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# BOUNDS FOR THE NORMALIZED DETERMINANT OF HADAMARD PRODUCT OF TWO POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators $A$ on a Hilbert space $H$ and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp \langle \ln Ax, x \rangle$. In this paper we obtain upper and lower bounds for the determinant $\Delta_x (A \circ B)$ of the Hadamard product of two operators under some natural assumptions such as $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$, where $m_i, M_i$ $(i = 1, 2)$ are constants.

## 1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space $H$, and $I$ stands for the identity operator on $H$. An operator $A$ in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that $A$ is positive and invertible. For a pair $A$, $B$ of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [7], [8], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators $A$ on a Hilbert space $H$ and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x \rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [7].

For each unit vector $x \in H$, see also [12], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x \rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x \rangle$;
(iii) *continuous mean*: $\langle A^p x, x \rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x \rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting $A$ and $B$;
(viii) *Ky Fan type inequality*: $\Delta_x((1 - \alpha) A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers $a$, $b$ by

$$L(a, b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ a \text{ if } b = a. \end{cases}$$

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In [7] the authors obtained the following additive reverse inequality for the operator $A$ which satisfy the condition $0 < mI \leq A \leq MI$, where $m, M$ are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m, M)\left[\ln L(m, M) + \frac{M\ln m - m\ln M}{M - m} - 1\right] \tag{1}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0, 1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \tag{2}$$

with equality if and only if $a = b$. The inequality (2) is also called *$\nu$-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [15]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

In [8], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (4) for $A^{-1}$ we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right) \tag{5}$$

for $x \in H$, $\|x\| = 1$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{6}$$

The function $K$ is decreasing on $(0, 1)$ and increasing on $[1, \infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^r\left(\frac{b}{a}\right)a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \leq K^R\left(\frac{b}{a}\right)a^{1-\nu}b^{\nu} \tag{7}$$

where $a$, $b > 0$, $\nu \in [0, 1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (7) was obtained by Zuo et al. in [19] while the second by Liao et al. [14].

Recall the *geometric operator mean* for the positive operators $A$, $B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

where $t \in [0, 1]$ and

$$A\# B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B)\otimes(B\#A) = (A\otimes B)\#(B\otimes A).$$

In 2007, S. Wada [17] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B)\otimes(A\#B) &\le \frac{1}{2}\left[(A\#_{\alpha}B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_{\alpha}B)\right] \\ &\le \frac{1}{2}(A\otimes B+B\otimes A)\end{aligned} \tag{8}$$

for $A,\ B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of $A$ and $B$ in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\langle (A\circ B)e_j, e_j\rangle = \langle Ae_j, e_j\rangle\langle Be_j, e_j\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space $H$.

It is known that, see [5], we have the representation

$$A\circ B = \mathcal{U}^{*}(A\otimes B)\mathcal{U} \tag{9}$$

where $\mathcal{U}: H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If $f$ *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [11, p. 173]

$$f(A\circ B)\ge(\le)\, f(A)\circ f(B) \text{ for all } A,\ B\ge 0. \tag{10}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\le\left(\frac{A+B}{2}\right)\circ 1 \text{ for } A,\ B\ge 0$$

and *Fiedler inequality*

$$A\circ A^{-1}\ge 1 \text{ for } A>0. \tag{11}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\le\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2} \text{ for } A,\ B\ge 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B\le\left(A^2\circ B^2\right)^{1/2} \text{ for } A,\ B\ge 0.$$

It has been shown in [13] that $\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2}$ and $\left(A^2\circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices $A$ and $B$.

Motivated by the above results, we establish in this paper the following upper and lower bounds for the determinant $\Delta_x (A \circ B)$

$$
\begin{aligned}
&(m_1 m_2)^{\frac{M_1 M_2 - \langle (A\circ B)x,x\rangle}{M_1 M_2 - m_1 m_2}} (M_1 M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1 m_2}{M_1 M_2 - m_1 m_2}} \\
&\leq K\left(\frac{M_1 M_2}{m_1 m_2}\right)^{\left(\frac{1}{2} - \frac{1}{M_1 M_2 - m_1 m_2}\left\langle \mathcal{U}^*\left(\left|A\otimes B - \frac{m_1 m_2 + M_1 M_2}{2}\right|\right)\mathcal{U}x,x\right\rangle\right)} \\
&\times (m_1 m_2)^{\frac{M_1 M_2 - \langle (A\circ B)x,x\rangle}{M_1 M_2 - m_1 m_2}} (M_1 M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1 m_2}{M_1 M_2 - m_1 m_2}} \\
&\leq \exp\left[\langle \mathcal{U}^* (\ln (A\otimes B))\,\mathcal{U}x,x\rangle\right] \\
&\leq \Delta_x (A\circ B) \\
&\leq K\left(\frac{M_1 M_2}{m_1 m_2}\right)^{\left(\frac{1}{2} + \frac{1}{M_1 M_2 - m_1 m_2}\left\langle \left|A\circ B - \frac{m_1 m_2 + M_1 M_2}{2}\right| x,x\right\rangle\right)} \\
&\times (m_1 m_2)^{\frac{M_1 M_2 - \langle (A\circ B)x,x\rangle}{M_1 M_2 - m_1 m_2}} (M_1 M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1 m_2}{M_1 M_2 - m_1 m_2}} \\
&\leq K\left(\frac{M_1 M_2}{m_1 m_2}\right) (m_1 m_2)^{\frac{M_1 M_2 - \langle (A\circ B)x,x\rangle}{M_1 M_2 - m_1 m_2}} (M_1 M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1 m_2}{M_1 M_2 - m_1 m_2}} \\
&\leq K\left(\frac{M_1 M_2}{m_1 m_2}\right) \exp\langle \mathcal{U}^* (\ln (A\otimes B))\,\mathcal{U}x,x\rangle,
\end{aligned}
$$

provided that $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$.

## 2. Main Results

We start to the following operator inequalities involved positive operators and positive linear maps:

**Theorem 1.** *Assume that the selfadjoint operator $P$ satisfies the condition $0 < m \leq P \leq M$ for some constants, $m$, $M$ and $\Phi$ a unital positive linear map from $B(H)$ into $B(K)$. Then*

$$
\begin{aligned}
&\ln m \frac{M - \Phi(P)}{M - m} + \ln M \frac{\Phi(P) - m}{M - m} \qquad (12)\\
&\leq \ln K\left(\frac{M}{m}\right)\left(\frac{1}{2} - \frac{1}{M - m}\left|\Phi(P) - \frac{m + M}{2}\right|\right) \\
&+ \ln m \frac{M - \Phi(P)}{M - m} + \ln M \frac{\Phi(P) - m}{M - m} \\
&\leq \ln \Phi(P) \\
&\leq \ln K\left(\frac{M}{m}\right)\left(\frac{1}{2} + \frac{1}{M - m}\left|\Phi(P) - \frac{m + M}{2}\right|\right) \\
&+ \ln m \frac{M - \Phi(P)}{M - m} + \ln M \frac{\Phi(P) - m}{M - m}
\end{aligned}
$$

*and*

$$\begin{aligned}
&\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m} \qquad (13)\\
&\le \ln K\left(\frac{M}{m}\right)\left(\frac{1}{2}-\frac{1}{M-m}\Phi\left(\left|P-\frac{m+M}{2}\right|\right)\right)\\
&+\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}\\
&\le \Phi(\ln P)\\
&\le \ln K\left(\frac{M}{m}\right)\left(\frac{1}{2}+\frac{1}{M-m}\Phi\left(\left|T-\frac{m+M}{2}\right|\right)\right)\\
&+\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}.
\end{aligned}$$

*Proof.* Assume that $0<a<b$. If we take $\nu=\frac{t-a}{b-a}\in[0,1]$ for $t\in[a,b]$ and observe that

$$r=\min\left\{\frac{t-a}{b-a},\frac{b-t}{b-a}\right\}=\frac{1}{2}-\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|,$$

$$R=\max\left\{\frac{t-a}{b-a},\frac{b-t}{b-a}\right\}=\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|$$

and

$$(1-\nu)a+\nu b=\frac{b-t}{b-a}a+\frac{t-a}{b-a}b=t.$$

By utilizing (7) we get

$$K^{\frac{1}{2}-\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|}\left(\frac{b}{a}\right)a^{\frac{b-t}{b-a}}b^{\frac{t-a}{b-a}}\le t\le K^{\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|}\left(\frac{b}{a}\right)a^{\frac{b-t}{b-a}}b^{\frac{t-a}{b-a}} \qquad (14)$$

for all $t\in[a,b]$.

If we take the logarithm in (14), then we get

$$\begin{aligned}
&\ln a\frac{b-t}{b-a}+\ln b\frac{t-a}{b-a} \qquad (15)\\
&\le \ln K\left(\frac{b}{a}\right)\left(\frac{1}{2}-\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right)+\ln a\frac{b-t}{b-a}+\ln b\frac{t-a}{b-a}\\
&\le \ln t\\
&\le \ln K\left(\frac{b}{a}\right)\left(\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right)+\ln a\frac{b-t}{b-a}+\ln b\frac{t-a}{b-a}
\end{aligned}$$

for all $t\in[a,b]$.

By utilizing the continuous functional calculus for selfadjoint operators $T$ with spectra $\lim Sp(T)\subseteq[a,b]$, we obtain from (15) that

$$\begin{aligned}
&\ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a} \qquad (16)\\
&\le \ln K\left(\frac{b}{a}\right)\left(\frac{1}{2}-\frac{1}{b-a}\left|T-\frac{a+b}{2}\right|\right)+\ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a}\\
&\le \ln T\\
&\le \ln K\left(\frac{b}{a}\right)\left(\frac{1}{2}+\frac{1}{b-a}\left|T-\frac{a+b}{2}\right|\right)+\ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a}.
\end{aligned}$$

Now if $0 < m \le P \le M$, then $0 < m \le \Phi(P) \le M$ and by (16) we get for $T = \Phi(P)$, $a = m$ and $b = M$ the inequality (12). If we take $T = P$, $a = m$ and $b = M$ in (16) and then apply $\Phi$ we also obtain (13). $\square$

**Corollary 1.** *With the assumptions of Theorem 1 we have the chain of inequalities*

$$\begin{aligned}
&\ln m \frac{M - \Phi(P)}{M - m} + \ln M \frac{\Phi(P) - m}{M - m} \qquad (17)\\
&\le \ln K\left(\frac{M}{m}\right)\left(\frac{1}{2} - \frac{1}{M - m}\Phi\left(\left|P - \frac{m + M}{2}\right|\right)\right)\\
&+ \ln m \frac{M - \Phi(P)}{M - m} + \ln M \frac{\Phi(P) - m}{M - m}\\
&\le \Phi(\ln P) \le \ln \Phi(P)\\
&\le \ln K\left(\frac{M}{m}\right)\left(\frac{1}{2} + \frac{1}{M - m}\left|\Phi(P) - \frac{m + M}{2}\right|\right)\\
&+ \ln m \frac{M - \Phi(P)}{M - m} + \ln M \frac{\Phi(P) - m}{M - m}\\
&\le \ln K\left(\frac{M}{m}\right) + \ln m \frac{M - \Phi(P)}{M - m} + \ln M \frac{\Phi(P) - m}{M - m}\\
&\le \ln K\left(\frac{M}{m}\right) + \Phi(\ln P).
\end{aligned}$$

*Proof.* Third inequality follows by Jensen's operator inequality for the operator concave function $\ln$. The fifth inequality follows by the fact that

$$\left|\Phi(P) - \frac{m + M}{2}\right| \le \frac{1}{2}(M - m),$$

while the last inequality follows by the fact that

$$\ln m \frac{M - \Phi(P)}{M - m} + \ln M \frac{\Phi(P) - m}{M - m} \le \Phi(\ln P)$$

from the first part of (17). $\square$

**Theorem 2.** *Assume that* $0 < m_1 \leq A \leq M_1$ *and* $0 < m_2 \leq B \leq M_2$, *then*

$$
\begin{aligned}
&(m_1m_2)^{\frac{M_1M_2-\langle (A\circ B)x,x\rangle}{M_1M_2-m_1m_2}}(M_1M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2-m_1m_2}} \qquad (18)\\
&\leq K\left(\frac{M_1M_2}{m_1m_2}\right)^{\left(\frac{1}{2}-\frac{1}{M_1M_2-m_1m_2}\left\langle \mathcal{U}^*\left(\left|A\otimes B-\frac{m_1m_2+M_1M_2}{2}\right|\right)\mathcal{U}x,x\right\rangle\right)}\\
&\times (m_1m_2)^{\frac{M_1M_2-\langle (A\circ B)x,x\rangle}{M_1M_2-m_1m_2}}(M_1M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2-m_1m_2}}\\
&\leq \exp\left[\langle \mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}x,x\rangle\right]\\
&\leq \Delta_x\left(A\circ B\right)\\
&\leq K\left(\frac{M_1M_2}{m_1m_2}\right)^{\left(\frac{1}{2}+\frac{1}{M_1M_2-m_1m_2}\left\langle \left|A\circ B-\frac{m_1m_2+M_1M_2}{2}\right|x,x\right\rangle\right)}\\
&\times (m_1m_2)^{\frac{M_1M_2-\langle (A\circ B)x,x\rangle}{M_1M_2-m_1m_2}}(M_1M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2-m_1m_2}}\\
&\leq K\left(\frac{M_1M_2}{m_1m_2}\right)(m_1m_2)^{\frac{M_1M_2-\langle (A\circ B)x,x\rangle}{M_1M_2-m_1m_2}}(M_1M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2-m_1m_2}}\\
&\leq K\left(\frac{M_1M_2}{m_1m_2}\right)\exp\left\langle \mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}x,x\right\rangle,
\end{aligned}
$$

*for* $x \in H$, $\|x\| = 1$.

*Proof.* Since $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$, then $0 < m_1m_2 \leq P = A \otimes B \leq M_1M_2$. From (17) for $m = m_1m_2$, $M = M_1M_2$, $\Phi(P) = \mathcal{U}^*(A\otimes B)\mathcal{U} = A \circ B$ we get

$$
\begin{aligned}
&\ln\left(m_1m_2\right)\frac{M_1M_2-A\circ B}{M_1M_2-m_1m_2}+\ln\left(M_1M_2\right)\frac{A\circ B-m_1m_2}{M_1M_2-m_1m_2} \qquad (19)\\
&\leq \ln K\left(\frac{M_1M_2}{m_1m_2}\right)\\
&\times\left(\frac{1}{2}-\frac{1}{M_1M_2-m_1m_2}\mathcal{U}^*\left(\left|A\otimes B-\frac{m_1m_2+M_1M_2}{2}\right|\right)\mathcal{U}\right)\\
&+\ln\left(m_1m_2\right)\frac{M_1M_2-A\circ B}{M_1M_2-m_1m_2}+\ln\left(M_1M_2\right)\frac{A\circ B-m_1m_2}{M_1M_2-m_1m_2}\\
&\leq \mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}\leq \ln\left(A\circ B\right)\\
&\leq \ln K\left(\frac{M_1M_2}{m_1m_2}\right)\left(\frac{1}{2}+\frac{1}{M_1M_2-m_1m_2}\left|A\circ B-\frac{m_1m_2+M_1M_2}{2}\right|\right)\\
&+\ln\left(m_1m_2\right)\frac{M_1M_2-A\circ B}{M_1M_2-m_1m_2}+\ln\left(M_1M_2\right)\frac{A\circ B-m_1m_2}{M_1M_2-m_1m_2}\\
&\leq \ln K\left(\frac{M_1M_2}{m_1m_2}\right)\\
&+\ln\left(m_1m_2\right)\frac{M_1M_2-A\circ B}{M_1M_2-m_1m_2}+\ln\left(M_1M_2\right)\frac{A\circ B-m_1m_2}{M_1M_2-m_1m_2}\\
&\leq \ln K\left(\frac{M_1M_2}{m_1m_2}\right)+\mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}.
\end{aligned}
$$

If we take the inner product for $x \in H$, $\|x\| = 1$, then we get

$$\begin{aligned}
&\ln(m_1 m_2)\frac{M_1M_2 - \langle (A\circ B)x,x\rangle}{M_1M_2 - m_1m_2} + \ln(M_1M_2)\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2 - m_1m_2}\\
&\le \ln K\left(\frac{M_1M_2}{m_1m_2}\right)\\
&\times\left(\frac{1}{2} - \frac{1}{M_1M_2 - m_1m_2}\left\langle \mathcal{U}^*\left(\left|A\otimes B - \frac{m_1m_2 + M_1M_2}{2}\right|\right)\mathcal{U}x,x\right\rangle\right)\\
&+\ln(m_1m_2)\frac{M_1M_2 - A\circ B}{M_1M_2 - m_1m_2} + \ln(M_1M_2)\frac{A\circ B - m_1m_2}{M_1M_2 - m_1m_2}\\
&\le \langle \mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x,x\rangle \le \langle \ln(A\circ B)x,x\rangle\\
&\le \ln K\left(\frac{M_1M_2}{m_1m_2}\right)\\
&\times\left(\frac{1}{2} + \frac{1}{M_1M_2 - m_1m_2}\left\langle \left|A\circ B - \frac{m_1m_2 + M_1M_2}{2}\right|x,x\right\rangle\right)\\
&+\ln(m_1m_2)\frac{M_1M_2 - \langle (A\circ B)x,x\rangle}{M_1M_2 - m_1m_2} + \ln(M_1M_2)\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2 - m_1m_2}\\
&\le \ln K\left(\frac{M_1M_2}{m_1m_2}\right)\\
&+\ln(m_1m_2)\frac{M_1M_2 - \langle (A\circ B)x,x\rangle}{M_1M_2 - m_1m_2} + \ln(M_1M_2)\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2 - m_1m_2}\\
&\le \ln K\left(\frac{M_1M_2}{m_1m_2}\right) + \langle \mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x,x\rangle,
\end{aligned}$$

namely

$$\begin{aligned}
&\ln\left[(m_1m_2)^{\frac{M_1M_2 - \langle (A\circ B)x,x\rangle}{M_1M_2 - m_1m_2}}(M_1M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2 - m_1m_2}}\right] \qquad (20)\\
&\le \ln\left[K\left(\frac{M_1M_2}{m_1m_2}\right)\right]^{\left(\frac{1}{2} - \frac{1}{M_1M_2 - m_1m_2}\left\langle \mathcal{U}^*\left(\left|A\otimes B - \frac{m_1m_2 + M_1M_2}{2}\right|\right)\mathcal{U}x,x\right\rangle\right)}\\
&+\ln\left[(m_1m_2)^{\frac{M_1M_2 - \langle (A\circ B)x,x\rangle}{M_1M_2 - m_1m_2}}(M_1M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2 - m_1m_2}}\right]\\
&\le \langle \mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x,x\rangle \le \langle \ln(A\circ B)x,x\rangle\\
&\le \ln\left[K\left(\frac{M_1M_2}{m_1m_2}\right)\right]^{\left(\frac{1}{2} + \frac{1}{M_1M_2 - m_1m_2}\left\langle \left|A\circ B - \frac{m_1m_2 + M_1M_2}{2}\right|x,x\right\rangle\right)}\\
&+\ln(m_1m_2)\frac{M_1M_2 - \langle (A\circ B)x,x\rangle}{M_1M_2 - m_1m_2} + \ln(M_1M_2)\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2 - m_1m_2}\\
&\le \ln K\left(\frac{M_1M_2}{m_1m_2}\right)\\
&+\ln\left[(m_1m_2)^{\frac{M_1M_2 - \langle (A\circ B)x,x\rangle}{M_1M_2 - m_1m_2}}(M_1M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2 - m_1m_2}}\right]\\
&\le \ln K\left(\frac{M_1M_2}{m_1m_2}\right) + \langle \mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x,x\rangle,
\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

If we take the exponential in (20), then we get (18). □

## 3. Connection to Oppenheim's Inequalities

In the finite dimensional case, if we consider the matrices $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{M}_n(\mathbb{C})$, then $A \circ B$ has an associated matrix $A \circ B = (a_{ij}b_{ij})$ in $\mathbb{M}_n(\mathbb{C})$.

Recall Hadamard determinant inequality [18, p. 218] for $A \geq 0$

$$\det A \leq \det(A \circ 1) \ \ (= \prod_{i=1}^{n} a_{ii})$$

and Oppenheim's inequality [18, p. 242] for $A,\ B \geq 0$

$$\det A \det B \leq \det(A \circ B) \leq \det(A \circ 1)\det(B \circ 1) \ \ \left(= \prod_{i=1}^{n} a_{ii}b_{ii}\right).$$

In the recent paper [12] the authors obtained the following Oppenheim's type inequalities

$$\frac{1}{S(h_1)\,S(h_2)} \leq \frac{\Delta_x(A \circ B)}{\Delta_x(A \circ 1)\,\Delta_x(1 \circ B)} \leq S(h_1 h_2) \tag{21}$$

for $x \in H$, $\|x\| = 1$, provided that $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$.

We have the following inequalities:

**Proposition 1.** *With the assumptions of Theorem 2 we have the determinant inequalities*

$$\frac{1}{K(h_1)\,K(h_2)} \leq \frac{\Delta_x(A \circ B)}{\Delta_x(A \circ 1)\,\Delta_x(1 \circ B)} \leq K(h_1 h_2) \tag{22}$$

*where* $h_1 = \frac{M_1}{m_1} > 1$, $h_2 = \frac{M_2}{m_2} > 1$.

*Proof.* By the properties of the tensorial product, we have that

$$A \otimes B = (A \otimes 1)(1 \otimes B)$$

where $A \otimes 1$ and $1 \otimes B$ are commutative operators.

Therefore

$$\ln(A \otimes B) = \ln[(A \otimes 1)(1 \otimes B)] = \ln(A \otimes 1) + \ln(1 \otimes B)$$

and

$$\begin{aligned} \mathcal{U}^*(\ln(A \otimes B))\,\mathcal{U} &= \mathcal{U}^*[\ln(A \otimes 1) + \ln(1 \otimes B)]\,\mathcal{U} \\ &= \mathcal{U}^*(\ln(A \otimes 1))\,\mathcal{U} + \mathcal{U}^*(\ln(1 \otimes B))\,\mathcal{U}. \end{aligned}$$

Using Jensen's operator inequality for the operator concave function ln, we also have

$$\mathcal{U}^*(\ln(A \otimes 1))\,\mathcal{U} \leq \ln(\mathcal{U}^*(A \otimes 1)\,\mathcal{U}) = \ln(A \circ 1)$$

and

$$\mathcal{U}^*(\ln(1 \otimes B))\,\mathcal{U} \leq \ln(\mathcal{U}^*((1 \otimes B))\,\mathcal{U}) = \ln(1 \circ B).$$

These imply for $x \in H$, $\|x\| = 1$ that

$$\begin{aligned} \exp\langle \mathcal{U}^*(\ln(A \otimes B))\,\mathcal{U}x, x\rangle &\leq \exp[\langle \ln(A \circ 1)\,x, x\rangle + \langle \ln(1 \circ B)\,x, x\rangle] \\ &= \exp[\langle \ln(A \circ 1)\,x, x\rangle]\exp[\langle \ln(1 \circ B)\,x, x\rangle] \\ &= \Delta_x(A \circ 1)\,\Delta_x(1 \circ B) \end{aligned}$$

and by the second part of (18) we derive the second inequality in (22).

From (17) we have

$$\ln \Phi(P) \leq \ln K\left(\frac{M}{m}\right) + \Phi(\ln P)$$

provided that $0 < m \le P \le M$.

Now, if we take in this inequality $0 < m_1 \le P = A \otimes 1 \le M_1$, then we get for $\Phi(P) = \mathcal{U}^*(A \otimes 1)\mathcal{U} = A \circ 1$ that

$$\ln(A \circ 1) \le \ln K\left(\frac{M_1}{m_1}\right) + \mathcal{U}^*(\ln(A \otimes 1))\mathcal{U}$$

while for $0 < m_2 \le P = 1 \otimes B \le M_2$

$$\ln(1 \circ B) \le \ln K\left(\frac{M_2}{m_2}\right) + \mathcal{U}^*(\ln(1 \otimes B))\mathcal{U},$$

which gives, by addition, that

$$\begin{aligned}&\ln(A \circ 1) + \ln(1 \circ B) - \ln\left[K\left(\frac{M_1}{m_1}\right)K\left(\frac{M_2}{m_2}\right)\right]\\ &\le \mathcal{U}^*(\ln(A \otimes 1))\mathcal{U} + \mathcal{U}^*(\ln(1 \otimes B))\mathcal{U} = \mathcal{U}^*(\ln(A \otimes B))\mathcal{U}.\end{aligned}$$

By taking the inner product for $x \in H$, $\|x\| = 1$ we get that

$$\begin{aligned}&\langle \ln(A \circ 1)x, x\rangle + \langle \ln(1 \circ B)x, x\rangle - \ln\left[K\left(\frac{M_1}{m_1}\right)K\left(\frac{M_2}{m_2}\right)\right]\\ &\le \langle \mathcal{U}^*(\ln(A \otimes B))\mathcal{U}x, x\rangle\end{aligned}$$

and by taking the exponential, we derive

$$\frac{\exp\langle (A \circ 1)x, x\rangle \exp\langle \ln(1 \circ B)x, x\rangle}{K(h_1)K(h_2)} \le \exp\langle \mathcal{U}^*(\ln(A \otimes B))\mathcal{U}x, x\rangle$$

for $x \in H$, $\|x\| = 1$ and by the third inequality in (18) we obtain the first part of (22). □

**Remark 1.** *Since $K(h) \ge S(h)$ for $h > 0$ (see for instance* [10, p. 4]*), then the bounds for the ratio*

$$\frac{\Delta_x(A \circ B)}{\Delta_x(A \circ 1)\Delta_x(1 \circ B)}$$

*provided by (21) are better than the ones from (22).*

**Lemma 1.** *For all $h_1, h_2 \in (1, \infty)$ or $h_1, h_2 \in (0, 1)$ we have*

$$K(h_1 h_2) \ge K(h_1)K(h_2). \tag{23}$$

*If $h_1 \in (1, \infty)$ and $h_2 \in (0, 1)$ or $h_2 \in (1, \infty)$ and $h_1 \in (0, 1)$ then the sign of inequality reverses in (23).*

*Proof.* We have for $h_1, h_2 \in (0, \infty)$ that

$$\begin{aligned}K(h_1h_2) - K(h_1)K(h_2) &= \frac{(h_1h_2+1)^2}{4h_1h_2} - \frac{(h_1+1)^2}{4h_1}\frac{(h_2+1)^2}{4h_2}\\ &= \frac{1}{16h_1h_2}\left[4(h_1h_2+1)^2 - (h_1+1)^2(h_2+1)^2\right]\\ &= \frac{1}{16h_1h_2}\left[2(h_1h_2+1) + (h_1+1)(h_2+1)\right]\\ &\times\left[2(h_1h_2+1) - (h_1+1)(h_2+1)\right].\end{aligned}$$

Observe that

$$\begin{aligned}2(h_1h_2+1) - (h_1+1)(h_2+1) &= 2h_1h_2 + 2 - h_1h_2 - h_1 - h_2 - 1\\ &= h_1h_2 + 1 - h_1 - h_2 = (h_1-1)(h_2-1),\end{aligned}$$

which shows that the sign of $K(h_1h_2) - K(h_1)K(h_2)$ is the same with the one for $(h_1-1)(h_2-1)$, and this proves the lemma. □

**Corollary 2.** *With the assumptions of Theorem 2 we have the deteminant inequalities*

$$\frac{1}{K(h_1h_2)} \le \frac{\Delta_x(A \circ B)}{\Delta_x(A \circ 1)\Delta_x(1 \circ B)} \le K(h_1h_2). \tag{24}$$

The proof follows by (22) and (23).

## References


[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Lin. Alg. & Appl.* **26** (1979), 203-241.

[2] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, *Proc. Amer. Math. Soc.* **128** (2000), No. 7, 2075-2084.

[3] J. S. Aujila and H. L. Vasudeva, Inequalities involving Hadamard product and operator means, *Math. Japon.* **42** (1995), 265-272.

[4] A. Korányi. On some classes of analytic functions of several variables. *Trans. Amer. Math. Soc.,* **101** (1961), 520–554.

[5] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. *Math. Jpn.* **41** (1995), 531-535

[6] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[7] J. I. Fujii and Y. Seo, Determinant for positive operators, *Sci. Math.,* **1** (1998), 153–156.

[8] J. I. Fujii, S. Izumino and Y. Seo, Determinant for positive operators and Specht's Theorem, *Sci. Math.*, **1** (1998), 307–310.

[9] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** (2012), 46–49.

[10] S. Furuichi, Note on constants appearing in refined Young inequalities, Journal of Inequalities & Special Functions, 2019, Vol. 10 Issue 3, p1-8. 8p.

[11] T. Furuta, J. Mičić-Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Element, Croatia.

[12] S. Hiramatsu and Y. Seo, Determinant for positive operators and Oppenheim's inequality, *J. Math. Inequal.*, Volume 15 (2021), Number 4, 1637–1645.

[13] K. Kitamura and Y. Seo, Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities, *Scient. Math.* **1** (1998), No. 2, 237-241.

[14] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.

[15] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74** (1960), pp. 91-98.

[16] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.

[17] S. Wada, On some refinement of the Cauchy-Schwarz Inequality, *Lin. Alg. & Appl.* **420** (2007), 433-440.

[18] F. Zhang, *Matrix Theory - Basic Results and Techniques, Second edition*, Universitext, Springer, 2011.

[19] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.



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