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ON PHYSICAL AND MATHEMATICAL WAVE FRONTS IN TEMPERATURE WAVES

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Abstract. A rather "tenuous" existence of mathematical wavefronts in parabolic temperature waves is revealed to accompany a certain hyperbolicity dormant in these waves. The revelation is based on a proof that temperature waves do satisfying a certain new telegrapher's equation, equivalent to Fourier's heat conduction equation. This parabolic-equivalent hyperbolic heat equation happens to be similar to the famous Cattaneo-Vernotte non-Fourier heat conduction equation. A basic result of this work is that temperature waves can mathematically support proper wavefronts of infinite span. Physically, however they can support wavefronts only in "shortened" form. The paper reports also on an associated shrinkage of a triangle for detectable wavefronts of such waves, and on an unknown frequency dependence of the inclination of wavefronts in classical (parabolic) temperature waves. This, added to the strong spatial damping and significant dispersion of these waves, has been forming a pathological obstacle in the experimental verification of their support to conventional wavefronts.

1. Introduction and problem formulation

The term temperature wave in the title implies that this quantity possesses oscillatory, or discontinuous, behavior, with characteristics such as the wavefront (WF) and ray, [1-3], its reflection and refraction at boundaries, and some kind of polarization, [4] when orthogonal to other similar waves. Temperature waves (TWs) have been widely in use, [5-7], during the 20-th century for the determination of thermophysical properties of solids, especially at low temperatures. Afterwards, their additional applications have been found in areas like nondestructive testing, [8], medicare, [9], and tomography, [10].

Like diffusional neutron density waves, which are transverse, [11], TWs experience strong spatial attenuation and significant dispersion (frequency dependence of speed) during propagation, [11-13]. In the next section we shall explain how a 1-D temperature wave can physically have only a shortened wavefront (SWF), which is localized on the $z-t$ plane, and not an infinite line. What is unknown, moreover, and reported as Remark 2 in this work, is the frequency dependence of the positioning slope of the SWF for these waves. These facts, combined, make the experimental detection or verification of physical WFs in them quite a difficult task. This situation has led many physicists to question the proper wave nature of these temperature oscillations, which propagate without energy transfer. Some of them have went further, using rather unsharpened arguments, even to claim the nonexistence of physical WFs in them, [12-14]. Accordingly, it is a purpose of this work to settle all such arguments, by providing a sharp mathematical-physics proof of the existence of WFs in TWs.

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Classical (Fourier) heat conduction, [5-7], in a one dimensional semi-infinite medium, $\Re = [0, \infty)$, is normally characterized by a temperature distribution

$$T(z,t) = T_s + \theta(z,t) \tag{1}$$

around a steady-state, T_s, reference temperature. $T(z,t)$ is a solution to the following singular boundary value problem (BVP), with one additional Neumann boundary condition (BC), (34), encompassing the heat flux $I(z,t)$.

Solve:

$$\frac{1}{\alpha}\nabla_t T - \triangle_z T = 0, \tag{2}$$

subject to:

$$(i) - (ii) : T(z,t) < \infty, \ \forall z, t \in [0, \infty), \tag{3}$$

where α is the thermal diffusivity.

The general solution of the underlying parabolic heat conduction equation (PHE), (13), is well known not to support, [1], oscillatory forms. However, the presence of a periodic heat source, or boundary heat flux,

$$I(0,t) = J(t) = Re\left[\frac{J_0}{2}(1 + e^{i\omega t})\right], \tag{4}$$

with an amplitude of $\frac{J_0}{2}$ and frequency ω of time modulation, can generate travelling-wave particular solutions that had conventionally been named around 1921, [5], as temperature (or heat) waves. Unfortunately the strong spatial damping of these waves, which makes them almost periodic in space (though still periodic in time), together with their significant dispersion, complicates any experimental verification of their proper wave nature, [12], in the form of physical WFs or rays.

Revealing the existence, at least theoretically, of such rather tenuous physical WFs calls for a re-evaluation of the physics foundations for TWs. In actual fact, Fourier's law $\nabla_z T = -\frac{1}{k}I$, satisfies the second law of thermodynamics, [2]. Also when substituted in the first law of thermodynamics, it yields the PHE (2). Of major relevance here is the concept of relaxation time γ between $\nabla_z T$ and I, [15, 16], expressible via

$$I(z, t+\gamma) = -k\nabla_z T(z,t). \tag{5}$$

Taylor series expansion of this leads to the non-Fourier heat conduction Markovian, [15-17], law

$$\gamma\nabla_t I + I = -k\nabla_z T. \tag{6}$$

This γ characterizes the transition to a diffusion mechanism for heat propagation and can refer also to a certain relaxation time, [18-21].

Obviously, the parabolic HE is characterized by $\gamma = 0$ in (6), which leads to an infinite heat disturbance propagation speed $\varsigma = \sqrt{\frac{\alpha}{\gamma}}$. Moreover, BVPs based on it can generate $T(z,t)$ that travel with a phase speed $V_p = \sqrt{2\alpha\omega}$ which can unrelativistically reach ∞ when $\omega \to \infty$. Therefore Fourier's law is incompatible with special relativity. To eliminate this defect, a hyperbolic heat conduction equation (HHE), of the telegrapher's type, with finite heat disturbance propagation speed ς ($\gamma \neq 0$), was proposed by Cattaneo, [15], and Vernotte, [16], in 1958. BVPs similar to (2)-(3), based on the hyperbolic HE of Cattaneo and Vernotte (C-VHHE) turned out to generate a similar TW that travels with a phase speed V_p that tends to ς, i.e. to become independent of modulation as ω increases, [12, 21-23].

As to be demonstrated later, the expression for the hyperbolic TW, which supports physical parabolic WFs, turns out to be qualitatively identical, and quantitatively quite similar, to that for the parabolic TW. This should not be a surprise since in all common materials, at ambient temperatures, γ is quite short (of the order of $10^{-14} - 10^{-10}$ sec), [21-22], i.e. ≈ 0. Such a satisfactory performance of the parabolic HE happens also to hold for almost all heat engineering applications. These and other facts motivate the need for new methods for revealing the tenuously existing physical WFs in parabolic TWs.

At this point, it should be emphasized that for any periodic (or almost periodic) solution, on a $t-z$ plane, to a partial differential equation (PDE), one needs to distinguish between two loosely related WFs. These are namely, the physical and mathematical wavefronts (PWFs and MWFs), defined as follows.

Definition 1. *A physical wavefront is a locus of points, $\mathcal{F}$, on the $z-t$ plane, that stationarize the phase of a periodic (or almost periodic) wave. A corresponding mathematical wavefront, $\aleph$, is the locus of points of singular behavior of this wave, when conceived as a general solution to a partial differential equation. The $\aleph$ is the well-known PDE characteristic, [1-2].*

For an illustration of this definition, consider a wave PDE

$$\frac{1}{c^2} \triangle_t u - \triangle_z u = 0, \tag{7}$$

which is known to have the characteristic

$$\left(\frac{\partial u}{\partial t}\right)^2 - c^2 \left(\frac{\partial u}{\partial z}\right)^2 = 0. \tag{8}$$

This is equivalent to

$$c^2(dt)^2 - (dz)^2 = 0, \tag{9}$$

that yields the mathematical WF:

$$z \pm ct = G \mathrel{\hat=} \aleph. \tag{10}$$

Furthermore, the particular solution $u = cos(z \pm ct)$ to this PDE stationarizes its phase $(z \pm ct)$ via the physical WF:

$$z \pm ct = Q \mathrel{\hat=} \mathcal{F}. \tag{11}$$

Here G and Q are abitrary constants, and $\aleph = \mathcal{F}$. However, this should not be true for all boundary value problems, and the inter-relationship between $\mathcal{F}$ and $\aleph$ is strongly influenced by the hyperbolicity of the PDE, as shall be demonstrated in subsection 3.2.

The paper is organized as follows. A new concept of physically shortend WF is heuristically developed for the TW in Section 2. Section 3 reviews the derivation of the expression for parabolic TWs and identifies their basic unique features, such as their energy non-transfer and reducibility to a universal form. Section 4 studies the associated hyperbolic TW based on the C-V hyperbolic heat equation with an analysis of their associated WFs and their possible transformation as $\gamma \to 0$. The main result of this paper is reported in Section 5, which contains a constructive advance of a telegrapher's equation, equivalent to the parabolic heat equation, and similar (but different) from the non-Fourier C-V hyperbolic heat equation. In this section, we also provide a rigorous proof of the existence of mathematical WFs in parabolic TWs, with a frequency dependence of their inclination. The paper reports here on an associated shrinkage of a triangle for detectable wavefronts of all temperature waves. Then Section 6 reports on a possibility for an experimental verification of the existence of physically shortened WFs in TWs. The paper in concluded in Section 7.

2. Heuristic analysis of physical wavefronts

In the absence of a universal definition for a physical wavefront (PWF) that may cover temperature waves, we shall investigate in what follows this problem and the possibility for its resolution.

Let $\mathbb{R} = (-\infty, \infty)$ be the real line or a 1-D Euclidean space and $\mathbb{R}^+ = [0, \infty)$ to consider $(\mathbf{r}, t)$ as a 3-D position vector $\mathbf{r} = (x, y, z) \in \mathbb{R}^3$ at time $t \in \mathbb{R}^+$. In 1-D, $\mathbf{r} \triangleq z$, where z is a point with a propagation following a trajectory $(z, t) \in$ a curve $\mathcal{L} \subset \mathbb{R} \cup \mathbb{R}^+$. In 2-D, $\mathbf{r} \triangleq$ a curve following a trajectory $(\mathbf{r}, t) \in$ a surface $\subset \mathbb{R}^2 \cup \mathbb{R}^+$; whereas in in 3-D, $\mathbf{r} \triangleq$ a surface with a trajectory $(\mathbf{r}, t) \in$ a solid body $\subset \mathbb{R}^3 \cup \mathbb{R}^+$.

2.1. Nonharmonic undamped wave.

Consider a 1-D progressive wave

$$u(z, t) = f\left(\frac{z}{\mu} - \omega t + \frac{\pi}{4}\right). \tag{12}$$

where $\frac{1}{\mu}$ and ω are respectively spatial and temporal scaling factors, as in (36), with $f(\xi)$ as an arbitrary nonharmonic function.

The physical WF, $\mathcal{F}$, for $u(z, t)$ is defined as the set of (z, t) points satisfying

$$\frac{z}{\mu} - \omega t + \frac{\pi}{4} = C, \ \textit{a constant}. \tag{13}$$

Assumption of $\omega\mu = \upsilon$ in (13) transforms it to

$$z - \upsilon t = \left(C - \frac{\pi}{4}\right)\mu \triangleq \mathcal{L}. \tag{14}$$

The sketch of $u(z, t)$ in Figure 1, over an $\mathbb{R} \cup \mathbb{R}^+$ plane, illustrates that the WF is an infinite straight line $\mathcal{L}$ representing the locus of all (z, t) points of equal phase C, i.e.

$$\frac{z_0}{\mu} + \frac{\pi}{4} = \frac{z_1}{\mu} - \omega t_1 + \frac{\pi}{4} = \frac{z_i}{\mu} - \omega t_i + \frac{\pi}{4} = C, \quad \forall i = 1, 2, 3, ..., \tag{15}$$

where all the corresponding $u(z_i, t_i)$ are identical, for example, crests i.e.

$$u(z_0, 0) = f\left(\frac{z_0}{\mu} + \frac{\pi}{4}\right) = u(z_1, t_1) = f\left(\frac{z_1}{\mu} - \omega t_1 + \frac{\pi}{4}\right) = \cdots$$

$$\cdots = u(z_i, t_i) = f\left(\frac{z_i}{\mu} - \omega t_i + \frac{\pi}{4}\right) = f(C), \quad \forall i. \tag{16}$$

Let ψ be the angle between $\mathcal{L}$ and the t-axis. Since tan $\psi = \upsilon$, then $\upsilon = tan^{-1}\psi$ determines the slope of $\mathcal{L}$ with the t-axis. Moreover $t_0 = -\left(C - \frac{\pi}{4}\right)\frac{1}{\omega}$ and $z_0 = \left(C - \frac{\pi}{4}\right)\mu$ are the t and z intercepts of $\mathcal{L}$. Therefore, a 1-D WF represents a point z_0 on the z-axis that moves in time, along this axis, according to the trajectory $\mathcal{L}$.

2.2. **Temperature wave.** Consider now a planar temperature wave, [11-12] (see (36))

$$\theta(z,t) = Qe^{-\frac{z}{\mu}} cos\left(\frac{z}{\mu} - \omega t + \frac{\pi}{4}\right), \tag{17}$$

in which Q is the amplitude and $\mu = \mu(\omega,\alpha) = \sqrt{\frac{2\alpha}{\omega}}$. Its temporal frequency and period are $\omega = \frac{2\pi}{T}$ and $T = \frac{2\pi}{\omega}$, respectively $u(z,t)$ also has a spatial frequency $\frac{1}{\mu}$ with a spatial period $d = 2\pi\mu = 2\pi\sqrt{\frac{2\alpha}{\omega}}$. The T and d are minimal time and space intervals for which the following constraints

$$\theta(z,t) - \theta(z,t+T) = 0, \tag{18}$$

$$\theta(z,t) - \theta(z+d,t) = 0, \tag{19}$$

are satisfied, $\forall\ t, z$.

The WF for this $\theta(z,t)$ wave can also be defined by means of (13). Taking into consideration then that

$$\omega\mu = \upsilon = \sqrt{2\alpha\omega} \quad and \quad T = \frac{d^2}{4\pi\alpha}\ , \tag{20}$$

in (13) leads to

$$z - \upsilon t = \left(\frac{C}{2\pi} - \frac{1}{8}\right)\upsilon T = \frac{1}{4\pi\alpha}\left(\frac{C}{2\pi} - \frac{1}{8}\right)\upsilon d^2 \triangleq \mathcal{L}. \tag{21}$$

As in the case of nonharmonic $f(\xi)$, here also $\upsilon = tan^{-1}\psi$ determines the slope of $\mathcal{L}$ with the t-axis. Distinctively, however, the t and z intercepts of $\mathcal{L}$ are determined by both C and d^2,

$$t_0 = -\frac{1}{4\pi\alpha}\left(\frac{C}{2\pi} - \frac{1}{8}\right)d^2 \quad and \quad z_0 = \frac{1}{4\pi\alpha}\left(\frac{C}{2\pi} - \frac{1}{8}\right)\upsilon d^2. \tag{22}$$

The arguments of the exponential and cosinusoidal functions in (17) are obviously different. Accordingly, the concept of (z,t) points of equal phase is rather meaningless or inapplicable, when conceived globally over the entire domain for z (or effectively C) and t. This justifies questioning the global existence of a PWF $\mathcal{F}$ for such a wave, [23], but leaves the door open for its existence locally over some subdomain of the $z - t$ plane (Figure 1).

In this regard, we may heuristically advance the following localized analysis of the problem of points of equal phase for (17). The physical WF $\mathcal{F}$ can approximately similarly be defined via (13) where

$$\frac{z}{\mu} = C + \omega t - \frac{\pi}{4}\ , \tag{23}$$

with υ and d^2 playing the same role, as in (21), for $\mathcal{L}$, but in an unusual form for which

$$\theta(z,t) = Q\ e^{-\left(C+\omega t-\frac{\pi}{4}\right)} cos\ C\ , \tag{24}$$

is a nonconstant, though Figure 1 illustrates also $\theta(z,t)$ of (17) as a wave form oscillating in space and propagating in time.

This $\theta(z,t)$ can also be sketched, alternatively, as oscillating in time and propagating in space. The infinite line $\mathcal{L}$, representing the locus of all (z,t) points of (15) of equal phase C is also exhibited in this figure, where

$$\theta(z_0,0) = Q\ e^{-\left(C-\frac{\pi}{4}\right)} cos\ C \neq \theta(z_1,t_1) = Q\ e^{-\left(C+\omega t_1-\frac{\pi}{4}\right)} cos\ C \neq \cdots$$
$$\cdots \neq \theta(z_i,t_i) = Q\ e^{-\left(C+\omega t_i-\frac{\pi}{4}\right)} cos\ C, \quad \forall i. \tag{25}$$

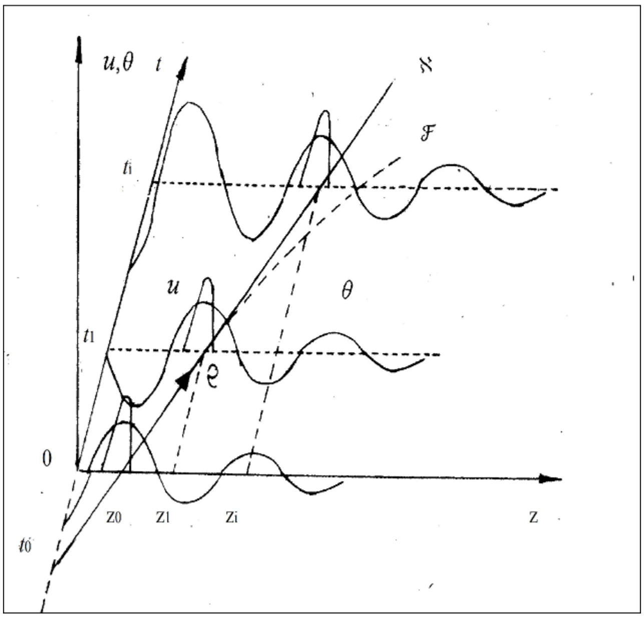

FIGURE 1. Sketch to illustrate the physical wafefront $\mathcal{L}$ for a nonharmonic undamped 1-D wave $u(z,t)$, the physical wavefront $\mathcal{F}$ for a temperature wave $\theta(z,t)$ and its associated shortened wavefront ϱ

This means that $\mathcal{F} = \mathcal{L}$ is not a locus of (z,t) points over which $\theta(z,t)$ is a constant. Hence, generally speaking, this $\mathcal{L}$ does not represent a physical WF for the rather pathological temperature wave. In a localized sense, however, it is possible to investigate the following three distinct subdomains of the $t - C$ plane.

i) If

$$\omega t_i \quad << \left(C - \frac{\pi}{4}\right), \ \forall i, \tag{26}$$

then

$$\theta(z_0, 0) \approx \theta(z_1, t_1) \approx \ \cdots \ \approx \theta(z_i, t_i) \approx Q \ e^{-\left(C - \frac{\pi}{4}\right)} cos \ C, \ \forall i, \tag{27}$$

to indicate that $\mathcal{F} \approx \varrho$ approximately exists as a locus ϱ of (z,t) points of the same $Q \ e^{-\left(C-\frac{\pi}{4}\right)} cos \ C$ temperature.

The rather uncertain size of ϱ can obviously be increased by lowering ω and/or t or by increasing C.

ii) For

$$\left(C - \frac{\pi}{4}\right) << \omega t_i \ , \ \forall i \geq k, \tag{28}$$

over some time interval $(t_k, \infty) \subset \mathbb{R}^+$. Here as t_i becomes $> t_k$,

$$\theta(z_k, t_k) \approx \theta(z_{k+1}, t_{k+1}) \ = \ \cdots \ = \theta(z_\infty, t_\infty) = 0. \tag{29}$$

This relation means that $\mathcal{F} \approx \varrho_*$ exists locally over a time interval $(t_k, \infty), \forall C$, as a locus ϱ_* of (z,t) points of exclusively null temperatures.

iii) In the intermediate subdomain

$$\left(C - \frac{\pi}{4}\right) << \omega t_i << \omega t_k \ , \tag{30}$$

the situation is seemingly more complex than for the subdomains of (i) or (ii). Here one can conjecture that υ plays the same role for $\mathcal{L}$ as in the case of an undamped harmonic wave $Qcos\xi$. Moreover, consistency with the reality of almost periodicity of the TW suggests, however, that the role played by d^2 in defining $\mathcal{L}$ should be played by an intrinsic almost period d_ε^2, in the sense of Besicovitch, [1,3], defined via satisfaction of

$$\|\theta(z,t) - \theta(z + d_\varepsilon, t)\| \leq \varepsilon \ , \tag{31}$$

which is a weak form of the (19) constraint.

The difference between d_ε and d should invoke a smooth change in the t and z intercepts of $\mathcal{L}$. This change can cause a bending of $\mathcal{L}$ with a gradual decay in temperature $\theta(z,t)$ towards a "WF" exclusively through the null values of $\theta(z,t)$. Such a remarkable bending, towards the z-axis when $d_\varepsilon > d$, appears to be essential for preserving continuity of $\mathcal{F}$ when connecting the two subdomains (i) and (ii) that have distinct temperatures. Furthermore, the approximate nature of (31), the possibility of existence of more than one d_ε for it, and its dependence on the magnitude of the rather uncertain $\varepsilon \in (0,1)$ can be a reason for blurring of the bending $\mathcal{F}$. Accordingly, the physical WF, $\mathcal{F} \approx \mathcal{L}$, for the pathological TW behaves, over the previous three subdomains of the $t - C$ plane, more like a finite "bird feather" than an infinite straight line $\mathcal{L}$, as sketched in Figure 1.

Definition 2. *The physical wavefront $\mathcal{F}$ of 1-D temperature wave exists in the form of a finite line segment $\varrho \subset \mathcal{L}$, only when $\omega t << \left(C - \frac{\pi}{4}\right)$. This $\varrho \subset \mathcal{F}$, which resembles a calamous in a finite "bird feather", is to be called a* ***shortened wavefront*** *(SWF).*

3. PARABOLIC TEMPERATURE WAVES

Since the wave-like nature of $T(z,t)$ is solely represented by its $\theta(z,t)$ component, then this can be termed as a TW, resulting from the solution of an associated with (1)-(4) similarly singular BVP:

$$\frac{1}{\alpha}\nabla_t\theta - \triangle_z\theta = 0, \tag{32}$$

subject to :

$$(i) - (ii) : \theta(z,t) < \infty, \ \forall z, t \in [0, \infty), \tag{33}$$

$$(iii) : -k\nabla_z\theta|_{z-0} = I(0,t) = g(t) = J(t) - \frac{J_0}{2} = Re\left[\frac{J_0}{2}e^{i\omega t}\right]. \tag{34}$$

Here k is the thermal conductivity of $\Re$.

A separated variables solution, $\theta(z,t) = X(z)e^{i\omega t}$, to (32), subjected to satisfaction of (33)-(34), when

$$\mu = \mu(\omega;\alpha) = \sqrt{\frac{2\alpha}{\omega}} \ and \ \varepsilon = \frac{k}{\sqrt{\alpha}}, \tag{35}$$

leads, see e.g. [7], [11-12], to the conventional parabolic TW,

$$\theta(z,t) = \frac{J_0}{2\varepsilon\sqrt{\omega}} \ e^{-\frac{z}{\mu}}cos\left(\frac{z}{\mu} - \omega t + \frac{\pi}{4}\right). \tag{36}$$

It should be noted here that in some specific applications, [7], only normalized parabolic TWs

$$\vartheta(z,t) = \frac{T(z,t) - T_s}{J_0/2\varepsilon\sqrt{\omega}} = e^{-\frac{z}{\mu}}cos\left(\frac{z}{\mu} - \omega t + \frac{\pi}{4}\right), \tag{37}$$

with a unitary amplitude, are of practical interest.

As expected, nonetheless, both $\theta(z,t)$ and $\vartheta(z,t)$ travel with the same phase speed

$$V_p = \omega\mu = \sqrt{2\alpha\omega}, \tag{38}$$

or the same group speed $V_g = \left\{\frac{\partial}{\partial\omega}\left(\frac{1}{\mu}\right)\right\}^{-1} = 2V_p = 2\sqrt{2\alpha\omega}$ which may tend to ∞, when $\omega \to \infty$, in a way contradicting special relativity. Also both $\theta(z,t)$ and $\vartheta(z,t)$ are periodic in t, but only almost-periodic in z.

3.1. **Fuzziness of wavefronts.** A basic feature of any proper wave motion in space-time is its associated physical WF. In one spatial dimension, a physical parabolic WF is a curve, on the $z-t$ space, of constant phase. Although physical WFs depend on the geometry of the wave source, they are invariably spaced by T, and propagate at the same V_p speed of the wave. Their nature is defined by the characteristics of the related PDE. In this respect, the parabolic HE has the characteristic equation (CE), see e.g. [2]: $\alpha\left(\frac{\partial\theta}{\partial t}\right)^2 = 0$, i.e.

$$(dt)^2 = 0, \tag{39}$$

with the unique *characteristic*

$$\aleph \triangleq t = C\ (constant),\ \forall z. \tag{40}$$

The independence of z in (40) implies that the general solution of the parabolic HE cannot support physical WFs. However, the parabolic TW, of (16), is a solution to a special BVP, employing the parabolic HE, and not a general solution for it. Incidentally, the physical WF is defined here by (16) as $\frac{z}{\mu} - \omega t + \frac{\pi}{4} = C$, which is equivalent to

$$\mathcal{F} \approx \varrho \triangleq z - V_p t = C, \tag{41}$$

a clear indication that $\mathcal{F} \neq \aleph$ for a parabolic temperature wave.

On another note, for any oscillatory function of two variables to be a proper wave (i.e. supporting physical WFs), it is sufficient (but not necessary) that the function is periodic in both variables. The sufficiency-only nature of this assertion is due to its applicability also to the, similar to (36), hyperbolic TW, of (52), which is better qualified to support physical WFs. In actual fact TWs of all kinds are spatially damped and therefore periodic only in one variable and almost-periodic in the other.

3.2. **Unique features of the parabolic temperature wave.** The wave-like solution $\theta(z,t)$ of the BVP (32)-(34) has additionally the following three unique features, that deserve downlisting.

a) The $\frac{\pi}{4}$ phase in (36) for $\theta(z,t)$ indicates that it is a spatially damped superposition of two space-time oscillations, namely $cos\left(\frac{z}{\mu} - \omega t\right)$ and $\left[-\ sin\left(\frac{z}{\mu} - \omega t\right)\right]$.

b) The analytical form of (36) suggests two generic mechanisms for frequency "multi-resolution" of parabolic TWs. The first mechanism is one of spatial scaling $\sim \sqrt{\omega}$,

$$\varkappa = \varkappa(z;\omega,\alpha) = \frac{z}{\mu} = \sqrt{\frac{\omega}{2\alpha}}z. \tag{42}$$

The second is of temporal dilation $\sim \frac{1}{\omega}$,

$$\nu = \nu(z;\omega) = t - \frac{\pi}{4\omega}. \tag{43}$$

As in wavelet analysis, [24], such a reversed multi-resolution in the $z-t$ space, brings (16) back to a universal parabolic TW (UPTW)

$$\theta(\varkappa,\nu) = \frac{J_0}{2\varepsilon\sqrt{\omega}}e^{-\varkappa}cos\left(\varkappa - \omega\nu\right), \tag{44}$$

that travels with a unit V_p in the $\varkappa - \nu$ space, with a $\frac{1}{\sqrt{\omega}}$-dependent amplitude. A fact to be exploited, in Section 5, for introducing (in (83)) a new hyperbolic HE, equivalent to the parabolic HE.

c) It is worthwhile to mention that the heat flux associated with $\theta(z,t)$ of (36) happens to satisfy

$$I(z,t) = -k \; \nabla_z \; \theta = \frac{J_0}{2} e^{-\frac{z}{\mu}} cos\left(\omega t - \frac{z}{\mu}\right). \tag{45}$$

Moreover, the possibility for energy, E_ω, temporal transfer by such a wave over a period P, at any point $z = z_o$, is a crucial property for it. In this regard, it is remarkable that for the parabolic TW,

$$E_\omega \sim \|I\|_{L^1} = \frac{1}{2} e^{-\frac{z_0}{\mu}} \int_b^{b+P} cos(\omega t - \frac{z_0}{\mu}) dt = 0, \quad \forall b \in [0, \infty). \tag{46}$$

This means that parabolic TWs *do not transfer energy* while propagating in time. This fact happens, luckily, not to contradict special relativity, even when V_p does.

Overlooking any of these features can naturally lead to, not uncommon, misconceptions of the nature of TWs. It should be underlined, moreover, that both the hyperbolic heat flux, $I(z,t)$ of (56), and parabolic $I(z,t)$ of (45) are asymptotically vanishing quantities. Hence any contemplated hyperbolicity of $I(z,t)$ appears to be irrelevant to revealing the wavefronts of TWs.

4. HYPERBOLIC TEMPERATURE WAVES

The BVP generating the hyperbolic TW employs the Cattaneo-Vernotte, [14-17], hyperbolic heat conduction equation (C-VHHE):

$$\frac{1}{\varsigma^2} \triangle_t \theta + \frac{1}{\alpha} \nabla_t \theta = \triangle_z \theta, \tag{47}$$

subject to :

$$(i) - (ii) : \theta(z,t) < \infty, \; \forall z, t \in [0, \infty), \tag{48}$$

$$(iii) : (\gamma \nabla_t I + I)_{z=0} = -k \nabla_z \theta \,|_{z=0} \;, \; non-Fourier \; law, \tag{49}$$

with $I(0,t) = g(t)$ of (34).

Compared with the parabolic HE, (47) is a telegrapher's PDE, which contains an additional second-order term $\frac{1}{\varsigma^2} \triangle_t \theta$, with "viscous damping" coefficient $\frac{\varsigma^2}{\alpha} = \frac{1}{\gamma}$ and zero "restoration" coefficient. All coefficients of (47) are independent of ω, but critically depend on γ and α. Clearly, in the limit of $\gamma \to 0$, $\varsigma \to \infty$, and the entire BVP (47)-(49) reverts back to the parabolic BVP (32)-(34).

$\theta(z,t)$ is obtained here, as in [12], also via separation of variables: $\theta(z,t) = Y(z) \; e^{i\omega t}$. In application to (47), this leads to the Helmholtz equation

$$Y'' - æ^2 Y = 0,$$

with

$$æ = \sqrt{\frac{\gamma}{\alpha}} \; \omega \; \sqrt{\frac{1}{\gamma\omega} i - 1} \; = \; \pm \; (\mathcal{A} + i\mathcal{B}), \tag{50}$$

where

$$\binom{\mathcal{A}}{\mathcal{B}} = \frac{1}{\mu} \sqrt{\left(\sqrt{\gamma^2\omega^2 + 1} \; \mp \gamma\omega\right)}. \tag{51}$$

$\theta(z,t)$ is then subjected to satisfaction of (48) to yield the hyperbolic TW

$$\theta(z,t) = \left(J_0 \,/ 2k(\mathcal{A}^2 + \mathcal{B}^2)\right) e^{-\mathcal{A}z} \left[(\gamma\omega\mathcal{B} + \mathcal{A}) cos(\mathcal{B}z - \omega t) \; + (\gamma\omega\mathcal{A} - \mathcal{B}) sin\,(\mathcal{B}z - \omega t)\right]. \tag{52}$$

For low ω, when $\gamma\omega << 1$, it may be shown that

$$(\gamma\omega\mathcal{B}+\mathcal{A}) = -(\gamma\omega\mathcal{A}-\mathcal{B}) \approx \frac{1}{\mu}, \tag{53}$$

and (52) reduces to the form

$$\theta(z,t) = \frac{J_0}{2\varepsilon\sqrt{\omega}}\frac{1}{\sqrt{\gamma^2\omega^2+1}}e^{-\mathcal{A}z}cos\left(\mathcal{B}z-\omega t+\frac{\pi}{4}\right), \tag{54}$$

which tends to the parabolic TW (36) when $\gamma \to 0$.

Moreover, at high frequencies, when $\gamma\omega >> 1$, it may be shown that

$$\mathcal{A} = \frac{1}{2\gamma\varsigma} = \frac{1}{2\sqrt{\alpha\gamma}} \approx 0,$$

$$\mathcal{B} = \frac{\omega}{\varsigma},$$

and relation (52) becomes

$$\theta(z,t) = \frac{J_0\sqrt{\gamma}}{2\varepsilon}e^{-\frac{1}{2\sqrt{\alpha\gamma}}z}cos\left(\frac{\omega}{\varsigma}z-\omega t\right). \tag{55}$$

Here the HTW and PTW clearly differ, as the hyperbolic TW will propagate nearly undamped at the finite speed $\varsigma = \sqrt{\frac{\alpha}{\gamma}}$.

4.1. **The heat flux.** According to the non-Fourier law (49), the heat flux $I(z,t)$ should satisfy, when $\gamma\omega << 1$, for which $\mathcal{A} = \mathcal{B} \approx \frac{1}{\mu}$, the initial value problem (IVP):

$$\gamma\nabla_t I + I = \frac{J_0}{2}\frac{1}{\sqrt{\gamma^2\omega^2+1}}e^{-\mathcal{A}z}\cos\left(\omega t-\mathcal{B}z\right).$$

Subject to:

$$I(0,0) = g(0) = \frac{J_0}{2}.$$

Straightforwardly, the solution to this IVP is

$$I(z,t) = \frac{J_0}{2\gamma}\frac{1}{\sqrt{\gamma^2\omega^2+1}}e^{-\frac{t}{\gamma}}\int_0^t e^{-(\mathcal{A}z-\frac{\tau}{\gamma})}\cos\left(\omega\tau-\mathcal{B}z\right)d\tau + \frac{J_0}{2}e^{-\frac{t}{\gamma}}, \tag{56}$$

which yields $I(0,t) = g(t) = Re\left[\frac{J_0}{2}e^{i\omega t}\right]$.

The temporal energy transfer, E_ω, by the associated $\theta(z,t)$, over a period P, at $z=0$, for example, is

$$E_\omega \sim \|I\|_{L^1} = \int_b^{b+P} I(0,t)dt = 0, \quad \forall b \in [0,\infty). \tag{57}$$

Hence, like the parabolic TW, the hyperbolic TW also *does not transfer energy*, when $\gamma\omega << 1$, while propagating in time. It can be shown, moreover, that the same energy nontransfer happens to hold when $\gamma\omega >> 1$.

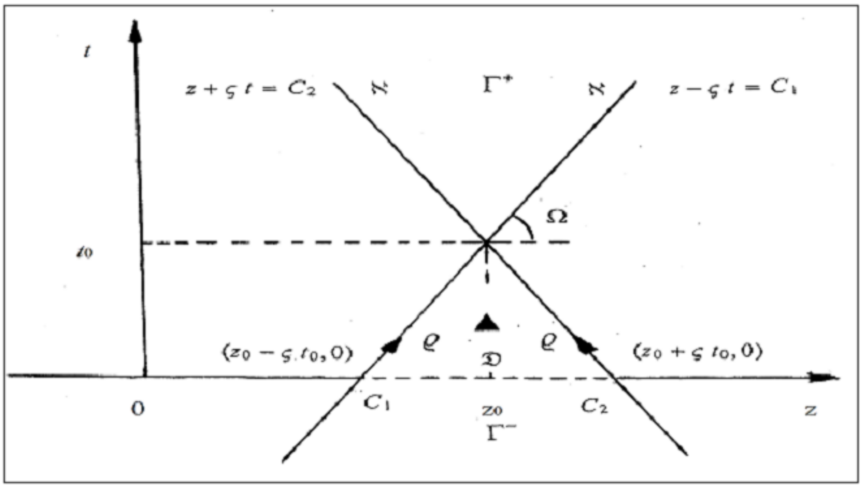

FIGURE 2. Sketch to illustrate wafefronts and the characteristic triangle for temperature waves

4.2. **Wavefronts.** As in (7)-(8) the Cattaneo-Vernotte hyperbolic heat equation C-VHHE (32) has a characteristic equation

$$\left(\frac{\partial\theta}{\partial t}\right)^2 - \varsigma^2\left(\frac{\partial\theta}{\partial z}\right)^2 = 0,$$

which is equivalent to

$$\varsigma^2(dt)^2 - (dz)^2 = 0. \tag{58}$$

Obviously, the solution characteristics, of the Cattaneo-Vernotte hyperbolic heat equation C-VHHE, are

$$\left.\begin{aligned} z - \varsigma t = C_1 \\ z + \varsigma t = C_2 \end{aligned}\right\} \triangleq \aleph. \tag{59}$$

These represent two physical WFs, [1-2], inclined by Ω, where $\tan\Omega = \pm\frac{1}{\varsigma} = \pm\sqrt{\frac{\gamma}{\alpha}}$, freely of ω, as illustrated in Figure 2.

Incidentally, the mathematical WF is defined , when $\gamma\omega >> 1$, by (40) as $\frac{\omega}{\varsigma}z - \omega t \ = C$, which is equivalent to

$$\aleph \triangleq \quad z - \varsigma t = C, \tag{60}$$

a clear indication that $\varrho = \aleph$ for a hyperbolic TW.

Moreover, the solution characteristic passing through any point (z_o, t_o) in an initial-value problem (IVP) (Cauchy problem, [2]): Solve a PDE (47), subject to $\theta(z, 0) = \theta_0(z)$ and $\nabla_t\theta\,|_{t=0} \ = \theta_1(z))$ is

$$\varsigma^2(t - t_o)^2 - (z - z_o)^2 = 0,$$

where the set $\{\theta_0(z), \theta_1(z)\}$ represents the data of the Cauchy problem.

This relation defines the Cauchy problem solvability boundaries

$$\left.\begin{aligned} \Gamma^+ &\triangleq \ \varsigma(t - t_o) > |z - z_o|\,, \quad \text{future cone} \\ \Gamma^- &\triangleq -\ \varsigma(t - t_o) > |z - z_o|\,, \quad \text{past cone} \end{aligned}\right\}, \tag{61}$$

with the $\pm\frac{1}{\varsigma}$ positioning slopes which define the characteristic triangle $\blacktriangle \subset \Re$ for the Cattaneo-Vernotte hyperbolic heat equation C-VHHE. The base segment of $\blacktriangle$:

$$\mathfrak{D} = \{z \ \mid (z_o - \varsigma t_o) \leq z \leq (z_o + \varsigma t_o)\} \tag{62}$$

which is cut out of the initial $t = 0$ line by the cone Γ^- issuing from (z_o, t_o), is called the *domain of dependence* of the solution of the Cauchy problem, i.e. the domain for its data set. This solution is uniquely determined in the whole interior $\Re$ of $\blacktriangle$ by the Cauchy data on $\mathfrak{D}$. Furthermore, relations (61) guarantee causality for the pertaining TW, since any disturbance at (z_o, t_o) can affect only those points z at future time t inside Γ^+.

Remark 1. *A corner stone of the present analysis is that under the limiting* $\gamma = 0$ *condition (leading to the PTW)* $\varsigma \to \infty$ *and* $\Omega = \pm \ \tan^{-1}\frac{1}{\varsigma} \to 0$. *Consequently, the domain* $\mathfrak{D} \to \varnothing$ *(null set), and* $\blacktriangle$ *converts rotationally "degenerates" to a horizontal straight line,*

$$t = 0, \quad \forall z, \tag{63}$$

which remarkably agrees with the non-WF characteristic (40), weakly related to the parabolic TW.

4.3. **Cauchy problem for the Cattaneo-Vernotte hyperbolic heat equation C-VHHE.** Consider now the map

$$\phi(z,t) = e^{\frac{\varsigma^2}{2\alpha}t}\theta(z,t), \tag{64}$$

to transform the posing Cattaneo-Vernotte hyperbolic heat equation C-VHHE Cauchy problem to the equivalent IVP:

$$\triangle_z\phi - \frac{1}{\varsigma^2}\triangle_t\phi + \frac{\varsigma^2}{4\alpha^2}\phi = 0, \tag{65}$$

$$\phi_0(z) = \phi(z,0) = \theta_0(z), \tag{66}$$

$$\phi_1(z) = \nabla_t\phi(z,0) = \frac{\varsigma^2}{2\alpha}\theta_0(z) + \theta_1(z). \tag{67}$$

Without loss of generality, for the special case of (37), when $\gamma\omega << 1$, and $\mathcal{A} \approx \mathcal{B} \approx \frac{1}{\mu}$, relation (39) takes the form

$$\theta(z,t) = Q\ e^{-\mathcal{A}z} cos\left(\mathcal{B}z - \omega t + \frac{\pi}{4}\right), \tag{68}$$

in which

$$Q = \frac{J_0}{2\varepsilon\sqrt{\omega}} \cdot \frac{1}{\sqrt{\gamma^2\omega^2 + 1}}. \tag{69}$$

Accordingly,

$$\theta_0(z) = Q\ e^{-\mathcal{A}z} cos\left(\mathcal{B}z + \frac{\pi}{4}\right),$$
$$\theta_1(z) = \omega Q\ e^{-\mathcal{A}z}\sin\left(\mathcal{B}z + \frac{\pi}{4}\right). \tag{70}$$

The hyperbolic TW (55) solution to the BVP (47)-(49) for the C-VHHE can also be generated as a solution to an equivalent Cauchy problem, with explicit dependence on the WFs, according to a result that follows.

Theorem 1. *Unlike the shortened physical wave front* $\varrho \subset \mathcal{F}$, *the mathematical wavefront* $\aleph$ *of the hyperbolic temperature wave is everywhere active.*

Proof. Since the vertex (x_0, t_0) of the characteristic triangle ▲ is arbitrary, so is its surface $\Re$. Let us integrate then both sides of (65) over $\Re$ to write

$$\iint_{\Re} \left(\triangle_t \phi - \varsigma^2 \triangle_z \phi\right) d\Re = \frac{\varsigma^4}{2\alpha^2} \iint_{\Re} \phi d\Re, \tag{71}$$

where $d\Re = dx\ dt = d\xi\ d\tau$.

Application of Green's theorem, [1], and the Duhamel principle, [25], to both sides of (71) can be shown to yield for $\phi(z,t)$ a rather novel decomposition,

$$\phi(z,t) = U(z \pm \varsigma t) + V\ [\phi(z,t)], \tag{72}$$

in which

$$\begin{aligned} U(z \pm \varsigma t) &= \frac{1}{2}\Big[\phi_0(z - \varsigma t) + \phi_0(z + \varsigma t)\Big] + \frac{1}{2\varsigma}\int_{z-\varsigma t}^{z+\varsigma t} \phi_1(s) ds \\ &= \frac{1}{2}\Big[\theta_0(z - \varsigma t) + \theta_0(z + \varsigma t)\Big] + \frac{1}{2\varsigma}\int_{z-\varsigma t}^{z+\varsigma t} \left[\frac{\varsigma^2}{2\alpha}\theta_0(s) + \theta_1(s)\right] ds, \end{aligned} \tag{73}$$

is the familiar d'Alembert solution, and the integral

$$V[\phi(z,t)] = \frac{\varsigma^3}{8\alpha^2}\int_0^t \int_{z-\varsigma(t-\tau)}^{z+\varsigma(t-\tau)} \phi(\xi,\tau) d\xi d\tau. \tag{74}$$

The claimed novelty of (72)-(74) consists in revealing that the effect of the ingredients $z \mp \varsigma t$ of $\aleph$ is most pronounced in the first term of $U(z \pm \varsigma t)$, while it is smoothed by single integration in its second term. Moreover, this effect is apparently least pronounced in $V[\phi(z,t)]$, due to its enhanced smoothing by iterated double integration. Substitute then (64) in (72)-(74) to obtain the nonhomogeneous second-kind linear Volterra integral equation

$$\theta(z,t) = e^{-\frac{\varsigma^2}{2\alpha}t}U(z \pm \varsigma t) + \frac{\varsigma^3}{8\alpha^2}e^{-\frac{\varsigma^2}{2\alpha}t}\int_0^t e^{\frac{\varsigma^2}{2\alpha}\tau}\int_{z-\varsigma(t-\tau)}^{z+\varsigma(t-\tau)} \theta(\xi,\tau) d\xi d\tau, \tag{75}$$

which is equivalent to the BVP (47)-(49).

Consideration of relations (69)-(70) for $\theta_0(z)$ and $\theta_1(z)$ illustrates that the negative exponential factor in $e^{-\frac{\varsigma^2}{2\alpha}t}U(z \pm \varsigma t)$ is compensated, for all t, by a reflexive positive exponential factor inside $U(z \pm \varsigma t)$. Furthermore, being a well-posed, [26], integral equation, (75) is solvable iteratively

$$^{<n>}\theta(z,t) = e^{-\frac{\varsigma^2}{2\alpha}t}U(z \pm \varsigma t) + \frac{\varsigma^3}{8\alpha^2}e^{-\frac{\varsigma^2}{2\alpha}t}\int_0^t e^{\frac{\varsigma^2}{2\alpha}\tau}\int_{z-\varsigma(t-\tau)}^{z+\varsigma(t-\tau)} {}^{<n-1>}\theta(\xi,\tau) d\xi d\tau, \tag{76}$$

with $n = 1, 2, 3, ...$ and for any $^{<0>}\theta(z,t) \neq 0$. Also convergence of this iterative process, i.e. $\lim\limits_{n\to\infty} {}^{<n>}\theta(z,t) \to \theta(z,t)$ of (68), should not be questionable.

Clearly, (47) and (65) happen to have the same characteristics. However, unlike the previous situation with $\aleph$, the ingredients $z \mp \varsigma t$ of $\mathcal{F}$ are inactive on $\theta(z,t)$ when $(C - \frac{\pi}{4}) << \omega t_i$, $\forall i \geq k$, i.e. over the subdomain (iii) of the $z - t$ plane, defined by (29). Here the proof completes. □

4.4. **Shrinkage of the triangle for detectable wavefronts.** Let us revisit the $\mathfrak{D}$ domain of dependence, (37), of the solution to the Cauchy problem of Theorem 1. For any $z_0 \in \mathfrak{D}$, there exists a pair of parameters t_ϱ and $\mathfrak{D}_\varrho$ that are associated with the length l_ϱ of the *calamous* ϱ of $\mathcal{F}$. These are namely,

$$t_\varrho = l_\varrho / \sqrt{1 + \varsigma^2} \tag{77}$$

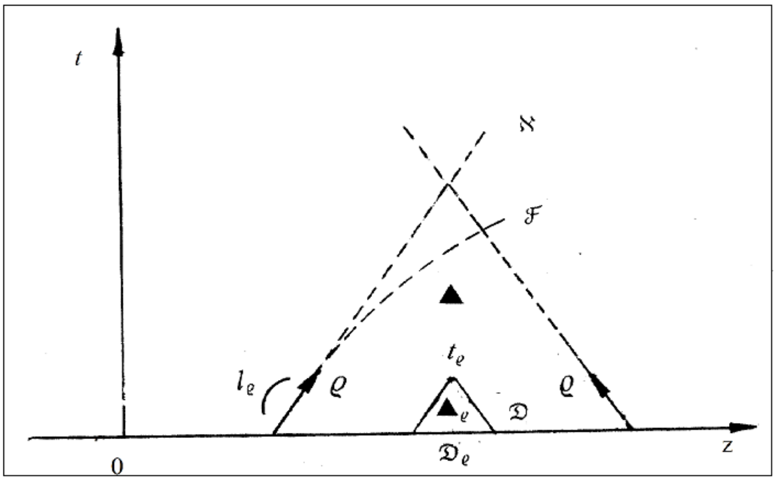

FIGURE 3. Sketch to illustrate the shrinked triangle for detectable temperature wavefronts

and

$$\mathfrak{D}_\varrho = \{z \ \mid (z_o - \varsigma t_\varrho) \le z \le (z_o + \varsigma t_\varrho)\}. \tag{78}$$

Figure 3 illustrates how these parameters happen to define a shrinked triangle $\blacktriangle_\varrho$ with a surface $\Re_\varrho \subset \Re$, and a vertex (z_0, t_ϱ) opposite to $\mathfrak{D}_\varrho$.

Proposition 1. *For any $z_0 \in \mathfrak{D}$, the physical wavefront $\mathcal{F}$ is detectable at any $t_0 \in [0, t_\varrho]$. Outside this interval, i.e. at $t_0 > t_\varrho$, $\mathcal{F}$ is either blurred or undetectable.*

Proof. By assuming the validity of arguments for dividing the $t-z$ plane into the three subdomains of (26)-(30). □

It should be noted here that while above $\blacktriangle$ there might be a futur cone, (61), it is certain, however, that above $\blacktriangle_\varrho$ there is no such cone.

5. MAIN RESULT

The previous analysis demonstrates the asymptotic $\gamma\omega >> 1$ discrepancy between the behavior of a hyperbolic and parabolic temperature waves. Also the independence of relation (62) for the hyperbolic TW, of $f = \frac{\omega}{2\pi}$, as of about $f_L = \frac{1}{\gamma}$, is unique to this wave. Apart from that, the next facts motivate us to derive a mathematical physics result, that follows, on WFs in the parabolic TW.

i) The structural similarity between the travelling wave expressions (36) and (54 or 55) for the respective parabolic TW and hyperbolic TW.

ii) Both TWs do not transfer energy.

iii) The correct $\gamma = 0$ limiting transition, of Remark 1, by the WF for the diverging hyperbolic TW to an anticipated non-WF characteristic (63) or (40).

iv) Partial universalization, of the form (44), for all TWs by a dilating mapping of the $z - t$ space to a $z - \nu$ space.

Let us revisit the parabolic TW (36) and temporally dilate it according to (43), $t = \nu + \frac{\pi}{4\omega}$, to convert it to

$$\theta(z, \nu) = Q\ e^{-\frac{z}{\mu}} cos\left(\frac{z}{\mu} - \omega\nu\right),\ Q = \frac{J_0}{2\varepsilon\sqrt{\omega}}.$$

Define then the auxiliary wave

$$\phi(z,\nu) = e^{\omega\nu}\theta(z,\nu) = Qe^{-(\frac{z}{\mu}-\omega\nu)}cos\left(\frac{z}{\mu}-\omega\nu\right), \tag{79}$$

for which

$$\phi(z,t) = Qe^{-(\frac{z}{\mu}-\omega t+\frac{\pi}{4})}cos\left(\frac{z}{\mu}-\omega t+\frac{\pi}{4}\right) = e^{(\omega t-\frac{\pi}{4})}\theta(z,t) \tag{80}$$

to state below a basic result of this paper.

Lemma 1. *The auxiliary wave $\phi(z,t)$ of (80) has the same wave fronts as the $\theta(z,t)$ TW of (36).*

Proof. It is straightforward to demonstrate that $\phi(z,\nu)$ satisfies

$$2\alpha\omega \triangle_z \phi - \triangle_\nu \phi = 0. \tag{81}$$

This happens to be the same as saying $\phi(z,t)$ of (80) satisfies the wave equation

$$2\alpha\omega \triangle_z \phi - \triangle_t \phi = 0. \tag{82}$$

Substitution of (80) in (82) leads to

$$2\alpha\omega \triangle_z \theta - \omega^2\theta - 2\omega\nabla_t\theta - \triangle_t\theta = 0.$$

This is also the same as saying that $\theta(z,t)$ of (80) turns out to satisfy the remarkable telegrapher's partial differential equation

$$\frac{1}{2\alpha\omega} \triangle_t \theta + \frac{1}{\alpha}\nabla_t\theta + \frac{\omega}{2\alpha}\theta = \triangle_z\theta, \tag{83}$$

subject to satisfaction of the initial conditions

$$\theta(z,0) = \theta_0(z) = Qe^{-\frac{z}{\mu}}cos\left(\frac{z}{\mu}+\frac{\pi}{4}\right), \tag{84}$$

$$\theta_1(z) = \nabla_t\theta(z,0) = \omega Qe^{-\frac{z}{\mu}}sin\left(\frac{z}{\mu}+\frac{\pi}{4}\right). \tag{85}$$

Since both PDEs, (82) and (83), have the same CE

$$2\alpha\omega(dt)^2 - (dz)^2 = 0.$$

i.e. the same characteristics, then they have the same WFs for their respective $\phi(z,t)$ and $\theta(z,t)$ waves. Here the proof ends. □

Definition 3. *The constructively introduced parabolic-equivalent hyperbolic heat equation (P-EHHE), (83), is a unique Telegrapher's-type PDE with a Cauchy problem solution identical to the TW solution (36) of the BVP (32)-(34) for the parabolic heat equation.*

A distinguishing composition of this parabolic-equivalent hyperbolic heat equation P-EHHE includes its $\frac{1}{2\alpha\omega} \triangle_t \theta$ term, in which $2\alpha\omega = V_p^2$. It has a 2ω "viscous damping" coefficient and a "restoration" coefficient of ω^2. Both the P-EHHE and the entirely different Cattaneo-Vernotte hyperbolic heat equation C-VHHE are invariant under the time-dilation transformation (49). The IVP (83)-(85) is obviously the Cauchy problem for the parabolic-equivalent hyperbolic heat equation P-EHHE.

Theorem 2. *A parabolic TW has the same wave fronts as a hyperbolic TW of the parabolic-equivalent hyperbolic heat equation P-EHHE, but for ς replaced by V_p, $\forall\varsigma \in [0,\infty)$.*

Proof. By invoking Lemma 1 and realizing that the parabolic-equivalent hyperbolic heat equation P-EHHE (83) and the Cattaneo-Vernotte hyperbolic heat equation C-VHHE (47) have the same respective CEs,

$$2\alpha\omega(dt)^2 - (dz)^2 = 0, \tag{86}$$

$$\varsigma^2(dt)^2 - (dz)^2 = 0, \tag{87}$$

which both turn out to produce the same characteristics (i.e. same WFs) when ς^2 and $2\alpha\omega = V_p^2$ are interchanged. □

5.1. **The parabolic temperature wave tenuous wavefronts.** By relations (86) and (87) of Theorem 2, the solution characteristics of the P-E hyperbolic heat equation are

$$\left.\begin{array}{l} z - V_p t = C_1 \\ z + V_p t = C_2 \end{array}\right\} \triangleq \aleph, \tag{88}$$

which obviously coincide with $\aleph$, and are two WFs, inclined by

$$\Omega = \pm \tan^{-1}\frac{1}{V_p} = \pm \tan^{-1}\frac{1}{\sqrt{2\alpha\omega}}. \tag{89}$$

Remark 2. *A distinctive feature of the tenuous WFs of the parabolic TW from the WFs of the hyperbolic TW is the frequency dependence in (89) for Ω in the first WFs. Furthermore, any practical refraction of the parabolic TW can possibly cause a blurring of the WF edges. This is perhaps a major reason for the experimental fuzziness of parabolic WFs, in comparison to hyperbolic WFs.*

As for Cauchy problems related to parabolic WFs, passing through a point (z_o, t_o), the parabolic WFs should satisfy

$$V_p^2(t - t_o)^2 - (z - z_o)^2 = 0,$$

with the Cauchy problem solvability boundaries

$$\left.\begin{array}{l} \Gamma^+ \triangleq \sqrt{2\alpha\omega}(t - t_o) > |z - z_o|\,, \text{ future cone} \\ \Gamma^- \triangleq -\sqrt{2\alpha\omega}(t - t_o) > |z - z_o|\,, \text{ past cone} \end{array}\right\},$$

that are also explicitly frequency dependent.

Theorem 2 is clearly a sharp proof of the fact that: the parabolic heat equation-for parabolic TWs- cannot support mathematical WFs in its general solution, should not mean that the parabolic temperature waves also cannot have them (as is sometimes wrongly reported in the literature). The reason is simply because parabolic TWs are particular solutions to the parabolic HE, that are periodically constrained in an associated BVP, and are not general solutions to it. In addition to this theorem, one cannot overlook that the parabolic TW is, in general, a good approximation to the solution of the C-V hyperbolic BVP (47)-(49), which unquestionably supports WFs. The proved possibility for the parabolic TW to be a general solution to the parabolic-equivalent hyperbolic heat equation P-EHHE of (83) makes the existence of WFs in it also unquestionable.

Theorem 3. *The solution of the Cauchy problem for the parabolic-equivalent hyperbolic heat equation P-EHHE (83), with the data set $\{\theta_0(z), \theta_1(z)\}$ of (84)-(85) is*

$$\theta(z,t) = e^{-\omega t + \frac{\pi}{4}}\left\{\frac{1}{2}\Big[\theta_0(z - V_p t) + \theta_0(z + V_p t)\Big] + \frac{1}{2V_p}\int_{z-V_p t}^{z+V_p t}\big[\theta_1(s) + \omega\theta_0(s)\big]ds\right\}$$

$$= \frac{J_0}{2\varepsilon\sqrt{\omega}}e^{-\frac{z}{\mu}}cos(\frac{z}{\mu} - \omega t + \frac{\pi}{4}) \tag{90}$$

and admits the wave fronts, $\aleph$, of (88).

Proof. Consider the map (79) to transform (83) with its Cauchy data to the equivalent IVP:

$$\triangle_z\phi - \frac{1}{2\alpha\omega}\triangle_t \phi = 0, \tag{91}$$

$$\phi_0(z) = \phi(z,0) = \theta_0(z)e^{-\frac{\pi}{4}} = \frac{J_0}{2\varepsilon\sqrt{\omega}}e^{-(\frac{z}{\mu}+\frac{\pi}{4})}cos\left(\frac{z}{\mu}+\frac{\pi}{4}\right), \tag{92}$$

$$\begin{aligned}\phi_1(z) &= \nabla_t\phi(z,0) = [\theta_1(z)+\omega\theta_0(z)]e^{-(\frac{z}{\mu}+\frac{\pi}{4})}\\ &= \omega\frac{J_0}{2\varepsilon\sqrt{\omega}}e^{-(\frac{z}{\mu}+\frac{\pi}{4})}\left[cos\left(\frac{z}{\mu}+\frac{\pi}{4}\right)+sin\left(\frac{z}{\mu}+\frac{\pi}{4}\right)\right].\end{aligned} \tag{93}$$

The result (90) is the well-known d'Alembert's solution of (91)-(93) when subjected to reversed mapping via (79) with $V_p^2 = 2\alpha\omega$. In distinction from the situation of the hyperbolic IVP of Theorem 1, the $e^{-\omega t+\frac{\pi}{4}}$ factor in (90) shall be compensated in some of its terms by $e^{\frac{V_p}{\mu}-\frac{\pi}{4}} = e^{\omega t-\frac{\pi}{4}}$ that emerge from (92) and (93). Moreover, (83) and (92) happen to have the same characteristics. Clearly, as in Theorem 1, the ingredients $z \pm V_p t$ of $\aleph$ can impact $\theta(z,t)$ for any t. Here the proof completes. □

Remark 3. *Theorem 2 and Theorem 3 are effective illustrations that for temperature waves although the mathematical* $\aleph$ *is of infinite span, the physical* $\mathcal{F}$ *is in shortened,* ϱ*, form.*

Remark 4. *The close similarity between the C-VHHE (47) and the P-EHHE (83) should not lead to covering up the structural differences between their IVP solutions. On one hand, the C-VHHE IVP (65)-(67) is solvable only as a Volterra integral equation (75) with a d'Alembertian nonhomogeneous term. On the other hand, the P-EHHE IVP (83)-(85) closed form solution (90) is entirely d'Alembertian.*

Finally, the situation on the mathematical side of WFs appears therefore to be equally fine for the parabolic TW and the hyperbolic TW. Conversely, the experimental side of wavefronts, remains , however, to be difficult, and calls perhaps for more-refined, frequency-filtered, temperature and temperature gradient measurement instrumentation.

6. EXPERIMENTAL VERIFICATION OF TEMPERATURE WAVEFRONTS

Relation (89) on the $\Omega \sim \frac{1}{\sqrt{\omega}}$ dependence can cause a blur in wavefronts of temperature waves. This situation, when added to the shortened nature of wavefronts (SWF), $\varrho \subset \mathcal{F}$, happens to be consistent with the seemingly reflectionless-refractionless nature, [23], of these waves. Indeed, specific simulations, [23, 27, 28], show that reflections and Snell's law can be adequate approximations only under near-normal incidence conditions.

As a way out of such difficult experiments, and as a proposal for a possible inroad to physically sense the existence of SWFs, ϱ, consider, for $\omega t << (C - \frac{\pi}{4})$, near $t = 0$, i.e. $t < t_\varrho$ of (77), a compactly supported initial temperature distribution $\theta(z,0)$ with a support only on an interval $\mathfrak{D} = [a,b]$ of the z-axis. $|\mathfrak{D}| = b - a$, when $b > a$, on a one-dimensional heat conducting wedge of heat diffusivity α. This can be represented as

$$\theta(z,0) = \theta_0(z) = f(z)\left[u(z-a) - u(z-b)\right], \tag{94}$$

where $u(z)$ is Heaviside's unit step function. Clearly then,

$$\theta_0(z) = \begin{cases} f(z) = Qe^{-\frac{z}{\mu}}cos(\frac{z}{\mu}+\frac{\pi}{4}),\ z\in[a,b] \\ 0,\ z\notin[a,b]\end{cases} \tag{95}$$

Similarly

$$\nabla_t \phi(z,0) = \theta_1(z) = \omega \widetilde{f}(z)[u(z-a) - u(z-b)], \tag{96}$$

with

$$\theta_1(z) = \begin{cases} \omega \widetilde{f}(z) = \omega Q e^{-\frac{z}{\mu}} cos(\frac{z}{\mu} - \frac{\pi}{4}), \ z \in [a,b] \\ 0, \ z \notin [a,b] \end{cases} \tag{97}$$

$\theta_0(z) = f(z)[u(z-a) - u(z-b)]$ can be generated on $[a,b]$ at $t = 0$ by a set of heating lasers. Simultaneously at the same moment $t = 0$ one can use a set of pulsed laser heating beams, with power proportional to ω, over a short period δt to generate a temperature increment $\delta\theta$ distribution

$$\delta\theta = \omega \widetilde{f}(z)[u(z-a) - u(z-b)] \ \delta t. \tag{98}$$

The experiment comprises the following two steps.

i)On a wedge supporting TWs of frequency ω, we have $|\mathfrak{D}_\varrho| = 2V_p t_\varrho = 2\sqrt{2\alpha\omega} \ t_\varrho$. Therefore at a time $t_0 \leq t_\varrho$ we take two simultaneous measurements, $\widehat{\theta}(z_0, t_0), \widehat{\theta}(z_*, t_0)$, of the corresponding instantaneous temperatures, respectively at $z_0 \in \mathfrak{D}_\varrho$ and $z_* \notin \mathfrak{D}_\varrho$. Such a z_* means that (z_*, t_0) lies outside the detectable domain of determinacy of $\mathfrak{D}_\varrho$.

ii) If

$$\widehat{\theta}(z_0, t_0) = Q e^{-\frac{z_0}{\mu}} cos(\frac{z_0}{\mu} - \omega t_0 + \frac{\pi}{4}), \tag{99}$$

and

$$\widehat{\theta}(z_*, t_0) = 0, \tag{100}$$

then the existence of WFs in TWs is asserted in a necessary (but not sufficient) sense. Indeed, violation of (99) and/or (100), outside the scale of experimental and systematic error of $\widehat{\theta}$, $\delta\theta$ and δt measurements, indicates a possibility for nonexistence of ϱ, given by (90).

7. CONCLUSION

The basic result of this paper has been a revelation of a tenuous nature of mathematical wavefronts $\aleph$ in parabolic temperature waves and a shortened ϱ-nature of their physical wavefronts $\mathcal{F}$. It is developed along the landmarks that follow.

(i) Introduction, in Definition 1, of the concept of a shortened wavefront $(\varrho \subset \mathcal{L}) \approx \mathcal{F}$, when $\omega t << \left(C - \frac{\pi}{4}\right)$, exclusively.
(ii) Remark 1, on the rotational conversion (63) of wavefronts for the C-V hyperbolic heat equation.
(iii) Constructive derivation of the parabolic-equivalent hyperbolic heat equation P-EHHE in the proof of Lemma 1.
(iv) Remark 2, on the frequency dependence of the inclination of wavefronts for the parabolic temperature wave.
(v) Proposition 1, on shrinkage of the triangle for detectable wavefronts.
(vi) Theorem 3, on the travelling wave solution to the Cauchy problem for the parabolic-equivalent hyperbolic heat equation P-EHHE.
(vii) Remark 3, on shrinkage of physical $\mathcal{F}$, while the mathematical $\aleph$ is of infinite span.
(viii) Remark 4, on the difference between the IVPs of the C-VHHE and the P-EHHE.
(ix) The experiment, proposed in Section 6, for verifying the existence of temperature physical wavefronts.

The reported analysis, which reviewed some unique features for parabolic TWs in subsection 3.2, calls for intensifying efforts towards an experimental detection and identification of physical wavefronts in both parabolic TWs and hyperbolic TWs. Despite the insignificant difference between these, there are specific heat conduction applications where the hyperbolic TW is expected to perform better than the parabolic temperature wave. This should particularly be true at low temperatures when γ can be significant, [29-30], and in periodic heat fluxes of laser heating, [31-32], or other phenomena on the nanoscale.

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