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Gg-CONVEX FUNCTIONS

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Abstract. In this paper, the concept of Gg-convex function is given the first time in the literature. Some inequalities of Hadamard's type for Gg-convex functions are given. Some algebraic properties of Gg-convex functions and special cases are discussed. In addition, we establish some new integral inequalities for Gg-convex functions by using an integral identity.

1. Introduction

Convexity theory plays a central and fundamental role in the fields of mathematical finance, economics, engineering, management sciences, and optimization theory. In recent years, the concept of convexity has been extended and generalized in several directions using the novel and innovative ideas; see, for example, [2, 3, 4, 5, 7, 8, 9] and the references therein.

Definition 1. *A function $f : I \subseteq \mathbf{R} \to \mathbf{R}$ is said to be convex if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$. This definition is well known in the literature. Denote by $C(I)$ the set of the convex functions on the interval I.

Definition 2. *$f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions.

In [6], Kadakal and İşcan gave the concept of the Ag-convexity as follow:

Definition 3. *Let $I \subset \mathbf{R}$ be an interval, $f : I \to \mathbf{R}$, $g : J \to \mathbf{R}$, $J \supset f(I)$. f is said to be Ag-convex if the inequality $f(tx + (1-t)y) \leq tg(f(x)) + (1-t)g(f(y))$ is valid for all $x, y \in I$ and $t \in [0,1]$. Denote by $AgC(I)$ the set of the Ag-convex functions on the interval I.*

Definition 4 ([10])**.** *A function $f : I \subseteq \mathbf{R}_0 = [0,\infty)$ is said to be geometric arithmetically convex (or GA-convex) on I if $f\left(x^t y^{1-t}\right) \leq tf(x) + (1-t)f(y)$ holds for all $x, y \in I$ and $t \in [0,1]$, where $x^t y^{1-t}$ and $tf(x) + (1-t)f(y)$ are, respectively, the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$. We will denote by $GAC(I)$ the set of the GA-convex functions on the interval I.*

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In [1], authors proved the following lemma and established new inequalities of Hermite-Hadamard type for GA-convex functions:

Lemma 1. *Let $f : I \subset \mathbf{R}_+ = (0,\infty) \to \mathbf{R}$ be a differentiable mapping on I° and $a,b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then the following identity holds:*

$$\begin{aligned} bf(b) - af(a) - \int_a^b f(x)dx &= (\ln x - \ln a)\int_0^1 x^{2t}a^{2(1-t)}f'\left(x^t a^{1-t}\right)dt \\ &\quad - (\ln x - \ln b)\int_0^1 x^{2t}b^{2(1-t)}f'\left(x^t b^{1-t}\right)dt \end{aligned}$$

for all $x \in [a,b]$.

The main purpose of this paper is to give a new class of convex functions called as Gg-convex function (or (GA,g)-convex) and establish both the Hermite-Hadamard type integral inequalities and in addition, we establish some new integral inequalities for Gg-convex functions by using an integral identity. The results obtained in special cases are reduced to the results obtained in the literature.

2. Main results for Gg-convex functions

Definition 5. *Let $I \subset (0,\infty)$ be an interval, $f : I \to \mathbf{R}$, $g : J \to \mathbf{R}$, $J \supset f(I)$. f is said to be Gg-convex (or (GA,g)-convex) function if the inequality*

$$f\left(x^t y^{1-t}\right) \leq tg(f(x)) + (1-t)g(f(y)) \tag{2}$$

is valid for all $x,y \in I$ and $t \in [0,1]$. If the inequality (2) is reversed, then the function is said to be Gg-concave. We will denote by $GgC(I)$ the set of the Gg-convex functions on the interval I.

Proposition 1. *Let $I \subset (0,\infty)$ be an interval, $f : I \to \mathbf{R}$, $g : J \to \mathbf{R}$ and $J \supset f(I)$. If f is Gg-convex, then $y \leq g(y)$ for every $y \in f(I)$.*

Proof. Let $y \in f(I)$ be arbitrary. Then, there exists a $x \in I$ such that $y = f(x)$. If we take $a \in I \setminus \{x\}$ as a constant, then since the function f is Gg-convex, for every $t \in [0,1]$

$$f\left(x^t a^{1-t}\right) \leq t(g \circ f)(x) + (1-t)(g \circ f)(a).$$

For $t = 1$, $f(x) \leq g(f(x))$, that is $y \leq g(y)$. This show us that $y \leq g(y)$ for every $y \in f(I)$. □

Remark 1. *Let $I \subset (0,\infty)$ be an interval, $f : I \to \mathbf{R}$, $g : J \to \mathbf{R}$ and $J \supset f(I)$.*

i) If the function g satisfies the condition $y \leq g(y), y \in f(I)$ and the function f is GA-convex, then f is Gg-convex function. Indeed, for every $t \in [0,1]$ and every $a,b \in I$

$$\begin{aligned} f\left(a^t b^{1-t}\right) \leq tf(a) + (1-t)f(b) &\leq tg(f(a)) + (1-t)g(f(b)) \\ &\leq t(g \circ f)(a) + (1-t)(g \circ f)(b). \end{aligned}$$

ii) If the function g satisfies the condition $g(y) \leq y, y \in f(I)$ and the function f is Gg-convex, then f is GA-convex function. Indeed, for every $t \in [0,1]$ and every $a,b \in I$

$$f\left(a^t b^{1-t}\right) \leq tg(f(a)) + (1-t)g(f(b)) \leq tf(a) + (1-t)f(b).$$

iii.) It is obvious that $GgC(I) = GAC(I) \Leftrightarrow g(x) = x$.

Theorem 1. *Let $I \subset (0,\infty)$ be an interval and $c \in [0,\infty)$. If $f \in GgC(I)$ and g is linear, then cf is Gg-convex function.*

Proof. For $c \in [0, \infty)$,

$$(cf)\left(x^t y^{1-t}\right) \leq= tg\left(cf(x)\right) + (1-t)g(cf(y)) \quad = \quad t\left(g \circ (cf)\right)(x) + (1-t)\left(g \circ (cf)\right)(y)$$

This completes the proof of theorem. □

Theorem 2. *Let $I \subset (0, \infty)$ be an interval. If the functions $f, h \in GgC(I)$ and g is linear, then $f + h \in GgC(I)$.*

Proof. For $x, y \in I$ and $t \in [0, 1]$,

$$\begin{aligned}(f+h)\left(x^t y^{1-t}\right) &\leq [tg(f(x)) + (1-t)g(f(y))] + [tg(h(x)) + (1-t)g(h(y))] \\ &= t\left[g(f(x)) + g(h(x))\right] + (1-t)\left[g(f(y)) + g(h(y))\right] \\ &= tg\left(f(x) + h(x)\right) + (1-t)g\left(f(y) + h(y)\right) \\ &= t\left(g \circ (f+h)\right)(x) + (1-t)\left(g \circ (f+h)\right)(y).\end{aligned}$$

This completes the proof of theorem. □

Theorem 3. *Let $I \subset (0, \infty)$ be an interval. If the function $f \in GgC(I)$ and monotone increasing, and h is GG-convex, then $f \circ h \in GgC(I)$.*

Proof. For $x, y \in I$ and $t \in [0, 1]$,

$$\begin{aligned}(foh)\left(x^t y^{1-t}\right) \leq f\left(h^t(x)h^{1-t}(y)\right) &\leq tg\left(f(h(x))\right) + (1-t)g\left(f(h(y))\right) \\ &\leq t\left(g \circ (f(h))\right)(x) + (1-t)\left(g \circ (f(h))\right)(y).\end{aligned}$$

This completes the proof of theorem. □

Theorem 4. *Let $I \subset (0, \infty)$ be an interval, $f, h : I \to \mathbf{R}$ are both nonnegative, monotone (increasing or decreasing) and $g : J \to \mathbf{R}$, $J \supset f(I)$, is monotone (increasing or decreasing) and satisfies the condition $g(u)g(v) \leq g(uv)$. If $f, h \in GgC(I)$, then $fh \in GgC(I)$.*

Proof. If $x \leq y$ ($y \leq x$ runs in the same fashion) then $[g(f(x)) - g(f(y))]\,[g(h(y)) - g(h(x))] \leq 0$ which implies

$$g(f(x))g(h(y)) + g(f(y))g(h(x)) \leq g(f(x))g(h(x)) + g(f(y))g(h(y)). \tag{3}$$

On the other hand for $x, y \in I$ and $t \in [0, 1]$,

$$\begin{aligned}(fh)\left(x^t y^{1-t}\right) &= f\left(x^t y^{1-t}\right) h\left(x^t y^{1-t}\right) \\ &\leq [tg(f(x)) + (1-t)g(f(y))]\,[tg(h(x)) + (1-t)g(h(y))] \\ &= t^2 g(f(x))g(h(x)) + t(1-t)g(f(x))g(h(y)) \\ &\quad + t(1-t)g(f(y))g(h(x)) + (1-t)^2 g(f(y))g(h(y)).\end{aligned}$$

Using now (3), we obtain,

$$\begin{aligned}&(fh)\left(x^t y^{1-t}\right) \\ \leq\ & t^2 g(f(x))g(h(x)) + (1-t)^2 g(f(y))g(h(y)) + t(1-t)\left[g(f(x))g(h(x)) + g(f(y))g(h(y))\right] \\ \leq\ & t\left[t + (1-t)\right] g(f(x))g(h(x)) + (1-t)\left[t + (1-t)\right] g(f(y))g(h(y)) \\ =\ & tg(f(x))g(h(x)) + (1-t)g(f(y))g(h(y)) \\ \leq\ & tg(fh)(x) + (1-t)g(fh)(y).\end{aligned}$$

This completes the proof of theorem. □

3. Hermite-Hadamard inequality for Gg-convex functions

Theorem 5. *Let $I \subset (0,\infty)$ be an interval. If $f \in GgC(I)$, $g : J \to \mathbf{R}$, $J \supset f(I)$, $a,b \in I$ with $a<b$ and $g \circ f \in L[a,b]$. The following inequality holds.*

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(u)}{u}du \le \frac{(g\circ f)(a) + (g \circ f)(b)}{2}.$$

Proof. By the definition of Gg-convexity of the function f on $[a,b]$, we write $f\left(a^t b^{1-t}\right) \le tg(f(a)) + (1-t)g(f(b))$. Now, if we take integral the last inequality on $t \in [0,1]$, we get

$$\int_0^1 f\left(a^t b^{1-t}\right)dt \le \int_0^1 \left[tg(f(a)) + (1-t)g(f(b))\right]dt.$$

By changing variable as $u = a^t b^{1-t}$, we obtain

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(u)}{u}du \le \frac{(g\circ f)(a) + (g \circ f)(b)}{2}$$

□

Remark 2. *If we take $g(x) = x$ in the Theorem 5, then we obtain the right side of the Hermite-hadamard inequality for the GA-convex functions.*

Theorem 6. *Let $I \subset (0,\infty)$ be an interval. If $f \in GgC(I)$, $g : J \to \mathbf{R}$, $J \supset f(I)$, $a,b \in I$ with $a<b$ and $f \in L[a,b]$, then the following inequality holds:*

$$f\left(\sqrt{ab}\right) \le \frac{1}{\ln b - \ln a}\int_a^b \frac{g(f(u))}{u}du.$$

Proof. By the Gg-convexity of the function f, we write

$$f\left(\sqrt{ab}\right) = f\left(\sqrt{a^{1-t}b^t a^t b^{1-t}}\right) \le \frac{g\left(f(a^{1-t}b^t)\right) + g\left(f(a^t b^{1-t})\right)}{2}.$$

for $t \in [0,1]$. Now, if we take integral the last inequality on $t \in [0,1]$, we get

$$f\left(\sqrt{ab}\right) \le \frac{1}{2}\left[\int_0^1 g\left(f(a^{1-t}b^t)\right)dt + \int_0^1 g\left(f(a^t b^{1-t})\right)dt\right].$$

Since

$$\int_0^1 g\left(f(a^{1-t}b^t)\right)dt = \int_0^1 g\left(f(a^t b^{1-t})\right)dt = \frac{1}{\ln b - \ln a}\int_a^b \frac{g(f(u))}{u}du,$$

we obtain the desired result. This completes the proof of theorem. □

Remark 3. *If we take $g(x) = x$ in the Theorem 6, then we obtain the left side of the Hermite-hadamard inequality for the GA-convex functions.*

Theorem 7. *Let f and h be real-valued, nonnegative and Gg-convex functions on interval $[a,b]$. Then, the following inequalities*

$$\begin{aligned} &4f\left(\sqrt{ab}\right)h\left(\sqrt{ab}\right) \qquad (4)\\ \le\ & \frac{2}{\ln b - \ln a}\left[\int_a^b \frac{(g\circ f)(x)(g\circ h)(x)}{x}dx + \int_a^b \frac{(g\circ f)(x)(g\circ h)\left(\frac{ab}{x}\right)}{x}dx\right].\end{aligned}$$

are valid for all $x,y \in [a,b]$ and $t \in [0,1]$.

Proof. Since both functions f and g are Gg-convex, for every two points $x, y \in [a,b]$, the following inequalities are valid

$$f\left(\sqrt{xy}\right) \leq \frac{(g\circ f)(x) + (g\circ f)(y)}{2}, h\left(\sqrt{xy}\right) \leq \frac{(g\circ h)(x) + (g\circ h)(y)}{2}$$

Multiplying the above inequalities, we have the following

$$\begin{aligned} 4f\left(\sqrt{xy}\right)h\left(\sqrt{xy}\right) &\leq (g\circ f)(x)(g\circ h)(x) + (g\circ f)(y)(g\circ h)(y) \\ &\quad + (g\circ f)(x)(g\circ h)(y) + (g\circ f)(y)(g\circ h)(x). \end{aligned}$$

Let $x = a^t b^{1-t}$ and $y = b^t a^{1-t}$, then we have

$$\begin{aligned} 4f\left(\sqrt{ab}\right)h\left(\sqrt{ab}\right) &\leq (g\circ f)(a^tb^{1-t})(g\circ h)(a^tb^{1-t}) + (g\circ f)(b^ta^{1-t})(g\circ h)(b^ta^{1-t}) \\ &\quad + (g\circ f)(a^tb^{1-t})(g\circ h)(b^ta^{1-t}) + (g\circ f)(b^ta^{1-t})(g\circ h)(a^tb^{1-t}). \end{aligned}$$

Both sides of the above inequality are integrable with respect to t on $[0,1]$,

$$\begin{aligned} &4f\left(\sqrt{ab}\right)h\left(\sqrt{ab}\right) \qquad (5) \\ \leq\ &\frac{2}{\ln b - \ln a}\left[\int_a^b \frac{(g\circ f)(x)(g\circ h)(x)}{x}dx + \int_a^b \frac{(g\circ f)(x)(g\circ h)\left(\frac{ab}{x}\right)}{x}dx\right]. \end{aligned}$$

□

Remark 4. *In the above Theorem, if $g\circ h$ is symmetric with respect to $\sqrt{ab}$, (i.e. $(g\circ h)\left(\frac{ab}{x}\right) = (g\circ h)(x)$, for all $x \in [a,b]$), then we have*

$$f\left(\sqrt{ab}\right)h\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{(g\circ f)(x)(g\circ h)(x)}{x}dx. \qquad (6)$$

4. Some new inequalities for Gg-convex functions

The main aim of this paper is to prove some new integral inequalities for Gg-convex functions by using the above integral identity.

Theorem 8. *Let $f : I \subset (0,\infty) \to \mathbf{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. If $|f'(x)|$ is Gg-convex function on the interval $[a,b]$, then the following inequality holds for all $x \in [a,b]$.*

$$\begin{aligned} &\left|bf(b) - af(a) - \int_a^b f(x)dx\right| \qquad (7) \\ \leq\ &\frac{1}{2}\left[\left|g\left(f'(x)\right)\right|\left(x^2 - A(a,x)L(a,x)\right) + \left|g\left(f'(a)\right)\right|\left(A(x,a)L(a,x) - a^2\right)\right] \\ &+\frac{1}{2}\left[\left|g\left(f'(x)\right)\right|\left(A(x,b)L(x,b) - x^2\right) + \left|g\left(f'(b)\right)\right|\left(b^2 - A(x,b)L(x,b)\right)\right]. \end{aligned}$$

Proof. From Lemma 1 and by using the Gg-convexity of $|f'(x)|$, that is

$$\begin{aligned} \left|f'\left(x^ta^{1-t}\right)\right| &\leq t\left|g\left(f'(x)\right)\right| + (1-t)\left|g\left(f'(a)\right)\right| \\ \left|f'\left(x^tb^{1-t}\right)\right| &\leq t\left|g\left(f'(x)\right)\right| + (1-t)\left|g\left(f'(b)\right)\right|, \end{aligned}$$

we have

$$
\begin{aligned}
&\left|bf(b)-af(a)-\int_a^b f(x)dx\right| \\
\leq\;& (\ln x-\ln a)\int_0^1 x^{2t}a^{2(1-t)}\left[t\left|g\left(f'(x)\right)\right|+(1-t)\left|g\left(f'(a)\right)\right|\right]dt \\
&+(\ln b-\ln x)\int_0^1 x^{2t}b^{2(1-t)}\left[t\left|g\left(f'(x)\right)\right|+(1-t)\left|g\left(f'(b)\right)\right|\right]dt \\
=\;& (\ln x-\ln a)\left[\left|g\left(f'(x)\right)\right|\int_0^1 tx^{2t}a^{2(1-t)}dt+x^{2t}\left|g\left(f'(b)\right)\right|\int_0^1(1-t)x^{2t}a^{2(1-t)}dt\right] \\
&+(\ln b-\ln x)\left[\left|g\left(f'(x)\right)\right|\int_0^1 tx^{2t}b^{2(1-t)}dt+\left|g\left(f'(b)\right)\right|\int_0^1(1-t)x^{2t}b^{2(1-t)}dt\right]. \quad (8)
\end{aligned}
$$

By sample calculation give us that

$$\int_0^1 tx^{2t}a^{2(1-t)}dt = \frac{1}{2}\frac{x^2-A(a,x)L(a,x)}{\ln x-\ln a} \quad (9)$$

$$\int_0^1 (1-t)x^{2t}a^{2(1-t)}dt = \frac{1}{2}\frac{A(a,x)L(a,x)-a^2}{\ln x-\ln a} \quad (10)$$

$$\int_0^1 tx^{2t}b^{2(1-t)}dt = \frac{1}{2}\frac{A(x,b)L(x,b)-x^2}{\ln b-\ln x} \quad (11)$$

$$\int_0^1 (1-t)x^{2t}b^{2(1-t)}dt = \frac{1}{2}\frac{b^2-A(x,b)L(x,b)}{\ln b-\ln x}, \quad (12)$$

where A, L are arithmetic and Logarithmeac means, respectively. Substituting (9)-(12) in (8), the required result is obtained. This completes the proof of the theorem. □

Remark 5. *If we take $g(x)=x$ in the Theorem 8, then we obtain the following inequality for the GA-convex functions.*

$$
\begin{aligned}
&\left|bf(b)-af(a)-\int_a^b f(x)dx\right| \leq \frac{1}{2}\left|f'(x)\right|\left[A(x,b)L(x,b)-A(a,x)L(a,x)\right] \\
&+\frac{1}{2}\left(\left[\left|f'(a)\right|\left(A(x,a)L(a,x)-a^2\right)\right]+\left[\left|f'(b)\right|\left(b^2-A(x,b)L(x,b)\right)\right]\right).
\end{aligned}
$$

Theorem 9. *Let $f:I\subset(0,\infty)\to\mathbf{R}$ be a differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$ and $f'\in L[a,b]$. If $|f'(x)|^q$ is Gg-convex function on the interval $[a,b]$, then the following inequality holds for all $x\in[a,b]$ and $q\geq 1$:*

$$
\begin{aligned}
&\left|bf(b)-af(a)-\int_a^b f(x)dx\right| \leq (\ln x-\ln a)\left(\frac{1}{2}\right)^{\frac{1}{q}}L^{1-\frac{1}{q}}\left(a^2,x^2\right) \\
&\times\left(\left|g\left(f'(x)\right)\right|^q\left[\frac{x^2-A(x,a)L(x,a)}{\ln x-\ln a}\right]+\left|g\left(f'(a)\right)\right|^q\left[\frac{A(x,a)L(x,a)-a^2}{\ln x-\ln a}\right]\right)^{\frac{1}{q}} \\
&+(\ln b-\ln x)\left(\frac{1}{2}\right)^{\frac{1}{q}}L^{1-\frac{1}{q}}\left(x^2,b^2\right) \\
&\times\left(\left|g\left(f'(x)\right)\right|^q\left[\frac{A(x,b)L(x,b)-x^2}{\ln b-\ln x}\right]+\left|g\left(f'(b)\right)\right|^q\left[\frac{b^2-A(x,b)L(x,b)}{\ln b-\ln x}\right]\right)^{\frac{1}{q}}.
\end{aligned}
$$

Proof. From Lemma 1 and by using the Gg-convexity of $|f'(x)|$, that is,

$$\begin{aligned} \left|f'\left(x^t a^{1-t}\right)\right|^q &\le t\left|g\left(f'(x)\right)\right|^q+(1-t)\left|g\left(f'(a)\right)\right|^q \\ \left|f'\left(x^t b^{1-t}\right)\right|^q &\le t\left|g\left(f'(x)\right)\right|^q+(1-t)\left|g\left(f'(b)\right)\right|^q, \end{aligned}$$

and well known the Hölder integral inequality, we obtain

$$\begin{aligned} &\left|bf(b)-af(a)-\int_a^b f(x)dx\right| \\ \le\ & (\ln x-\ln a)\left(\int_0^1 x^{2t}a^{2(1-t)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 x^{2t}a^{2(1-t)}\left|f'\left(x^t a^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}} \\ & +(\ln b-\ln x)\left(\int_0^1 x^{2t}b^{2(1-t)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 x^{2t}b^{2(1-t)}\left|f'\left(x^t b^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}} \\ =\ & (\ln x-\ln a)\left(\int_0^1 x^{2t}a^{2(1-t)}dt\right)^{1-\frac{1}{q}} \\ & \times\left(\left|g\left(f'(x)\right)\right|^q\int_0^1 tx^{2t}a^{2(1-t)}dt+\left|g\left(f'(a)\right)\right|^q\int_0^1(1-t)x^{2t}a^{2(1-t)}dt\right)^{\frac{1}{q}} \\ & +(\ln b-\ln x)\left(\int_0^1 x^{2t}b^{2(1-t)}dt\right)^{1-\frac{1}{q}} \\ & \times\left(\left|g\left(f'(x)\right)\right|^q\int_0^1 tx^{2t}b^{2(1-t)}dt+\left|g\left(f'(b)\right)\right|^q\int_0^1(1-t)x^{2t}b^{2(1-t)}dt\right)^{\frac{1}{q}}, \quad (13) \end{aligned}$$

where

$$\int_0^1 x^{2t}a^{2(1-t)}dt = \frac{x^2-a^2}{2\left(\ln x-\ln a\right)}=L\left(a^2,x^2\right), \quad (14)$$

$$\int_0^1 x^{2t}b^{2(1-t)}dt = \frac{a^2-b^2}{2\left(\ln b-\ln a\right)}=L\left(x^2,b^2\right). \quad (15)$$

Sunstituting (14) and (15) in the inequality (13), we obtain the desired result. This completes the proof of the theorem. □

Remark 6. *If we take $g(x)=x$ in the Theorem 9, then we obtain the following inequality for the GA-convex functions.*

$$\begin{aligned} &\left|bf(b)-af(a)-\int_a^b f(x)dx\right| \le (\ln x-\ln a)\left(\frac{1}{2}\right)^{\frac{1}{q}}L^{1-\frac{1}{q}}\left(a^2,x^2\right) \\ &\times\left(\left|f'(x)\right|^q\left[\frac{x^2-A\left(x,a\right)L\left(x,a\right)}{\ln x-\ln a}\right]+\left|f'(a)\right|^q\left[\frac{A\left(x,a\right)L\left(x,a\right)-a^2}{\ln x-\ln a}\right]\right)^{\frac{1}{q}} \\ &+(\ln b-\ln x)\left(\frac{1}{2}\right)^{\frac{1}{q}}L^{1-\frac{1}{q}}\left(x^2,b^2\right)\left(\left|f'(x)\right|^q\left[\frac{A\left(x,b\right)L\left(x,b\right)-x^2}{\ln b-\ln x}\right]+\left|f'(b)\right|^q\left[\frac{b^2-A\left(x,b\right)L\left(x,b\right)}{\ln b-\ln x}\right]\right)^{\frac{1}{q}}. \end{aligned}$$

Corollary 1. *If we take $q=1$ in the Theorem 9, we have the following inequality:*

$$\begin{aligned} \left|bf(b)-af(a)-\int_a^b f(x)dx\right| \le\ & \frac{1}{2}\left[\left|g\left(f'(x)\right)\right|\left(x^2-A\left(x,a\right)L\left(x,a\right)\right)+\left|g\left(f'(a)\right)\right|\left(A\left(x,a\right)L\left(x,a\right)-a^2\right)\right] \\ & +\frac{1}{2}\left[\left|g\left(f'(x)\right)\right|\left(A\left(x,b\right)L\left(x,b\right)-x^2\right)+\left|g\left(f'(b)\right)\right|\left(b^2-A\left(x,b\right)L\left(x,b\right)\right)\right]. \end{aligned}$$

This inequality coincides with the inequality (7) in Theorem 8.

Theorem 10. *Let $f : I \subset (0,\infty) \to \mathbf{R}$ be a differentiable mapping on I° and $a,b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. If $|f'(x)|^q$ is Gg-convex function on the interval $[a,b]$, then the following inequality holds for all $x \in [a,b]$ and $q > 1$:*

$$\begin{aligned}\left|bf(b)-af(a)-\int_a^b f(x)dx\right| &\leq (\ln x-\ln a)\,L^{1-\frac{1}{q}}\left(a^{\frac{2q}{q-1}},x^{\frac{2q}{q-1}}\right)A^{\frac{1}{q}}\left(\left|g\left(f'(a)\right)\right|^q,\left|g\left(f'(x)\right)\right|^q\right)\\ &\quad+(\ln b-\ln x)\,L^{1-\frac{1}{q}}\left(x^{\frac{2q}{q-1}},b^{\frac{2q}{q-1}}\right)A^{\frac{1}{q}}\left(\left|g\left(f'(x)\right)\right|^q,\left|g\left(f'(b)\right)\right|^q\right).\end{aligned}$$

Proof. From Lemma 1 and by using the Gg-convexity and well known the Hölder integral inequality, we obtain

$$\begin{aligned}&\left|bf(b)-af(a)-\int_a^b f(x)dx\right|\\ \leq\ &(\ln x-\ln a)\left(\int_0^1 a^{\frac{2q}{q-1}}\left(\frac{x}{a}\right)^{\frac{2qt}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f'\left(x^ta^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}\\ &+(\ln b-\ln x)\left(\int_0^1 b^{\frac{2q}{q-1}}\left(\frac{x}{b}\right)^{\frac{2qt}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|f'\left(x^tb^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}\\ \leq\ &(\ln x-\ln a)\left(\int_0^1 a^{\frac{2q}{q-1}}\left(\frac{x}{a}\right)^{\frac{2qt}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left[t\left|g\left(f'(x)\right)\right|^q+(1-t)\left|g\left(f'(a)\right)\right|^q\right]dt\right)^{\frac{1}{q}}\\ &+(\ln b-\ln x)\left(\int_0^1 b^{\frac{2q}{q-1}}\left(\frac{x}{b}\right)^{\frac{2qt}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left[t\left|g\left(f'(x)\right)\right|^q+(1-t)\left|g\left(f'(b)\right)\right|^q\right]dt\right)^{\frac{1}{q}}\\ =\ &(\ln x-\ln a)\,L^{1-\frac{1}{q}}\left(a^{\frac{2q}{q-1}},x^{\frac{2q}{q-1}}\right)A^{\frac{1}{q}}\left(\left|g\left(f'(a)\right)\right|^q,\left|g\left(f'(x)\right)\right|^q\right)\\ &+(\ln b-\ln x)\,L^{1-\frac{1}{q}}\left(x^{\frac{2q}{q-1}},b^{\frac{2q}{q-1}}\right)A^{\frac{1}{q}}\left(\left|g\left(f'(x)\right)\right|^q,\left|g\left(f'(b)\right)\right|^q\right),\end{aligned}$$

where

$$\begin{aligned}\int_0^1 a^{\frac{2q}{q-1}}\left(\frac{x}{a}\right)^{\frac{2qt}{q-1}}dt &= \frac{x^{\frac{2q}{q-1}}-a^{\frac{2q}{q-1}}}{\ln x^{\frac{2q}{q-1}}-\ln a^{\frac{2q}{q-1}}}=L\left(a^{\frac{2q}{q-1}},x^{\frac{2q}{q-1}}\right)\\ \int_0^1 b^{\frac{2q}{q-1}}\left(\frac{x}{b}\right)^{\frac{2qt}{q-1}}dt &= \frac{b^{\frac{2q}{q-1}}-x^{\frac{2q}{q-1}}}{\ln b^{\frac{2q}{q-1}}-\ln x^{\frac{2q}{q-1}}}=L\left(x^{\frac{2q}{q-1}},b^{\frac{2q}{q-1}}\right)\end{aligned}$$

This completes the proof of the theorem. □

Remark 7. *If we take $g(x) = x$ in the Theorem 10, then we obtain the following inequality for the GA-convex functions.*

$$\begin{aligned}\left|bf(b)-af(a)-\int_a^b f(x)dx\right| &\leq (\ln x-\ln a)\,L^{1-\frac{1}{q}}\left(a^{\frac{2q}{q-1}},x^{\frac{2q}{q-1}}\right)A^{\frac{1}{q}}\left(\left|f'(a)\right|^q,\left|f'(x)\right|^q\right)\\ &+\ (\ln b-\ln x)\,L^{1-\frac{1}{q}}\left(x^{\frac{2q}{q-1}},b^{\frac{2q}{q-1}}\right)A^{\frac{1}{q}}\left(\left|f'(x)\right|^q,\left|f'(b)\right|^q\right).\end{aligned}$$

Theorem 11. *Let $f : I \subset (0,\infty) \to \mathbf{R}$ be a differentiable mapping on I° and $a,b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. If $|f'(x)|^q$ is Gg-convex function on the interval $[a,b]$, then* *the following inequality holds for all $x \in [a,b]$ and $q \geq 1$:*

$$\left| bf(b) - af(a) - \int_a^b f(x)dx \right| \leq (\ln x - \ln a)^{1-\frac{1}{q}} \left(\frac{1}{2q}\right)^{\frac{1}{q}}$$

$$\times \left(|g(f'(x))|^q \left[x^{2q} - L\left(a^{2q}, x^{2q}\right)\right] + |g(f'(a))|^q \left[x^{2q} - a^{2q} + L\left(a^{2q}, x^{2q}\right)\right]\right)^{\frac{1}{q}} + (\ln b - \ln x)^{1-\frac{1}{q}} \left(\frac{1}{2q}\right)^{\frac{1}{q}}$$

$$\times \left(|g(f'(x))|^q \left[L\left(x^{2q}, b^{2q}\right) - x^{2q}\right] + |g(f'(b))|^q \left[b^{2q} - x^{2q} - L\left(x^{2q}, b^{2q}\right)\right]\right)^{\frac{1}{q}}.$$

Proof. By a similar argument to the proof of teh Theorem 10, since $|f'(x)|^q$ is Gg-convex function on the interval $[a,b]$, from Lemma 1 and by using a different version of Hölder integral inequality, we obtain

$$\begin{aligned}
& \left| bf(b) - af(a) - \int_a^b f(x)dx \right| \qquad (16) \\
\leq\ & (\ln x - \ln a) \int_0^1 x^{2t} a^{2(1-t)} \left| f'\left(x^t a^{1-t}\right)\right| dt + (\ln b - \ln x) \int_0^1 x^{2t} b^{2(1-t)} \left| f'\left(x^t b^{1-t}\right)\right| dt \\
\leq\ & (\ln x - \ln a) \left(\int_0^1 dt\right)^{1-\frac{1}{q}} \left(\int_0^1 x^{2t} a^{2(1-t)} \left| f'\left(x^t a^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}} \\
& + (\ln b - \ln x) \left(\int_0^1 dt\right)^{1-\frac{1}{q}} \left(\int_0^1 x^{2t} b^{2(1-t)} \left| f'\left(x^t b^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & (\ln x - \ln a) \left(\int_0^1 x^{2qt} a^{2q(1-t)} \left[t\, |g(f'(x))|^q + (1-t)\, |g(f'(a))|^q\right] dt\right)^{\frac{1}{q}} \\
& + (\ln b - \ln x) \left(\int_0^1 x^{2qt} b^{2q(1-t)} \left[t\, |g(f'(x))|^q + (1-t)\, |g(f'(b))|^q\right] dt\right)^{\frac{1}{q}} \\
=\ & (\ln x - \ln a) \left(|g(f'(x))|^q \int_0^1 t x^{2qt} a^{2q(1-t)} dt + |g(f'(a))|^q \int_0^1 (1-t) x^{2qt} a^{2q(1-t)} dt \right)^{\frac{1}{q}} \\
& + (\ln b - \ln x) \left(|g(f'(x))|^q \int_0^1 t x^{2qt} b^{2q(1-t)} dt + |g(f'(b))|^q \int_0^1 (1-t) x^{2qt} b^{2q(1-t)} dt \right)^{\frac{1}{q}}
\end{aligned}$$

where

$$\int_0^1 t x^{2qt} a^{2q(1-t)} dt = \frac{x^{2q} - L\left(a^{2q}, x^{2q}\right)}{2q(\ln x - \ln a)} \qquad (17)$$

$$\int_0^1 (1-t) x^{2qt} a^{2q(1-t)} dt = \frac{x^{2q} - a^{2q} + L\left(a^{2q}, x^{2q}\right)}{2q(\ln x - \ln a)} \qquad (18)$$

$$\int_0^1 t x^{2qt} b^{2q(1-t)} dt = \frac{L\left(x^{2q}, b^{2q}\right) - x^{2q}}{2q(\ln b - \ln x)} \qquad (19)$$

$$\int_0^1 (1-t) x^{2qt} b^{2q(1-t)} dt = \frac{b^{2q} - x^{2q} - L\left(x^{2q}, b^{2q}\right)}{2q(\ln b - \ln x)} \qquad (20)$$

By using the identities (17)-(20) in the inequality (16), the required result is obtained. This completes the proof of the theorem. □

Remark 8. *If we take $g(x) = x$ in the Theorem 11, then we obtain the following inequality for the GA-convex functions.*

$$\left| bf(b) - af(a) - \int_a^b f(x)dx \right| \leq (\ln x - \ln a)^{1-\frac{1}{q}} \left(\frac{1}{2q} \right)^{\frac{1}{q}}$$

$$+ \left(|f'(x)|^q \left[x^{2q} - L\left(a^{2q}, x^{2q}\right) \right] + |f'(a)|^q \left[x^{2q} - a^{2q} + L\left(a^{2q}, x^{2q}\right) \right] \right)^{\frac{1}{q}}$$

$$+ (\ln b - \ln x)^{1-\frac{1}{q}} \left(\frac{1}{2q} \right)^{\frac{1}{q}} \left(|f'(x)|^q \left[L\left(x^{2q}, b^{2q}\right) - x^{2q} \right] + |f'(b)|^q \left[b^{2q} - x^{2q} - L\left(x^{2q}, b^{2q}\right) \right] \right)^{\frac{1}{q}}.$$

Corollary 2. *If we take $q = 1$ in the Theorem 11, we have the following inequality:*

$$\left| bf(b) - af(a) - \int_a^b f(x)dx \right| \leq \frac{L\left(x^2, b^2\right) - L\left(a^2, x^2\right)}{2} |g\left(f'(x)\right)|$$

$$+ \frac{1}{2} \left(|g\left(f'(a)\right)| \left[x^2 - a^2 + L\left(a^2, x^2\right) \right] + |g\left(f'(b)\right)| \left[b^2 - x^2 - L\left(x^2, b^2\right) \right] \right).$$

This inequality coincides with the inequality (7) in Theorem 8.

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