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NOTES ON THE N-TH POWER OF GENERALIZED (S,T)-PELL AND (S,T)-PELL LUCAS MATRIX SEQUENCES

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Abstract. In this study, we define two matrix sequences special 2×2 matrices are defined with elements that are generalizations of Pell and Pell-Lucas sequences. Some formulas for the nth powers of these special matrix sequences are established by determinant and trace of these matrices. By these formulas, some properties for these generalized number sequences are demonsrated. The results are also usable for the classic Pell and Pell Lucas numbers, if we substitute for$s = t = 1$.

1. Introduction

The goal of this article is to find the nth power of the (s,t)-Pell and the (s,t)- Pell Lucas matrix sequences. Useful properties of the number sequences with elements that are obtained the elements of the generalized matrix sequences are established by the results found. In the literature, the researchers intended to find the nth power of the matrices by different methods. For example, in [1], Williams investigated the nth power of a $2x2$ matrix using a simple method. Laughlin studied some properties obtained by the nth power of some matrices in [2, 3]. Belbachir investigated linear sequences and powers of any square matrices in [4]. In [5], the authors investigated sum formulas and products for recurrent sequences. The authors derived combinatorial identities by using the trace, the determinant, and the nth power of a special matrix whose entries are Horadam numbers in [10]. Among special integer sequences, the Pell and Pell Lucas numbers have been studied extensively in the last decade years. The Pell numbers $\{p_n\}_{n=0}^{\infty}$ are given by $p_n = 2p_{n-1}+p_{n-2}$ for $n \geq 2$, beginning with the values $p_0 = 0,\ p_1 = 1$. The first elements of the sequences are $1, 2, 5, 12, 29, 70...$ respectively. The Pell Lucas numbers $\{q_n\}_{n=0}^{\infty}$ are defined by $q_n = 2q_{n-1} + q_{n-2}$ for $n \geq 2$, with the initial conditions $q_0 = 2,\ q_1 = 2$ in [6]. In [12], the author investigate the generalized (r, s) sequence in detail.

Definition 1. *Let s, t are real numbers. The $(s,t)-$Pell and the $(s,t)-$Pell Lucas sequence are demonstrated by the following second order relations*

$$\begin{aligned} p_n(s,t) &= 2sp_{n-1}(s,t) + tp_{n-2}(s,t), \quad (p_0(s,t) = 0,\ p_1(s,t) = 1) \\ q_n(s,t) &= 2sq_{n-1}(s,t) + tq_{n-2}(s,t), \quad (q_0(s,t) = 2,\ q_1(s,t) = 2s) \end{aligned} \tag{1}$$

where $s > 0,\ t \neq 0$ and $s^2 + t > 0$ [8, 11] .

Binet formula enables us to find any element of the number and matrix sequences easily. It can be clearly obtained from the roots r_1 and r_2 of the characteristic equation as the form $x^2 = 2sx + t$, where

$$r_1 = s + \sqrt{s^2 + t},\ r_2 = s - \sqrt{s^2 + 8t}. \tag{2}$$

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The Binet formula for the $(s,t)-$ Pell numbers is

$$p_n(s,t)=\frac{r_1^n-r_2^n}{r_1-r_2} \tag{3}$$

and the Binet formula for the$(s,t)-$ Pell Lucas numbers is given by

$$q_n(s,t)=r_1^n+r_2^n. \tag{4}$$

Definition 2. *In* [8, 11]*, Let $n\geq 1$, the matrix sequence with $(s,t)-$ Pell numbers is denoted by the following recurrent relation*

$$P_n(s,t)=2sP_{n-1}(s,t)+tP_{n-2}(s,t) \tag{5}$$

with the initial conditions $P_0(s,t)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $P_1(s,t)=\begin{pmatrix}2s&1\\t&0\end{pmatrix}$, *and the matrix sequence with $(s,t)-$Pell Lucas numbers is similarly given as*

$$Q_n(s,t)=2sQ_{n-1}(s,t)+tQ_{n-2}(s,t) \tag{6}$$

with the initial conditions $Q_0(s,t)=\begin{pmatrix}2s&2\\2t&-2s\end{pmatrix}$ *and* $Q_1(s,t)=\begin{pmatrix}4s^2+2t&2s\\2st&2t\end{pmatrix}$.

Proposition 1. *Some important properties for the $(s,t)-$Pell and the $(s,t)-$Pell Lucas matrix sequences are given as in* [8, 11]

a) $P_n(s,t)=\begin{pmatrix}p_{n+1}(s,t)&p_n(s,t)\\tp_n(s,t)&tp_{n-1}(s,t)\end{pmatrix}$..
b) $Q_n(s,t)=\begin{pmatrix}q_{n+1}(s,t)&q_n(s,t)\\tq_n(s,t)&tq_{n-1}(s,t)\end{pmatrix}$
c) $P_{m+n}(s,t)=P_m(s,t)P_n(s,t)$
d) $P_n(s,t)=P_1^n(s,t)$
e) $Q_{n+1}(s,t)=Q_1(s,t)P_n(s,t)$
f) $Q_n(s,t)=2sP_n(s,t)+2tP_{n-1}(s,t)$

In [1]*, Williams, gave a well-known formula that if* $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$*,then*

$$A^n=\begin{cases}\frac{r_1^n-r_2^n}{r_1-r_2}A-\frac{r_1^{n-1}-r_2^{n-1}}{r_1-r_2}I_2, & r_1\neq r_2\\ nr^{n-1}A-(n-1)\det(A)r^{n-2}I_2, & r_1=r_2\end{cases} \tag{7}$$

r_1,r_2 *being the roots of the equation*

$$r^2-(a+d)r+\det(A)=0.$$

Corollary 1. *The nth power of P_1 and Q_1 are*

$$P_1^n(s,t)=\frac{r_1^n-r_2^n}{r_1-r_2}\begin{pmatrix}2s&1\\t&0\end{pmatrix}-\frac{r_1^{n-1}-r_2^{n-1}}{r_1-r_2}I_2 \tag{8}$$

where $r_1=s+\sqrt{s^2+t}$, $r_2=s-\sqrt{s^2+t}$

$$Q_1^n(s,t)=\frac{s_1^n-s_2^n}{s_1-s_2}\begin{pmatrix}4s^2+2t&2s\\2st&2t\end{pmatrix}-\frac{s_1^{n-1}-s_2^{n-1}}{s_1-s_2}I_2 \tag{9}$$

where $s_1=2(s^2+t+s\sqrt{s^2+t})$, $s_2=2(s^2+t-s\sqrt{s^2+t})$*. If we choose $s=t=1$,we get the nth power of the matrix sequences with classic Pell and Pell Lucas numbers*

$$\begin{pmatrix}2&1\\1&0\end{pmatrix}^n=\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}\begin{pmatrix}2&1\\1&0\end{pmatrix}-\frac{(1+\sqrt{2})^{n-1}-(1-\sqrt{2})^{n-1}}{2\sqrt{2}}I_2$$

$$\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}^n = \frac{(4+2\sqrt{2})^n - (4-2\sqrt{2})^n}{4\sqrt{2}} \begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix} - \frac{(4+2\sqrt{2})^{n-1} - (4-2\sqrt{2})^{n-1}}{4\sqrt{2}} I_2$$

Proof. The proof is obtained by the eigenvalues of $P_1(s,t)$ and $Q_1(s,t)$ and (5), (6), (7). □

Corollary 2. *The determinant of $P_1^n(s,t)$ and $Q_1^n(s,t)$ are*

$$\begin{aligned} \det(P_1^n(s,t)) &= t^n \\ \det(Q_1^n(s,t)) &= (4t)^n(s^2+t)^n \end{aligned}$$

Proof. By the well-known property of the determinant of a matrix, we know that the determinant is the product of eigenvalues of the matrix. we get the determinant of $P_1(s,t)$ and $Q_1(s,t)$ as t and $4t(s^2+t)$ respectively. Because the determinant of the nth power of a matrix is the nth power of the product of the eigenvalues of the matrix. If we choose $s=t=1$, we get the well-known Pell and Pell Lucas matrix sequences and the determinants of them are obtained as $\det(P_1^n)=1$, and $\det(Q_1^n)=8^n$. □

Corollary 3. *The determinants of $P_1^n(s,t)$ and $Q_1^n(s,t)$ are*

$$\begin{aligned} \det(P_n(s,t)) &= t^n, \\ \det(Q_n(s,t)) &= (4t)^n(s^2+t)^n. \end{aligned}$$

Proof. By the property $P_n(s,t) = P_1^n(s,t)$ the first equality is easily seen. By the equality $Q_{n+1}(s,t) = Q_1(s,t)\,P_n(s,t)$, we get

$$\det(Q_n(s,t)) = \det(Q_1(s,t)\,P_{n-1}(s,t)) = 4t(s^2+t)t^{n-1}.$$

If we choose $s=t=1$, we get the classic Pell and Pell Lucas matrix sequences and the determinant of them are obtained as $\det(P_n)=1$, and $\det(Q_n)=8$.

Laughlin, in [3] gave if A is a 2×2 matrix as $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the nth power of A is given by

$$A^n = \begin{pmatrix} x_n - dx_{n-1} & bx_{n-1} \\ cx_{n-1} & x_n - ax_{n-1} \end{pmatrix} \tag{10}$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} T^{n-2i}(-D)^i$, $T=$trace of A, $D=$determinant of A. □

Corollary 4. *The nth power of P_1 and Q_1 are*

$$P_1^n(s,t) = \begin{pmatrix} x_n & x_{n-1} \\ tx_{n-1} & x_n - 2sx_{n-1} \end{pmatrix}, \tag{11}$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (2s)^{n-2i}(-t)^i$, *and*

$$Q_1^n(s,t) = \begin{pmatrix} y_n - 2ty_{n-1} & 2sy_{n-1} \\ 2sty_{n-1} & y_n - (4s^2+2t)y_{n-1} \end{pmatrix} \tag{12}$$

where $y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (4s^2+4t)^{n-i}(4t)^i(s^2+t)^i$.

If we substitute for $s = t = 1$ in (11), (12), we get the nth power of classic the Pell and Pell Lucas matrix sequences as follows

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} x_n & x_{n-1} \\ x_{n-1} & x_n - 2x_{n-1} \end{pmatrix}$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} 2^{n-2i}$, *and*

$$\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}^n = \begin{pmatrix} y_n - 2y_{n-1} & 2y_{n-1} \\ 2y_{n-1} & y_n - 6y_{n-1} \end{pmatrix}$$

where $y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} 8^n$.

Proof. The proof is obtained by (5), (6), (10). □

Corollary 5. *The nth elements of the (s,t)-Pell and the (s,t)-Pell Lucas matrix sequence are found by Proposition 1(d,e,f), (6), (10), (11)*

$$P_n(s,t) = \begin{pmatrix} x_n & x_{n-1} \\ tx_{n-1} & x_n - 2sx_{n-1} \end{pmatrix}$$

$$\begin{aligned} Q_n(s,t) &= Q_1(s,t) P_{n-1}(s,t) = \begin{pmatrix} 4s^2 + 2t & 2s \\ 2st & 2t \end{pmatrix} \begin{pmatrix} x_{n-1} & x_{n-2} \\ tx_{n-2} & x_{n-1} - 2sx_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} (4s^2 + 2t)x_{n-1} + 2stx_{n-2} & 2(sx_{n-1} + tx_{n-2}) \\ 2t(sx_{n-1} + tx_{n-2}) & 2t(x_{n-1} - sx_{n-2}) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} Q_n(s,t) &= 2sP_n(s,t) + 2tP_{n-1}(s,t) \\ &= 2s \begin{pmatrix} x_n & x_{n-1} \\ tx_{n-1} & x_n - 2sx_{n-1} \end{pmatrix} + 2t \begin{pmatrix} x_{n-1} & x_{n-2} \\ tx_{n-2} & x_{n-1} - 2sx_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} 2sx_n + 2tx_{n-1} & 2(sx_{n-1} + tx_{n-2}) \\ 2t(sx_{n-1} + tx_{n-2}) & 2s(x_n - 2sx_{n-1}) + 2t(x_{n-1} - 2sx_{n-2}) \end{pmatrix} \end{aligned}$$

where $x_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (2s)^{n-2i}(-t)^i$.

Theorem 1. *For any integer $n \geq 1$, the following property is satisfied*

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} \frac{n}{n-i} (2s)^{n-2i}(-t)^i = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} s^{n-2i}(s^2+t)^i. \quad (13)$$

Proof. The eigenvalues of P_1 are $r_1 = s + \sqrt{s^2+t}$, $r_2 = s - \sqrt{s^2+t}$. Therefore, the eigenvalues of $P_1^n(s,t)$ are r_1^n, r_2^n. Using (8, 11) it is obtained that $P_1^n(s,t) = \begin{pmatrix} x_n & x_{n-1} \\ tx_{n-1} & x_n - 2sx_{n-1} \end{pmatrix}$. The trace of $P_1^n(s,t)$ is $tr(P_1^n(s,t)) = 2x_n - 2sx_{n-1}$.

The sum of the eigenvalues is equal to the trace of the matrix, so $r_1^n + r_2^n = 2x_n - 2sx_{n-1}$.

$$\begin{aligned}
2x_n - 2sx_{n-1} &= 2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} T^{n-2i}(-D)^i \\
&\quad -2s\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} T^{n-1-2i}(-D)^i \\
&= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} (2s)^{n-2i}(-t)^i \left(\frac{n}{n-i}\right)
\end{aligned}$$

By binomial expansion, we get

$$\begin{aligned}
r_1^n + r_2^n &= \left(s + \sqrt{s^2+t}\right)^n + \left(s - \sqrt{s^2+t}\right)^n \\
&= \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} s^{n-2i}(s^2+t)^i.
\end{aligned}$$

The proof is completed by the equality of the results. If we choose $s = t = 1$, we get the same result for the classic Pell and Pell Lucas matrix sequences as

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} \frac{n}{n-i} 2^{n-2i} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} 2^i.$$

By the Binet formula for the nth element of the $(s,t)-$Pell Lucas sequence (4), it is obtained that

$$\begin{aligned}
q_n(s,t) &= \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} s^{n-2i}(s^2+t)^i, \\
q_n(s,t) &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} \frac{n}{n-i} (2s)^{n-2i} t^i.
\end{aligned}$$

□

Corollary 6. *The nth element of the $(s,t)-$Pell matrix sequence is also demonstrated using the elements of the $(s,t)-$Pell sequence*

$$P_n(s,t) = p_n(s,t) P_1(s,t) - p_{n-1}(s,t) I_2 \tag{14}$$

$$\begin{aligned}
P_n(s,t) &= \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} s^{n-1-2i}(s^2+t)^i P_1(s,t) \\
&\quad - \frac{1}{2^{n-2}} \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1}{2i+1} s^{n-2-2i}(s^2+t)^i I_2
\end{aligned}$$

Proof. By Proposition (1) and (7), it is obtained that

$$\begin{aligned} P_n(s,t) &= \begin{pmatrix} p_{n+1}(s,t) & p_n(s,t) \\ tp_n(s,t) & tp_{n-1}(s,t) \end{pmatrix} = P_1^n(s,t) \\ &= \frac{r_1^n - r_2^n}{r_1 - r_2} P_1(s,t) - \frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2} I_2 \\ &= p_n(s,t) P_1(s,t) - p_{n-1}(s,t) I_2. \end{aligned}$$

If we choose $s = t = 1$, we get the same result for the classic Pell and Pell Lucas matrix sequences as $P_n = p_n \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} - p_{n-1} I_2$.
By binomial expansion, it is derived that

$$\begin{aligned} \frac{r_1^n - r_2^n}{r_1 - r_2} &= \frac{1}{2\sqrt{s^2+t}} \left[\left(s + \sqrt{s^2+t}\right)^n - \left(s - \sqrt{s^2+t}\right)^n \right] \\ &= \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{pmatrix} n \\ 2i+1 \end{pmatrix} s^{n-2i-1} (s^2+t)^i. \end{aligned}$$

Then, we have

$$\begin{aligned} P_n(s,t) &= \frac{r_1^n - r_2^n}{r_1 - r_2} P_1(s,t) - \frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2} I_2 \\ &= \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{pmatrix} n \\ 2i+1 \end{pmatrix} s^{n-1-2i} (s^2+t)^i P_1(s,t) \\ &\quad - \frac{1}{2^{n-2}} \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \begin{pmatrix} n-1 \\ 2i+1 \end{pmatrix} s^{n-2-2i} (s^2+t)^i I_2. \end{aligned}$$

□

Lemma 1. [3] *Let A be an invertible 2×2 matrix with trace T and determinant D. Let g be a complex number such that $g^2 + Tg + D \neq 0$. It is easy to see this equality*

$$A = \frac{1}{g^2 + Tg + D}(A + gI)(gA + DI). \tag{15}$$

Assume that n be a positive integer and $g \neq 0$. Then

$$A^n = \left(\frac{gD}{g^2 + Tg + D}\right)^n \sum_{r=0}^{2n} \sum_{i=0}^{r} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n \\ r-i \end{pmatrix} \left(\frac{D}{g^2}\right)^i \left(\frac{g}{D}\right)^r A^r. \tag{16}$$

Corollary 7. *By Lemma 1, for $\forall g \in \mathbb{R}$ or $\mathbb{Z}$,*

$$\begin{aligned} P_1^n(s,t) &= \left(\frac{-tg}{g^2 + 2sg - t}\right)^n \sum_{r=0}^{2n} \sum_{i=0}^{r} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n \\ r-i \end{pmatrix} \left(\frac{t^{i-r}}{g^{2i-r}}\right) P_1^r(s,t), \\ Q_1^n(s,t) &= \left(\frac{4t(s^2+t)g}{g^2 + 4(s^2+t)g + 4t(s^2+t)}\right)^n \\ &\quad \sum_{r=0}^{2n} \sum_{i=0}^{r} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n \\ r-i \end{pmatrix} \left(\frac{(4t(s^2+t))^{i-r}}{g^{2i-r}}\right) Q_1^r(s,t). \end{aligned}$$

Example 1. *If we substitute for $s = t = 1$, we get the powers of the classic Pell and Pell Lucas matrix sequences. For example, if $n = 4$, it is obtained that*

$$\begin{aligned} P_1^4 &= \begin{pmatrix} p_5 & p_4 \\ p_4 & p_3 \end{pmatrix} \\ &= \left(\frac{-g}{g^2+2g-1}\right)^4 \sum_{r=0}^{8}\sum_{i=0}^{r} \begin{pmatrix} 4 \\ i \end{pmatrix} \begin{pmatrix} 4 \\ r-i \end{pmatrix} (-1)^{i-r} g^{r-2i} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^r \\ Q_1^4 &= \begin{pmatrix} c_5 & c_4 \\ c_4 & c_3 \end{pmatrix} \\ &= \left(\frac{8g}{g^2+8g+8}\right)^n \sum_{r=0}^{8}\sum_{i=0}^{r} \begin{pmatrix} 4 \\ i \end{pmatrix} \begin{pmatrix} 4 \\ r-i \end{pmatrix} \frac{8^{i-r}}{g^{2i-r}} \begin{pmatrix} 6 & 2 \\ 1 & 4 \end{pmatrix}^r . \end{aligned}$$

Theorem 2. *For any integer $n \geq 1$, it is satisfied that*

$$p_{nk+r} = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} \left[\begin{array}{c} (q_n(s,t))^{k-2i} (-t)^{in} \\ \left[p_r(s,t) + \frac{k-2i}{k-i} \frac{(-2t)^r p_{n-r}(s,t)}{q_n(s,t)} \right] \end{array} \right].$$

Proof. By the property $p_{n+1}(s,t) + tp_{n-1}(s,t) = q_n(s,t)$ and the Binet formula for the (s,t)-Pell sequence, it is obtained that

$$(P_1^n)^k = P_1^{nk} = P_{nk} = \begin{pmatrix} p_{nk+1}(s,t) & p_{nk}(s,t) \\ tp_{nk}(s,t) & tp_{nk-1}(s,t) \end{pmatrix}$$

$$\begin{aligned} (P_1^n)^k &= (P_n)^k = \begin{pmatrix} p_{n+1}(s,t) & p_n(s,t) \\ tp_n(s,t) & tp_{n-1}(s,t) \end{pmatrix}^k \\ &= \begin{pmatrix} x_k - tp_{n-1}(s,t)\, x_{k-1} & p_n(s,t)\, x_{k-1} \\ tp_n(s,t)\, x_{k-1} & x_k - p_{n+1}(s,t)\, x_{k-1} \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} x_k &= \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} T^{k-2i} (-D)^i \\ &= \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} \left[\begin{array}{c} (p_{n+1}(s,t) + tp_{n-1}(s,t))^{k-2i} \\ (t(p_{n+1}(s,t)\, p_{n-1}(s,t) - p_n^2(s,t))^i \end{array} \right] \\ &= \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} (q_n(s,t))^{k-2i} (-t)^{in}. \end{aligned}$$

By Proposition 1, 10 and [3]

$$P_1^n(s,t) = \begin{pmatrix} p_{n+1}(s,t) & p_n(s,t) \\ tp_n(s,t) & tp_{n-1}(s,t) \end{pmatrix}$$

and

$$P_1^{nk+r} = \begin{pmatrix} p_{nk+r+1}(s,t) & p_{nk+r}(s,t) \\ tp_{nk+r}(s,t) & tp_{nk+r-1}(s,t) \end{pmatrix}.$$

Then, we get

$$
\begin{aligned}
& P_1^{nk+r}(s,t) \\
= & \begin{pmatrix} p_{n+1}(s,t) & p_n(s,t) \\ tp_n(s,t) & tp_{n-1}(s,t) \end{pmatrix}^k \begin{pmatrix} p_{r+1}(s,t) & p_r(s,t) \\ tp_r(s,t) & tp_{r-1}(s,t) \end{pmatrix} \\
= & \begin{pmatrix} x_k - tp_{n-1}(s,t)\,x_{k-1} & p_n(s,t)\,x_{k-1} \\ tp_n(s,t)\,x_{k-1} & x_k - p_{n+1}(s,t)\,x_{k-1} \end{pmatrix} \cdot \begin{pmatrix} p_{r+1}(s,t) & p_r(s,t) \\ tp_r(s,t) & tp_{r-1}(s,t) \end{pmatrix}.
\end{aligned}
$$

By the equality of the matrices, it is verified that

$$
\begin{aligned}
p_{nk+r}(s,t) &= (x_k - tp_{n-1}(s,t)\,x_{k-1})\,p_r(s,t) + tp_n(s,t)\,x_{k-1}p_{r-1}(s,t) \\
&= p_r(s,t)\,x_k - t(p_{n-1}(s,t)\,p_r(s,t) - p_n(s,t)\,p_{r-1}(s,t))x_{k-1} \\
&= p_r(s,t)\,x_k + (-t)^r p_{n-r}(s,t)\,x_{k-1} \\
&= p_r(s,t)\sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} (q_n(s,t))^{k-2i}(-t)^{in} \\
&\quad +(-t)^r p_{n-r}(s,t)\sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \begin{pmatrix} k-1-i \\ i \end{pmatrix} (q_n(s,t))^{k-1-2i}(-t)^{in} \\
&= \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} \begin{bmatrix} (q_n(s,t))^{k-2i}(-t)^{in}. \\ \left[p_r(s,t) + \frac{k-2i}{k-i}\frac{(-2t)^r p_{n-r}(s,t)}{q_n(s,t)}\right] \end{bmatrix}.
\end{aligned}
$$

The desired result is completed. □

Let $s = t = 1$, we get the property of the Pell sequence

$$
p_{nk+r} = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \begin{pmatrix} k-i \\ i \end{pmatrix} (q_n)^{k-2i}(-1)^{in}\left[p_r + \frac{k-2i}{k-i}\frac{(-1)^r p_{n-r}}{q_n}\right].
$$

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