac{h(s)}{s} ds,$$

$n-1<r\leq n,\ n=[r]+1$, *where* $[r]$ *denotes the integer part of* r *and* $\log(\cdot)=\log_e(\cdot)$.

Definition 3. *([27]). Let* $AC^n_\delta[a,b]=\{g:[a,b]\to\mathbb{C},\ \{\delta^{n-1}g\in AC[a,b]\}$ *where* $\delta=t\frac{d}{dt}$ $0<a<b<\infty$ *and let* $\alpha\in\mathbb{C}$ *such that* $Re(\alpha)\geq 0$. *For a function* $g\in AC^n_\delta[a,b]$, *the Caputo type Hadamard derivative of fractional order* α *is defined as follows*

(i): *If* $\alpha\in\mathbb{N}$, *then* $({}^{CH}D^\alpha_a g)(t)=\frac{1}{\gamma(n-\alpha)}(t\frac{d}{dt})^n\int_a^t(\log\frac{t}{s})^{n-\alpha-1}\delta^n g(s)\frac{ds}{s}$,
$n-1<\alpha<n,\ \ n=[Re(\alpha)]+1$

(ii): *If* $\alpha=n\in\mathbb{N}$, *then* $({}^{CH}D^\alpha_a g)(t)=\delta^n g(t)$,

where $[Re(\alpha)]$ *denotes the integer part of the real number* $Re(\alpha)$.

Lemma 1. *Let* $y\in AC^n_\delta[a,b]$ *or* $C^n_\delta[a,b]$ *and* $\alpha\in\mathbb{C}$. *Then*

$$I^\alpha_a({}^{CH}D^\alpha_a y)(t)=y(t)-\sum_{k=0}^{n-1}\frac{\delta^k y(a)}{k!}\left(\log\frac{t}{a}\right)^k. \tag{5}$$

For convenience, we first recall the definition of the Kuratowski measure of noncompactness, and summarize the main properties of this measure.

Definition 4. *([6, 8]) Let* E *be a Banach space and let* Ω_E *be the family of bounded subsets of* E. *The Kuratowski measure of noncompactness is the map* $\alpha:\Omega_E\to[0,\infty)$ *defined by*

$$\alpha(B)=\inf\{\epsilon>0\ :B\subset\bigcup_{j=1}^m B_j\ \textit{and}\ diam(B_j)\leq\epsilon\},\ \textit{for}\ B\in\Omega_E.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for details, see [8],[6]).

(1) $\alpha(B)=0\Leftrightarrow\overline{B}$ is compact (B is relatively compact).
(2) $\alpha(B)=\alpha(\overline{B})$.
(3) $A\subset B\Rightarrow\alpha(A)\leq\alpha(B)$.
(4) $\alpha(A+B)\leq\alpha(A)+\alpha(B)$.
(5) $\alpha(cB)=|c|\alpha(B),c\in\mathbb{R}$.
(6) $\alpha(conB)=\alpha(B)$.

Here $\overline{B}$ and $conB$ denote the closure and the convex hull of the bounded set B, respectively.

Definition 5. *A multivalued map* $F:J\times E\to E$ *is said to be Carathéodory if*

(1) $t\to F(t,u)$ *is measurable for each* $u\in E$.
(2) $u\to F(t,u)$ *is upper semicontinuous for almost all* $t\in J$.

Let us now recall the Mönch's fixed point theorem and an important lemma.

Theorem 1. *([34],[3]) Let* D *be a bounded, closed and convex subset of a Banach space* E *such that* $0\in D$, *and let* N *be a continuous mapping of* D *into itself. If the implications*

$$V=\overline{co}N(V)\ \textit{or}\ V=N(V)\cup\{0\}\Longrightarrow\alpha(V)=0, \tag{6}$$

hold for every subset V *of* D, *then* N *has a fixed point.*

Lemma 2. *([21]) If* $V\subset C(J,E)$ *is a bounded and equicontinuous set, then*

(1) The function $t\to\alpha(V(t))$ *is continuous on* J.
(2)

$$\alpha\left(\left\{\int_J x(t)dt,x\in V\right\}\right)\leq\int_J\alpha(v(t))dt.$$

3. Main results

Consider the set of functions

$$PC(J,E) = \left\{ \begin{array}{l} y : J \to E,\ y \in C^2((t_k,t_{k+1}],\ E),\ k = 1,\dots,m, \text{ and there exist} \\ y(t_k^+) \text{ and } y(t_k^-),\ k = 1,\dots,m, \text{ with } y(t_k^-) = y(t_k). \end{array} \right\}$$

This set is a Banach space with the norm

$$\|y\|_{PC} = \sup\{||y(t)||_E : a \le t \le T\},$$

also,

$$PC'(J,\ E) = \left\{ \begin{array}{l} y : J \to E,\ y \in AC_\delta^2((t_k,t_{k+1}],\ E),\ k = 1,\dots,m, \text{ and there exist} \\ y(t_k^+) \text{ and } y(t_k^-),\ k = 1,\dots,m, \text{ with, } y(t_k^-) = y(t_k). \end{array} \right\}$$

This set is a Banach space with the norm

$$\|y\|_{PC'} = \sup\{||y(t)||_E : a \le t \le T\}$$

Set

$$J' := J \setminus \{t_1, ..., t_m\}.$$

Definition 6. *A function $y_2 \in PC(J,E) \Cap PC'(J,E)$ on J' is said to be a solution of (1)-(4) if y satisfies the differential equation ${}^{CH}D^r y(t) = f(t,y(t))$ on J', and satisfies conditions (2)-(4).*

To prove the existence of a solution to (1)-(4), we need the following auxiliary lemma.

Lemma 3. *Let $1 < r \le 2$ and let $\rho \in AC(J',\mathbb{R})$. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} y_1 + ay_2 \log\left(\frac{t}{a}\right) + \frac{1}{\Gamma(r)}\int_a^t \left(\log\frac{t}{s}\right)^{r-1}\rho(s)\frac{ds}{s}, & \text{if } t \in [a,t_1] \\ y_1 + ay_2 \log\left(\frac{t}{a}\right) + \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k\Gamma(r-1)}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-2}\rho(s)\frac{ds}{s} & \\ +\frac{1}{\Gamma(r)}\int_{t_k}^t \left(\log\frac{t}{s}\right)^{r-1}\rho(s)\frac{ds}{s} + \sum_{k=1}^m I_k(y(t_k^-)) & \\ +\sum_{k=1}^m t_k \log\left(\frac{t}{t_k}\right)\overline{I}_k(y(t_k^-)), & \text{if } t \in (t_k,t_{k+1}], k = 1,...,m, \end{cases} \tag{7}$$

if and only if y is a solution of the fractional IVP

$$ {}^{CH}D_a^r y(t) = \rho(t), \ \text{for each, } t \in J', \tag{8}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,...,m, \tag{9}$$

$$\Delta' y|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1,...,m, \tag{10}$$

$$y(a) = y_1, y'(a) = y_2. \tag{11}$$

Proof Let y be a solution of (8)-(11). Applying the Hadamard fractional integral of order r to both sides of (8), using conditions (9)-(11) and Lemma 1 we get,
For $t \in [a,t_1]$,

$$y(t) = c_1 + c_2\log\left(\frac{t}{a}\right) + \frac{1}{\Gamma(r)}\int_a^t\left(\log\frac{t}{s}\right)^{r-1}\rho(s)\frac{ds}{s}.$$

Hence, $c_1 = y(a) = y_1$ *and* $c_2 = y'(a) = y_2$, and so

$$y(t) = y_1 + ay_2\log\left(\frac{t}{a}\right) + \frac{1}{\Gamma(r)}\int_a^t\left(\log\frac{t}{s}\right)^{r-1}\rho(s)\frac{ds}{s}.$$

If $t \in (t_1,t_2]$,

$$y(t) = c_1 + c_2\log\left(\frac{t}{a}\right) + \frac{1}{\Gamma(r)}\int_{t_1}^t\left(\log\frac{t}{s}\right)^{r-1}\rho(s)\frac{ds}{s}. \tag{12}$$

We have

$$\Delta y|_{t=t_1} = y(t_1^+) - y(t_1^-),$$

and

$$\begin{aligned} I_1(y(t_1^-)) &= c_1 + c_2 \log\left(\tfrac{t_1}{a}\right) \\ &- \left(y_1 + ay_2 \log\left(\tfrac{t_1}{a}\right) + \tfrac{1}{\Gamma(r)} \int_a^{t_1} \left(\log \tfrac{t_1}{s}\right)^{r-1} \rho(s) \tfrac{ds}{s}\right). \end{aligned}$$

Hence,

$$c_1 + c_2 \log\left(\frac{t_1}{a}\right) = y_1 + ay_2 \log\left(\frac{t_1}{a}\right) + \frac{1}{\Gamma(r)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-1} \rho(s) \frac{ds}{s} + I_1(y(t_1^-).$$

We have

$$\Delta y'|_{t=t_1} = y'(t_1^+) - y'(t_1^-),$$

and

$$\overline{I}_1(y(t_1^-)) = \tfrac{c_2}{t_1} - \left(\tfrac{a}{t_1}\right) y_2 + \tfrac{1}{t_1\Gamma(r-1)} \int_a^{t_1} \left(\log \tfrac{t_1}{s}\right)^{r-2} \rho(s) \tfrac{ds}{s}.$$

Hence,

$$c_2 = ay_2 + \frac{1}{t_1\Gamma(r-1)} \int_a^{t_1} \left(\log \frac{t_1}{s}\right)^{r-2} \rho(s) \frac{ds}{s} + \overline{I}_1(y(t_1^-)), \tag{13}$$

and

$$\begin{aligned} c_1 &= y_1 - \tfrac{\log\left(\frac{t_1}{a}\right)}{t_1\Gamma(r-1)} \int_a^{t_1} \left(\log \tfrac{t_1}{s}\right)^{r-2} \rho(s) \tfrac{ds}{s} \\ &+ \tfrac{1}{\Gamma(r)} \int_a^{t_1} \left(\log \tfrac{t_1}{s}\right)^{r-1} \rho(s) \tfrac{ds}{s} + I_1(y(t_1^-)) - t_1 \log\left(\tfrac{t_1}{a}\right) \overline{I}_1(y(t_1^-). \end{aligned} \tag{14}$$

Then by (13)-(14) and (12), we have

$$\begin{aligned} y(t) &= y_1 + ay_2 \log\left(\tfrac{t}{a}\right) + \tfrac{\log\left(\frac{t}{t_1}\right)}{t_1\Gamma(r-1)} \int_a^{t_1} \left(\log \tfrac{t_1}{s}\right)^{r-2} \rho(s) \tfrac{ds}{s} \\ &+ \tfrac{1}{\Gamma(r)} \int_{t_1}^{t} \left(\log \tfrac{t}{s}\right)^{r-1} \rho(s) \tfrac{ds}{s} + I_1(y(t_1^-)) + t_1 \log\left(\tfrac{t}{t_1}\right) \overline{I}_1(y(t_1^-). \end{aligned}$$

If $t \in (t_k, t_{k+1}]$, then again from Lemma 1, we obtain (7). Conversely, assume that y satisfies the impulsive fractional integral equation (7). If $t \in [a, t_1]$, then $y(a) = y_1$, $y'(a) = y_2$, and using that ${}^{CH}D_a^r$ is the left inverse of I_a^r, we get

$${}^{CH}D_a^r y(t) = \rho(t), \text{ for all } t \in [a, t_1].$$

Let $t \in (t_k, t_{k+1}]$, $k = 1, ..., m$. We have ${}^{CH}D_a^r C = 0$, for any constant C, so

$${}^{CH}D_a^r y(t) = \rho(t), \text{ for all } t \in (t_k, t_{k+1}].$$

Also, we can easily show that

$$\Delta y|_{t=t_k} = I_k(y(t_k^-), \quad k = 1, ..., m,$$
$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, ..., m.$$

□

Theorem 2. *Assume the following hypotheses hold.*

(H1) *The function $f : J \times E \to E$ satisfies Carathéodory conditions.*

(H2) *There exists $p \in L^1(J, \mathbb{R}_+)$ such that*

$$\|f(t, y)\| \leq p(t)\|y\| \quad for \quad a.e.\ t \in J \quad and \quad each \quad y \in E.$$

(H3) *There exists a constant* $k^* > 0$ *such that*

$$\|I_k(y)\| \leq k^*\|y\| \quad for \quad each \quad y \in E.$$

(H4) *There exists a constant* $k_1^* > 0$ *such that*

$$\|\overline{I}_k(y)\| \leq k_1^*\|y\| \quad for \quad each \quad y \in E.$$

(H5) *For each bounded set* $B \subset E$*, we have*

$$\alpha(f(t,B)) \leq p(t)\alpha(B).$$

(H6) *For each bounded set* $B \subset E$*, we have*

$$\alpha(I_k(B)) \leq k^*\alpha(B), \quad k = 1, ..., m.$$

(H7) *For each bounded set* $B \subset E$*, we have*

$$\alpha(\overline{I}_k(B)) \leq k_1^*\alpha(B), \quad k = 1, ..., m.$$

Then the IVP (1)-(4) has at least one solution in $C(J,E)$*, provided that*

$$\frac{pm(\log T)^r}{\Gamma(r)} + \frac{p(logT)^r}{r\Gamma(r)} + mk^* + mT\log Tk_1^* < \frac{1}{2}. \tag{15}$$

where

$$p = \sup_{t\in J} p(t).$$

Proof: Transform the problem (1)-(4) into a fixed point problem. Consider the operator

$$Ny(t) = \begin{cases} y_1 + ay_2\log\left(\frac{t}{a}\right) + \frac{1}{\Gamma(r)}\int_a^t \left(\log\frac{t}{s}\right)^{r-1} f(s,y(s))\frac{ds}{s}, & \text{if } t \in [a,t_1] \\ y_1 + ay_2\log\left(\frac{t}{a}\right) + \sum_{k=1}^m \frac{\log\left(\frac{t}{t_k}\right)}{t_k\Gamma(r-1)}\int_{t_{k-1}}^{t_k}\left(\log\frac{t_k}{s}\right)^{r-2} f(s,y(s))\frac{ds}{s} \\ +\frac{1}{\Gamma(r)}\int_{t_k}^t \left(\log\frac{t}{s}\right)^{r-1} f(s,y(s))\frac{ds}{s} + \sum_{k=1}^m I_k(y(t_k^-)) \\ +\sum_{k=1}^m t_k\log\left(\frac{t}{t_k}\right)\overline{I}_k(y(t_k^-)), \ \text{if } t \in (t_k,t_{k+1}], k = 1,...,m, \end{cases} \tag{16}$$

Clearly, from Lemma 3, the fixed points of N are solutions to (1)-(4).
Let $R > 0$ with $|y_1| + |ay_2|\log T < \frac{R}{2}$ and consider the set

$$D_R = \{y \in C(J,E) \, : \|y\|_\infty \leq R\}.$$

We shall show that N satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in several steps.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,E)$. Then, for each $t \in J$,

$$\begin{aligned} \|N(y_n)(t) - N(y)(t)\| &\leq \frac{1}{\Gamma(r-1)}\sum_{k=1}^m \left|\frac{\log\left(\frac{t}{t_k}\right)}{t_k}\right| \int_{t_{k-1}}^{t_k}\left|\left(\log\frac{t_k}{s}\right)\right|^{r-2} \\ &\quad \|f(s,y_n(s)) - f(s,y(s))\|\frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)}\int_{t_k}^t \left|\left(\log\frac{t}{s}\right)\right|^{r-1}\|f(s,y_n(s)) - f(s,y(s))\|\frac{ds}{s} \\ &+ \sum_{k=1}^m \|I_k(y_n(t_k^-)) - I_k(y(t_k^-))\| \\ &+ \sum_{k=1}^m \left|t_k\log\left(\frac{t}{t_k}\right)\right| \|\overline{I}_k(y_n(t_k^-)) - \overline{I}_k(y(t_k^-))\|. \end{aligned}$$

Let $\rho > 0$ be such that

$$\|y_n\|_\infty \leq \rho \text{ and } \|y\|_\infty \leq \rho.$$

By (H2)-(H3) we have

$$\|f(s,y_n(s)) - f(s,y(s))\| \leq 2\rho p(s) := \sigma(s); \ \sigma \in L^1(J,\mathbb{R}_+).$$

Since $f,\ I_k\ \textit{and}\ \overline{I}_k,\ \ k=1,\ldots,m$, are Carathéodory functions, the Lebesgue dominated convergence theorem implies that

$$\|N(y_n)-N(y)\|_\infty \to 0 \text{ as } n\to\infty.$$

Step 2: *N maps D_R into itself.*
For each $y\in D_R$, by (H2),(H3),(H4) and (11), we have for each $t\in J$,

$$\begin{aligned}
\|(Ny)(t)\| \quad\leq\quad & |y_1|+|ay_2|\left|\log\left(\tfrac{t}{a}\right)\right|+\tfrac{1}{\Gamma(r-1)}\textstyle\sum_{k=1}^{m}\left|\frac{\log\left(\frac{t}{t_k}\right)}{t_k}\right| \\
& \textstyle\int_{t_{k-1}}^{t_k}\left|\left(\log\frac{t_k}{s}\right)\right|^{r-2}\|f(s,y(s)\|\frac{ds}{s} \\
& +\tfrac{1}{\Gamma(r)}\textstyle\int_{t_k}^{t}\left|\left(\log\frac{t}{s}\right)\right|^{r-1}\|f(s,y(s))\|\frac{ds}{s} \\
& +\textstyle\sum_{k=1}^{m}\|I_k(y(t_k^-))\| \\
& +\textstyle\sum_{k=1}^{m}\left|t_k\log\left(\frac{t}{t_k}\right)\right|\|\overline{I}_k(y(t_k^-))\| \\
\leq\quad & |y_1|+|ay_2|\log T+\tfrac{m\log T}{\Gamma(r-1)}\textstyle\int_{t_{k-1}}^{t_k}\left|\left(\log\frac{t_k}{s}\right)\right|^{r-2}p\|y\|\frac{ds}{s} \\
& +\tfrac{1}{\Gamma(r)}\textstyle\int_{t_k}^{t}\left|\left(\log\frac{t}{s}\right)\right|^{r-1}p\|y\|\frac{ds}{s}+mk^*\|y\|+mT\log Tk_1^*\|y\| \\
\leq\quad & |y_1|+|ay_2|\log T+\tfrac{pm(\log T)^r}{\Gamma(r)}\|y\|+\tfrac{p(logT)^r}{r\Gamma(r)}\|y\|+mk^*\|y\|+mT\log Tk_1^*\|y\| \\
\leq\quad & R
\end{aligned}$$

Step 3: *$N(D_R)$ is bounded and equicontinuous.*

By Step 2, it is obvious that $N(D_R)\subset C(J,E)$ is bounded. For the equicontinuity of $N(D_R)$, let $\lambda_1,\ \lambda_2\in J,\ \lambda_1<\ \lambda_2$, and $y\in D_R$. We have

$$\begin{aligned}
\|(Ny)(\lambda_2)-(Ny)(\lambda_1)\| \quad=\quad & \left|ay_2\log\left(\tfrac{\lambda_2}{\lambda_1}\right)\right| \\
& +\tfrac{1}{\Gamma(r-1)}\textstyle\sum_{0<t_k<(\lambda_2-\lambda_1)}\left|\frac{\log\left(\frac{\lambda_2}{t_k}\right)}{t^k}\right|\int_{t_{k-1}}^{t_k}\left|\left(\log\frac{t_k}{s}\right)\right|^{r-2}\|f(s,y(s)\|\frac{ds}{s} \\
& +\frac{\left|\frac{\log\left(\frac{\lambda_2}{\lambda_1}\right)}{s}\right|}{\Gamma(r-1)}\textstyle\sum_{0<t_k<\lambda_1}\int_{t_{k-1}}^{t_k}\left|\left(\log\frac{t_k}{s}\right)\right|^{r-2}\|f(s,y(s)\|\frac{ds}{s} \\
& +\tfrac{1}{\Gamma(r)}\textstyle\int_{t_k}^{t}\left|\left(\log\frac{t}{s}\right)\right|^{r-1}\|f(s,y(s))\|\frac{ds}{s} \\
& +\tfrac{1}{\Gamma(r)}\textstyle\int_{t_k}^{\lambda_1}\left[\left(\log\frac{\lambda_2}{s}\right)^{r-1}-\left(\log\frac{\lambda_1}{s}\right)^{r-1}\right]\|f(s,y(s))\|\frac{ds}{s} \\
& +\tfrac{1}{\Gamma(r)}\textstyle\int_{\lambda_1}^{\lambda_2}\left(\log\frac{\lambda_2}{s}\right)^{r-1}\|f(s,y(s))\|\frac{ds}{s} \\
& +\textstyle\sum_{0<t_k<(\lambda_2-\lambda_1)}\|I_k(y(t_k^-))\| \\
& +\textstyle\sum_{0<t_k<(\lambda_2-\lambda_1)}|t_k\log\left(\frac{\lambda_2}{t_k}\right)|\|\overline{I}_k(y(t_k^-))\| \\
& +\log\left(\tfrac{\lambda_2}{\lambda_1}\right)\textstyle\sum_{0<t_k<\lambda_1}|t_k|\|\overline{I}_k(y(t_k^-))\| \\
\leq\quad & \left|ay_2\log\left(\tfrac{\lambda_2}{\lambda_1}\right)\right| \\
& +\tfrac{1}{\Gamma(r-1)}\textstyle\sum_{0<t_k<(\lambda_2-\lambda_1)}\left|\frac{p\log\left(\frac{\lambda_2}{t_k}\right)}{t_k}\right|\int_{t_{k-1}}^{t_k}\left|\left(\log\frac{t_k}{s}\right)\right|^{r-2}\frac{ds}{s} \\
& +\frac{p\left|\frac{\log\left(\frac{\lambda_2}{\lambda_1}\right)}{s}\right|}{\Gamma(r-1)}\textstyle\sum_{0<t_k<\lambda_1}\int_{t_{k-1}}^{t_k}\left|\left(\log\frac{t_k}{s}\right)\right|^{r-2}\frac{ds}{s} \\
& +\tfrac{1}{\Gamma(r)}\textstyle\int_{t_k}^{t}\left|p\left(\log\frac{t}{s}\right)\right|^{r-1}\frac{ds}{s} \\
& +\tfrac{p}{\Gamma(r)}\textstyle\int_{t_k}^{\lambda_1}\left[\left(\log\frac{\lambda_2}{s}\right)^{r-1}-\left(\log\frac{\lambda_1}{s}\right)^{r-1}\right]\frac{ds}{s} \\
& +\tfrac{p}{\Gamma(r)}\textstyle\int_{\lambda_1}^{\lambda_2}\left(\log\frac{\lambda_2}{s}\right)^{r-1}\frac{ds}{s}
\end{aligned}$$

$$+\sum_{0<t_k<(\lambda_2-\lambda_1)}\|I_k(y(t_k^-))\|$$
$$+\sum_{0<t_k<(\lambda_2-\lambda_1)}|t_k\log\left(\frac{\lambda_2}{t_k}\right)|\|\overline{I}_k(y(t_k^-))\|$$
$$+\log\left(\frac{\lambda_2}{\lambda_1}\right)\sum_{0<t_k<\lambda_1}|t_k|\|\overline{I}_k(y(t_k^-))\|$$

As $\lambda_1 \longrightarrow \lambda_2$, the right-hand side of the above inequality tends to zero.

Now let V be a subset of D_R such that $V \subset \overline{co}(N(V)\cup\{0\})$. V is bounded and equicontinuous, and therefore the function $t \to \vartheta(t) := \alpha(V(t))$ is continuous on J. By (H2)-(H4), Lemma 2 , and the properties of the measure α we have for each $t \in J$,

$$\begin{aligned}
\vartheta(t) &\leq \alpha(N(V)(t)\cup\{0\})\\
&\leq \alpha(N(V)(t))\\
&\leq \int_a^t(\frac{p(s)m(\log T)^r\alpha(V(s))ds}{\Gamma(r)}+\frac{p(s)(\log T)^r\alpha(V(s))ds}{r\Gamma(r)}\\
&\quad +mk^*\alpha(V(t))+mT\log Tk_1^*\alpha(V(t)))\\
&\leq \|\vartheta\|_\infty\left[\frac{\|p\|_{L^\infty}m(\log T)^r}{\Gamma(r)}+\frac{\|p\|_{L^\infty}(\log T)^r}{r\Gamma(r)}+mk^*+mT\log Tk_1^*\right].
\end{aligned}$$

This means that

$$\|v\|_\infty\left[1-(\frac{\|p\|_{L^\infty}m(\log T)^r}{\Gamma(r)}+\frac{\|p\|_{L^\infty}(logT)^r}{r\Gamma(r)}+mk^*+mT\log Tk_1^*)\right]\leq 0.$$

By (15) it follows that $\|\vartheta\|_\infty = 0$, that is, $\vartheta = 0$ for each $t \in J$, and so $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D_R. Applying Theorem 1, we conclude that N has a fixed point which is a solution of the problem (1)-(4).

□

4. An Example

Let $E = l^1 = \{(y_1, y_2, ..., y_n, ...)\}, \sum_{i=1}^{+\infty}|y_i| < +\infty$, be our Banach space with the norm

$$\|y\|_E = \sum_{i=1}^{+\infty}|y_i|$$

We apply the main result of the paper Theorem 2 to the following system of fractional differential equations

$$^{CH}D^r y(t) = \frac{2}{9+e^t}|y_n(t)|, \tag{17}$$

for a.e. $t \in J = [a, e],\ a > 0,\ t \neq t_k,\ k = 1, ..., m,\ 1 < r \leq 2$,

$$\Delta y|_{t=\frac{3}{2}} = \frac{1}{3+|y_n(\frac{3}{2}^-)|}, \tag{18}$$

$$\Delta y'|_{t=\frac{3}{2}} = \frac{1}{5+|y_n(\frac{3}{2}^-)|}, \tag{19}$$

$$y(a) = 0,\ y'(a) = 0, \tag{20}$$

where

$$f_n(t,x) = \frac{x_n}{9+e^t}\ \ (t,x)\in J\times E,$$

and

$$I_k(x) = \frac{1}{3+x_n},$$
$$\overline{I}_k(x) = \frac{1}{5+x_n},$$
$$y = (y_1, ..., y_n, ...).$$

Set$f = (f_1, ..., f_n, ...)$, clearly conditions $(H1)$ and conditions $(H2) - (H5)$ hold with

$$p(t) = \frac{1}{9 + e^t},$$

and where the conditions $(H3) - (H5)$ and $(H6) - (H7)$ hold with $k^* = \frac{1}{3}$ and $k_1^* = \frac{1}{5}$. We shall that condition (15) is satisfied for some $r \in (1, 2]$. Then by Theorem 2, the problem (1)-(4) has a solution on $[a, e]$ where (15) is satisfied with $T = e$ and $m = 1$. Indeed

$$\frac{pm(\log T)^r}{\Gamma(r)} + \frac{p(logT)^r}{r\Gamma(r)} + mk^* + mT \log Tk_1^* < \frac{1}{2},$$

if

$$\Gamma(r) > \frac{30(r+1)}{r(9 + e^e)(5 - 6e)},$$

which is satisfied for some $r \in (1, 2]$. Then, by Theorem 2, the problem (17)-(20) has a solution on $[a, e]$ for values of r satisfying $\Gamma(r) > \frac{30(r+1)}{r(9+e^e)(5-6e)}$.

References

[1] Y. Adjabi,F. Jarad,D. Baleanu,and T. Abdeljawad, On Cauchy problems with Caputo Hadamard fractional derivatives, *Jour. Comput. Anal. and Appl.* **21**, (4)(2016), 661-681.

[2] R. P Agarwal, M. Benchohra and S. Hamani, Boundary value problems for fractional differential equations, *Adv. Stud. Contemp. Math.* **16**, (2)(2008), 181-196.

[3] R.P. Agarwal, M. Meehan and D. O'Regan, Fixed Point Theory and Applications,Cambridge University Press, Cambridge, UK, 2001.

[4] B. Ahmed and S.K. Ntouyas, Initial value problems for hybrid Hadamard fractional equations, *Electron. J. Diff. Equ.* **2014** (2014), No. 161, pp. 1-8.

[5] E. Ait Dads, M. Benchohra and S. Hamani, Impulsive fractional differential inclusions involving the Caputo fractional derivative, *Fract. Calc. Appl. Anal.* **12**, (1) (2009), 15-38.

[6] R.R. Akhmerov, M.I. Kamenski, A.S. Patapov, A.E. Rodkina, B.N. Sadovski, Measure of Noncompactness and Condensing Operators. Translated from the 1986 Russian original by A. Iacop. *Operator Theory: Advances and Applications,* **55**, Birhhauser Verlag, Bassel, 1992.

[7] D.D. Bainov, P. S. Simeonov, Systems with Impulsive Effect, Horwood, Chichester, 1989.

[8] J. Banas, K. Goebel, Measure of Noncompactness in Banach spaces, Marcel Dekker, New York.

[9] J. Banas, K. Sadarangani, On some measure of noncompactness in the space of continuous functions, *Nonlinear Anal.* **60**, no 2 (2008), 377-383.

[10] M. Benchohra, J.R. Graef, S. Hamani, Existence results for boundary value problems of nonlinear fractional differential equations with integral conditions, *Appl. Anal.* **87** (7) (2008), 851-863.

[11] M. Benchohra, S. Hamani, Boundary value problems for differential equations with fractional order and nonlocal conditions, *Nonlinear Anal.* **71** (2009), 2391-2396.

[12] M. Benchohra, S. Hamani, Nonlinear boundary value problems for defferential inclusions with Caputo fractional derivative, *Topol. Meth. Nonlinear Anal.* **32**, No 1 (2008), 115-130.

[13] M. Benchohra, S. Hamani, S.K. Ntouyas, Boundary value problems for differential equations with fractional order, *Surv. Math. Appl.* **3** (2008), 1-12.

[14] M. Benchohra, J. Henderson, S.K. Ntouyas, Impulsive Differential Equations and Inclusions, Hindawi Publishing Corporation, New York, 2006.

[15] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, *J. Math. Anal. Appl.* **338** (2008), 1340-1350.

[16] M. Benchohra, B.A. Slimani, Impulsive fractional differential equations, *Electron. J. Diff. Equats.* **2009**, (10) (2009), 11 pp.

[17] P.L. Butzer, A.A. Kilbas, J.J. Trujillo, Composition of Hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl.* **269** (2002), 387-400.

[18] P.L. Butzer, A.A. Kilbas, J.J. Trujillo, Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **269** (2002), 1-27.

[19] P.L. Butzer, A.A. Kilbas, J. J. Trujillo, Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **270** (2002), 1-15.

[20] Y.Y.Gambo, F.Jarad, D.Baleanu, T.Abdeljawad, On Caputo modification of the Hadamard fractional derivatives, *Adv. Diff. Equa.* **2014** Art. ID 10, 12 pages.

[21] D. Guo, V. Lakshmikantham, X. Liu , Nonlinear Integral Equations in Abstract Spaces, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[22] J. Hadamard, Essai sur l etude des fonctions donnees par leur development de Taylor, *J. Math. Pure Appl.* **8** (1892), 101-186.

[23] S. Hamani, A. Hammou, J. Henderson, Impulsive Fractional Differential Equations Involving The Hadamard Fractional Derivative,*Commu in Appl Nonl Anal.* **24** (2017), no 3, 48-58.

[24] A. Hammou, S. Hamani, J. Henderson, Impulsive Hadamard Fractional Differential Equations in Banach Spaces,*Commu in Appl Nonl. Anal.* **28** (2018), no 2, 52-62.

[25] N. Heymans, I. Podlubny, Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives, *Rheologica Acta* **45** (5) (2006), 765-772.

[26] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[27] F. Jarad, T. Abdeljawad, D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives, *Adv. Diff. Equa.* 2012:142, 8 pages, 2012.

[28] A.A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math Soc.* **38**, (6) (2001), 1191-1204.

[29] A.A. Kilbas, Hari M. Srivastava, Juan J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B.V,Amsterdam, 2006.

[30] M. Klimek, Sequential fractional differential equations with Hadamard derivative, *Commu Nonlinear Sci. Numer. Sim.* **16**, (12) (2011), 4689-4697.

[31] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differntial Equations, Worlds Scientific, Singapore, 1989.

[32] V. Lakshmikantham, S. Leela, Nonlinear Differential Equations in Abstract spaces, Pergamon Press, Oxford, UK, 1981.

[33] S.M. Momani, S.B. Hadid, Some comparison results for integro-fractional differential inequalities. *J. Fract. Calc.* **24** (2003), 37-44.

[34] H. Mönch, Boundary value problem for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **75**, No 5 (1980), 985-999.

[35] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[36] I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation, *Fract. Calculus Appl. Anal.* **5** (2002), 367-386.

[37] A.M. Samoilenko, N.A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.

[38] P. Thiramanus, S.K. Ntouyas, J. Tariboon, Existence and uniqueness results for Hadamard-type fractional differential equations with nonlocal fractional integral boundary conditions, *Abstr. Appl. Anal.* (2014), Art. ID 902054, 9 pp.

[39] S. Szufla, On the application of meusure of noncompacteness to existence theorems, *Rend Sem Math Della Univ Padova* **75** (1986), 1-14.

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