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NONHOLONOMIC NONLINEAR SYSTEMS: A GENERAL ENERGY EQUATION

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Abstract. Nonholonomic systems with nonlinear restrictions with respect to the velocities are considered. The mathematical problem is formulated by means of the Voronec equations extended to the nonlinear case. The double formulation both by means of the vector of accelerations and the Lagrangian function turns out to be convenient depending on the aspects and the properties to be pointed out. The main point of the paper is the balance of the mechanical energy induced by the equations of motion; the conservation of the energy on the basis of the tipology of the constraint equations is discussed. The special form of the energy equation allows to identify the categories of nonlinear constraints which entail the conservation of energy.

1. Introduction

Nonholonomic systems are frequently encountered in mechanical and engineering problems and attention to mathematical models concerning kinematic constraints is increasingly paid. The trait "nonholonomic" refers, in a general way, to restrictions imposed on a system which are not expressible solely in terms of one or more equations involving only the spatial coordinates which delineate the position of the system. In the wide range of restrictions with nonholonomic features, we focus the interest on the kinematic constraints, that is restrictions which can be formulated by equations containing the coordinates and components of the velocities. As a matter of fact, this kind of constraints show a more suitable relevance and evidence concerning applications and feasibility. A remarkable difference exists between the case of linear kinematic constraints and the case of nonlinear ones: even in literature, the starting point of the theory of linear constraints is distant in time ([4], [7]) and in parallel to the development of the lagrangian formalism, whereas the advancements in nonlinear constrained systems are somewhat recent. The not uniform progress is motivated both by modellistic reasons (actually, the linear case can be convincingly considered as a natural extent of the holonomic case, as we will see just below) and by practical reasons of realization of physical models exhibiting nonlinear constraints (such a question, arised in [1] and studied in [14], is still debated, see [2]).
In this paper we deal with systems submitted to nonlinear nonholonomic constraints. The first part (Section 2) consists in formulating the equations of motions by extending to the nonautonomous case the results proposed in ([11]); we base on an elementary principle, formulated along the lines of the linear nonholonomic case. The main requirements of this formulations are the explicit form of the constraint equations and the use of some of the generalized velocities as independent parameters. In this way, the formal path is simple and proceeds with extending the classical theory of mechanics. We will also point out the correspondence of the obtained equations of motion with models on the same topics present in literature, where the equations derive from more refined and formally

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complex methods. In literature, nonholonomic mechanics appear to be studied mainly via geometric methods, as Lagrangian systems on fibered manifolds (see, among others, [8]): the method proposed in the present paper is a handly approach developing in analogy to the basic concepts of a holonomic system.
The second part (Section 3) is devoted to the main result, regarding the expression for the rate of change of the mechanical energy of the system. The selected typology of the equations of motion turns ut to be appropriate both from the mathematical point of view and for the physical intepretation, since the variables involved are the real velocities. The energy balance is achieved via a somehow usual handling of the equations of motion: in spite of the spontaneity of the method, the resulting formula shows distintly each contribute in terms of energies and it is suitable in order to investigate which categories of constraints entail the conservation of generalized energy (Jacobi's first integral). In Section 4 we identify some cases of nonholonomic systems where the energy balance equation actually infers the conservation of the mechanical energy.

2. Mathematical model and notations

The starting point is a holonomic system $\mathbf{X}(q_1,\dots,q_n,t)$, where $\mathbf{X}$ locates the position of N material points P_i, with mass M_i, $i=1,\dots,N$. In addition to the $3N-n$ holonomic conditions, the system undergoes $k<n$ kinematic constraints possily depending explicitly on time t (rheonomic constraints)

$$\Phi_\nu(q_1,\dots,q_n,\dot q_1,\dots,\dot q_n,t)=0, \quad \nu=1,\dots,k \tag{1}$$

which can be linear or nonlinear and are assumed to be independent, in the sense that the rank of the Jacobian matrix $J_{(\dot q_1,\dots,\dot q_n)}(\Phi_1,\dots,\Phi_k)$ is the greatest value k. Under that assumption, if for istance $det\, J_{(\dot q_{m+1},\dots,\dot q_n)}(\Phi_1,\dots,\Phi_k)\neq 0$, $m=n-k$, conditions (1) can be made explicit:

$$\dot q_{m+\nu}=\alpha_\nu(q_1,\dots,q_n,\dot q_1,\dots,\dot q_m,t), \quad \nu=1,\dots,k. \tag{2}$$

with $m=n-k$. The case when all of (1) are linear is denoted by the functions $\alpha_{\nu,j}$ such that

$$\alpha_\nu=\sum_{j=1}^m a_{\nu,j}(q_1,\dots,q_n,t)\dot q_j, \quad \nu=1,\dots,k. \tag{3}$$

2.1. **Equations of motion.** Let $\mathbf{X}^{(\mathsf{M})}\in\mathbb{R}^{3N}$ list the n triplets M_iP_i, $i=1,\dots,N$, so that $\mathcal{Q}=\dot{\mathbf{X}}^{(\mathsf{M})}$ is the $3N$–vector of linear momenta; furthermore, let us denote by $\mathcal{F}=(\mathcal{F}_1,\dots,\mathcal{F}_N)$, $\mathcal{R}=(\mathcal{R}_1,\dots,\mathcal{R}_N)$ the list in $\mathbb{R}^{3N}$ of the active forces $\mathcal{F}_i$ and of the constraint forces $\mathcal{R}_i$ concerning each P_i, $i=1,\dots,N$. We initially present the equations of motion in the newtonian form. The key statement is

$$\left(\dot{\mathcal{Q}}-\mathcal{F}-\mathcal{R}\right)\cdot\widehat{\dot{\mathbf{X}}}=0, \qquad \widehat{\dot{\mathbf{X}}}=\sum_{r=1}^m \dot q_r\left(\frac{\partial\mathbf{X}}{\partial q_r}+\sum_{\nu=1}^k\frac{\partial\alpha_\nu}{\partial\dot q_r}\frac{\partial\mathbf{X}}{\partial q_{m+\nu}}\right) \tag{4}$$

where $\widehat{\dot{\mathbf{X}}}$ are all the possible displacements (see [2], [11]) consistent with the kinematic restrictions (2), for arbitrary $(\dot q_1,\dots,\dot q_m)\in\mathbb{R}^m$.
At this point, the assumption of ideal constraints demands $\mathcal{R}\cdot\widehat{\dot{\mathbf{X}}}=0$ for all possible $\widehat{\dot{\mathbf{X}}}$, so that (4) is equivalent to the equations

$$\left(\dot{\mathcal{Q}}-\mathcal{F}\right)\cdot\mathbf{X}_r=0, \quad r=1,\dots,m, \quad \mathbf{X}_r=\frac{\partial\mathbf{X}}{\partial q_r}+\sum_{\nu=1}^k\frac{\partial\alpha_\nu}{\partial\dot q_r}\frac{\partial\mathbf{X}}{\partial q_{m+\nu}} \tag{5}$$

which correspond explicitly to the m differential equations, to be joined with (2),

$$\sum_{r=1}^{m}\left(C_i^r\ddot{q}_r+\sum_{s=1}^{m}D_i^{r,s}\dot{q}_r\dot{q}_s+E_i^r\dot{q}_r\right)+G_i=\mathcal{F}^{(q_i)}+\sum_{\nu=1}^{k}\frac{\partial\alpha_\nu}{\partial\dot{q}_i}\mathcal{F}^{(q_{m+\nu})},\quad i=1,\dots,m \quad (6)$$

where $\mathcal{F}^{(q_j)}=\mathcal{F}\cdot\frac{\partial\mathbf{X}}{\partial q_j}$ for each $j=1,\dots,n$ is the generalized force, C_i^r, $D_i^{r,s}$, E_i^r and G_i, $i,r,s=1,\dots,m$, depend on $q_1,\dots,q_n$, $\dot{q}_1,\dots,\dot{q}_m$, t and are defined by

$$C_i^r=g_{i,r}+\sum_{\nu,\mu=1}^{k}(g_{i,m+\nu}\tfrac{\partial\alpha_\nu}{\partial\dot{q}_r}+g_{m+\nu,r}\tfrac{\partial\alpha_\nu}{\partial\dot{q}_i}+g_{m+\nu,m+\mu}\tfrac{\partial\alpha_\mu}{\partial\dot{q}_i}\tfrac{\partial\alpha_\nu}{\partial\dot{q}_r}),\quad D_i^{r,s}=\xi_{r,s,i}+\sum_{\nu=1}^{k}\xi_{r,s,m+\nu}\tfrac{\partial\alpha_\nu}{\partial\dot{q}_i},$$

$$E_i^r=\sum_{\nu,\mu=1}^{k}(2\xi_{r,m+\nu,m+\mu}\alpha_\nu+g_{m+\nu,m+\mu}\tfrac{\partial\alpha_\nu}{\partial q_r}+2\eta_{r,m+\mu})\tfrac{\partial\alpha_\mu}{\partial\dot{q}_i}+\sum_{\nu=1}^{k}(2\xi_{r,m+\nu,i}\alpha_\nu+g_{m+\nu,i}\tfrac{\partial\alpha_\nu}{\partial q_r})+2\eta_{r,i},$$

$$G_i=\sum_{\nu,\mu,p=1}^{k}(\xi_{m+\nu,m+\mu,m+p}\alpha_\nu\alpha_\mu+g_{m+\nu,m+p}\alpha_\mu\tfrac{\partial\alpha_\nu}{\partial q_{m+\mu}}+2\eta_{m+\nu,m+p}\alpha_\nu+g_{m+\nu,m+p}\tfrac{\partial\alpha_\nu}{\partial t}+\zeta_{m+p})\tfrac{\partial\alpha_p}{\partial\dot{q}_i}$$
$$+\sum_{\nu,\mu=1}^{k}(\xi_{m+\nu,m+\mu,i}\alpha_\nu\alpha_\mu+g_{m+\nu,i}\alpha_\mu\tfrac{\partial\alpha_\nu}{\partial q_{m+\mu}})+\sum_{\nu=1}^{k}(2\eta_{m+\nu,i}\alpha_\nu+g_{m+\nu,i}\tfrac{\partial\alpha_\nu}{\partial t})+\zeta_i.$$

by setting, for any $i,j,k=1,\dots,n$:

$$\begin{aligned}g_{i,j}(q_1,\dots,q_n,t)&=\frac{\partial\mathbf{X}^{(\mathsf{M})}}{\partial q_i}\cdot\frac{\partial\mathbf{X}}{\partial q_j},\quad \xi_{i,j,k}(q_1,\dots,q_n,t)=\frac{\partial^2\mathbf{X}^{(\mathsf{M})}}{\partial q_i\partial q_j}\cdot\frac{\partial\mathbf{X}}{\partial q_k},\\ \eta_{i,j}(q_1,\dots,q_n,t)&=\frac{\partial^2\mathbf{X}^{(\mathsf{M})}}{\partial q_i\partial t}\cdot\frac{\partial\mathbf{X}}{\partial q_j},\quad \zeta_i(q_1,\dots,q_n,t)=\frac{\partial^2\mathbf{X}^{(\mathsf{M})}}{\partial t^2}\cdot\frac{\partial\mathbf{X}}{\partial q_j}.\end{aligned}\quad (7)$$

The explicit form (5) is convenient in order to easily disclose the terms with the second derivatives $\ddot{q}_r$, $r=1,\dots,m$, which appear only in the linear terms with coefficients C_i^r. In case of scleronomic holonomous constraints $\mathbf{X}(\mathbf{q})$, in (7) it is $\eta_{i,j}=0$, $\zeta_i=0$ for any $i,j=1,\dots,n$.

Proposition 1. *The $m\times m$ matrix C with entries C_i^r, $r=1,\dots,m$, is positive definite.*

Proof: Indeed the terms originate from $C_i^r=(\frac{\partial\mathbf{X}^{(M)}}{\partial q_r}+\sum_{\nu=1}^{k}\frac{\partial\alpha_\nu}{\partial\dot{q}_r}\frac{\partial\mathbf{X}^{(M)}}{\partial q_{m+\nu}})\cdot(\frac{\partial\mathbf{X}}{\partial q_i}+\sum_{j=1}^{k}\frac{\partial\alpha_j}{\partial\dot{q}_i}\frac{\partial\mathbf{X}}{\partial q_{m+j}})$.
Setting $\mathbf{X}_r^{(\sqrt{M})}=\frac{\partial\mathbf{X}^{(\sqrt{M})}}{\partial q_r}+\sum_{\nu=1}^{k}\frac{\partial\alpha_\nu}{\partial\dot{q}_r}\frac{\partial\mathbf{X}^{(\sqrt{M})}}{\partial q_{m+\nu}}$, where $\mathbf{X}^{(\sqrt{M})}$ stands for $(\sqrt{m_1}P_1,\dots,\sqrt{m_n}P_n)$, one has $C_\nu^i=\mathbf{X}_\nu^{(\sqrt{M})}\cdot\mathbf{X}_i^{(\sqrt{M})}$, so that the matrix C is positive definite, since the vectors $\mathbf{X}_j^{(\sqrt{M})}$, $j=1,\dots,m$ are linearly independent, as it can be easily verified. □
Hence, the system of equations (6) is well posed, whatever the functions $\alpha_1,\dots,\alpha_\nu$ are; the already explicit version (2) does not require any further condition.

Remark 1. *In support of the form (5) of the equations of motion (in comparison with the more common Lagrangian form, recalled below) is that they are promptly attainable, whenever the coefficients (7) are simple; this is the case, for istance, of cartesian coordinates: if no holonomic constraint is present and $q_1,\dots,q_n$ are the cartesian coordinates $\mathbf{X}$ of the N points, $n=3N$, formulae are considerably simplified: actually, the only non zero coefficients in (7) are $g_{i,i}$ which write (in triplets) the masses $M_1,\dots,M_N$ and*

$$C_i^i=g_{i,i}+\sum_{\nu=1}^{k}g_{m+\nu,m+\nu}\left(\tfrac{\partial\alpha_\nu}{\partial\dot{q}_i}\right)^2,\quad C_i^r=C_r^i=\sum_{\nu=1}^{k}g_{m+\nu,m+\nu}\tfrac{\partial\alpha_\nu}{\partial\dot{q}_i}\tfrac{\partial\alpha_\nu}{\partial\dot{q}_r},\quad \textit{for } i\neq r$$
$$D_i^{r,s}=0,\quad E_i^r=\sum_{\nu}g_{m+\nu,m+\nu}\tfrac{\partial\alpha_\nu}{\partial q_r}\tfrac{\partial\alpha_\mu}{\partial\dot{q}_i},\quad G_i=\sum_{\nu=1}^{k}g_{m+\nu,m+\nu}\sum_{\mu=1}^{k}\alpha_\mu\tfrac{\partial\alpha_\nu}{\partial q_{m+\mu}}.$$

A more ordinary way to present the equations of motion starts from the kinetic energy $T(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) = \frac{1}{2}\mathcal{Q} \cdot \dot{\mathbf{X}}$, $\mathcal{Q} = \dot{\mathbf{X}}^{(M)}$, and to its restriction due to (2)

$$T^*(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, t) = T(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, \alpha_1(\cdot), \dots, \alpha_k(\cdot), t) \tag{8}$$

where each $\alpha_j(\cdot)$, $j = 1, \dots, k$ depends on $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m$ and t. In that case, the equations of motion assume the form, for each $i = 1, \dots, m$,

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_i} - \frac{\partial T^*}{\partial q_i} - \sum_{\nu=1}^{k} \frac{\partial T^*}{\partial q_{m+\nu}}\frac{\partial \alpha_\nu}{\partial \dot{q}_i} - \sum_{\nu=1}^{k} B_i^\nu \frac{\partial T}{\partial \dot{q}_{m+\nu}} = \mathcal{F}^{(q_i)} + \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial \dot{q}_i}\mathcal{F}^{(q_{m+\nu})} \tag{9}$$

joined with (2), where in $\frac{\partial T}{\partial \dot{q}_{m+\nu}}$ the variables $\dot{q}_{m+1}, \dots, \dot{q}_n$ are expressed in terms of $(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, t)$ by using (2) and the coefficients B_i^ν are

$$\begin{aligned} B_i^\nu &= \frac{d}{dt}\left(\frac{\partial \alpha_\nu}{\partial \dot{q}_i}\right) - \frac{\partial \alpha_\nu}{\partial q_i} - \sum_{\mu=1}^{k} \frac{\partial \alpha_\mu}{\partial \dot{q}_i}\frac{\partial \alpha_\nu}{\partial q_{m+\mu}} \\ &= \sum_{r=1}^{m}\left(\frac{\partial^2 \alpha_\nu}{\partial \dot{q}_i \partial q_r}\dot{q}_r + \frac{\partial^2 \alpha_\nu}{\partial \dot{q}_i \partial \dot{q}_r}\ddot{q}_r\right) - \frac{\partial \alpha_\nu}{\partial q_i} + \sum_{\mu=1}^{k}\left(\frac{\partial^2 \alpha_\nu}{\partial \dot{q}_i \partial q_{m+\mu}}\alpha_\mu - \frac{\partial \alpha_\mu}{\partial \dot{q}_i}\frac{\partial \alpha_\nu}{\partial q_{m+\mu}}\right) + \frac{\partial^2 \alpha_\nu}{\partial \dot{q}_i \partial t}. \end{aligned} \tag{10}$$

In order to achieve (9), the key point is the well known relation $\dot{\mathcal{Q}} \cdot \frac{\partial \mathbf{X}}{\partial q_i} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i}$.

2.2. **Some remarks.** Equations (5) or (9) hold for a system settled by n parameters q_1, $\dots$, q_n and undergoing the $m < n$ nonlinear nonholonomic constraints (2); it is worth it to dwell upon some significant points and remarks.

Equations (6) trace the Gibbs–Appell equations, since the left side of (6) corresponds to the calculus $\frac{\partial S}{\partial \ddot{q}_i}$, where $S = \frac{1}{2}\dot{\mathcal{Q}} \cdot \ddot{\mathbf{X}}$ is the acceleration energy (Gibbs–Appell function); actually $\frac{\partial S}{\partial \ddot{q}_i} = \dot{\mathcal{Q}} \cdot \frac{\partial \ddot{\mathbf{X}}}{\partial \ddot{q}_i} = \dot{\mathcal{Q}} \cdot (\frac{\partial \mathbf{X}}{\partial q_i} + \sum_{j=1}^{k} \frac{\partial \alpha_j}{\partial \dot{q}_i}\frac{\partial \mathbf{X}}{\partial q_{m+j}})$, which sends back to (4). On the other hand, equations (6) extend to the nonlinear case the Voronec equations (appeared in [13] and discussed in [9]) for the linear nonholonomic constraints (3); in the latter case the terms $\frac{\partial \alpha_\nu}{\partial \dot{q}_i}$ in (6) are simply $\alpha_{\nu,i}$ and the equations of motion (9) are, for each $i = 1, \dots, m$,

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_i} - \frac{\partial T^*}{\partial q_i} - \sum_{\nu=1}^{k} \alpha_{\nu,i}\frac{\partial T^*}{\partial q_{m+\nu}} - \sum_{\nu=1}^{k}\sum_{r=1}^{m} \beta_{ir}^\nu \dot{q}_r \frac{\partial T}{\partial \dot{q}_{m+\nu}} = \mathcal{F}^{(q_i)} + \sum_{\nu=1}^{k} \alpha_{\nu,i}\mathcal{F}^{(q_{m+\nu})} \tag{11}$$

where the coefficients (10) reduce to

$$B_i^\nu = \sum_{r=1}^{m} \beta_{ir}^\nu \dot{q}_r + \frac{\partial \alpha_{\nu,i}}{\partial t}, \quad \beta_{ir}^\nu = \frac{\partial \alpha_{\nu,i}}{\partial q_r} - \frac{\partial \alpha_{\nu,r}}{\partial q_i} + \sum_{\mu=1}^{k}\left(\frac{\partial \alpha_{\nu,i}}{\partial q_{m+\mu}}\alpha_{\mu,r} - \frac{\partial \alpha_{\nu,r}}{\partial q_{m+\mu}}\alpha_{\mu,i}\right). \tag{12}$$

Although the explicit dependence of $\alpha_{\nu,i}$ on t is absent in [9], the widening to the rheonomic case is trivial. Equations (9) correspond to the ones derived in [15], as the most general form of equations of motion in Poincaré–Chetaev variables extended to nonlinear nonholonomic systems; the Voronec's equations (9) are the same as the Voronec's equations pointed out in [15] as the special case of Poincaré's kinematic parameters chosen as the real generalized velocities. Also the geometric approach for nonholonomic machanical systems (Lagrangian systems on fibered manifolds) performed in [10] leads to the same equations of motion as (9). Concerning the dependence of (2) on the variables, a special case is

$$\begin{aligned} &\alpha_\nu = \alpha_\nu(q_1, \dots, q_m, \dot{q}_1, \dots, \dot{q}_m, t) \quad \text{for each } \nu = 1, \dots, k, \\ &T = T(q_1, \dots, q_m, \dot{q}_1, \dots, \dot{q}_n, t), \qquad \mathcal{F}_i = \mathcal{F}_i(q_1, \dots, q_m, \dot{q}_1, \dots, \dot{q}_n, t) \end{aligned} \tag{13}$$

that is the coordinates $q_{m+1},\dots,q_n$ corresponding to the dependent velocities do not occur; we may refer to these systems as nonlinear Čaplygin systems. In this case, system (9) reduces to (see also (10))

$$\begin{aligned} &\tfrac{d}{dt}\tfrac{\partial T^*}{\partial \dot q_i}-\tfrac{\partial T^*}{\partial q_i}-\underbrace{\sum_{\nu=1}^{k}\left(\sum_{r=1}^{m}\left(\frac{\partial^2\alpha_\nu}{\partial \dot q_i\partial q_r}\dot q_r+\frac{\partial^2\alpha_\nu}{\partial \dot q_i\partial \dot q_r}\ddot q_r\right)-\frac{\partial \alpha_\nu}{\partial q_i}+\frac{\partial^2\alpha_\nu}{\partial \dot q_i\partial t}\right)}_{\frac{d}{dt}\frac{\partial\alpha_\nu}{\partial \dot q_i}-\alpha_\nu}\tfrac{\partial T}{\partial \dot q_{m+\nu}} \\ &=\mathcal{F}^{(q_i)}+\sum_{\nu=1}^{k}\tfrac{\partial\alpha_\nu}{\partial \dot q_i}\mathcal{F}^{(q_{m+\nu})}. \end{aligned} \tag{14}$$

The remarkable advantage of the differential system (14) is that it contains only the unknown functions $q_1,\dots,q_m$ and it is disentangled from the constraints equations (3). Within assumption (13), the linear stationary case

$$\begin{aligned} &\alpha_{\nu,j}=\alpha_{\nu,j}(q_1,\dots,q_m) \qquad\qquad \text{for each } \nu=1,\dots,k \text{ and } j=1,\dots,m \\ &T=T(q_1,\dots,q_m,\dot q_1,\dots,\dot q_n),\quad \mathcal{F}_i=\mathcal{F}_i(q_1,\dots,q_m,\dot q_1,\dots,\dot q_n) \end{aligned} \tag{15}$$

leads to the Čaplygin's equations (see [9]), for each $i=1,\dots,m$:

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot q_i}-\frac{\partial T^*}{\partial q_i}-\sum_{\nu=1}^{k}\sum_{r=1}^{m}\left(\frac{\partial \alpha_{\nu,i}}{\partial q_r}-\frac{\partial \alpha_{\nu,r}}{\partial q_i}\right)\dot q_r\frac{\partial T}{\partial \dot q_{m+\nu}}=\mathcal{F}^{(q_i)}+\sum_{j=1}^{k}\alpha_{j,i}\mathcal{F}^{(q_{m+j})}. \tag{16}$$

We finally remark that whenever the kinetic energy is, consistently with the ordinary mechanical systems, $T=\frac12\sum\limits_{i,j=1}^{n}g_{i,j}\dot q_i\dot q_j+\sum\limits_{i=1}^{n}b_i\dot q_i+c$, $b_i(q_1,\dots,q_n,t)=\frac{\partial \mathbf{X}^{(\mathsf{M})}}{\partial q_i}\cdot\frac{\partial \mathbf{X}}{\partial t}$, $c(q_1,\dots,q_n,t)=\frac12\frac{\partial \mathbf{X}^{(\mathsf{M})}}{\partial t}\cdot\frac{\partial \mathbf{X}}{\partial t}$ ($g_{i,j}$ are defined in (7)) so that (8) writes

$$\begin{aligned} T^*=&\tfrac12\left(\sum_{r,s=1}^{m}g_{r,s}\dot q_r\dot q_s+\sum_{\nu,\mu=1}^{k}g_{m+\nu,m+\mu}\alpha_\nu\alpha_\mu\right) \\ &+\sum_{r=1}^{m}\sum_{\nu=1}^{k}g_{r,m+\nu}\dot q_r\alpha_\nu+\sum_{r=1}^{m}b_r\dot q_r+\sum_{\nu=1}^{k}b_{m+\nu}\alpha_\nu+c \end{aligned} \tag{17}$$

then rearranging the terms in (9) one can easily check (see [11]) that all the terms of $-\sum\limits_{\nu=1}^{k}\frac{\partial T}{\partial \dot q_{m+\nu}}B_i^\nu$, for each $i=1,\dots,m$, cancel out with part of the addends of $\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot q_i}\right)$, of $-\frac{\partial T^*}{\partial q_i}$ and of $-\sum\limits_{\nu=1}^{k}\frac{\partial T^*}{\partial q_{m+\nu}}\frac{\partial \alpha_\nu}{\partial \dot q_i}$. The remaining terms of (9) coincide precisely with the terms of (6).

3. Energy balance

Let us assume that the active forces depend only on $\mathbf{X}$ and t and come from a potential $\mathcal{U}$:

$$\mathcal{F}=\nabla_{\mathbf{X}}\mathcal{U}(\mathbf{X},t) \tag{18}$$

so that the restriction $U(q_1,\dots,q_n,t)=\mathcal{U}(\mathbf{X}(q_1,\dots,q_n,t),t)$ to the configuration manifold provides the generalized forces as follows:

$$\mathcal{F}^{(q_i)}=\frac{\partial U}{\partial q_i},\quad i=1,\dots,n \tag{19}$$

It is known that in case of holonomic systems $\mathbf{X}(q_1,\dots,q_\ell,t)$ (that is removing (1)) the equations of motion $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot q_i}-\frac{\partial\mathcal{L}}{\partial q_i}=0$, $i=1,\dots,n$, where $\mathcal{L}=T+U$ entail the energy balance $\frac{d}{dt}\left(\sum\limits_{i=1}^{n}\dot q_i\frac{\partial\mathcal{L}}{\partial\dot q_i}-\mathcal{L}\right)=-\frac{\partial\mathcal{L}}{\partial t}$, which supplies the conservation of the quantity in brackets, whenever $\mathcal{L}$ does not depend on t explicitly.
Now, if the constraints (1) are present, recalling T^* defined in (8) we set

$$\mathcal{L}^*(q_1,\dots,q_n,\dot q_1,\dots,\dot q_m,t)=T^*(q_1,\dots,q_n,\dot q_1,\dots,\dot q_m,t)+U(q_1,\dots,q_n,t) \tag{20}$$

as the Lagrangian in terms of the independent velocities. The following Proposition generalizes the just mentioned balance of holonomic systems.

Proposition 2. *The equations of motion (9) entail*

$$\frac{d}{dt}\left(\sum_{i=1}^{m}\dot q_i\frac{\partial\mathcal{L}^*}{\partial\dot q_i}-\mathcal{L}^*\right)-\sum_{\nu=1}^{k}(\overline{\alpha}_\nu-\alpha_\nu)\frac{\partial\mathcal{L}^*}{\partial q_{m+\nu}}-\sum_{\nu=1}^{k}\overline{B}_\nu\frac{\partial T}{\partial\dot q_{m+\nu}}=-\frac{\partial\mathcal{L}^*}{\partial t} \tag{21}$$

where

$$\overline{\alpha}_\nu(q_1,\dots,q_n,\dot q_1,\dots,\dot q_m,t)=\sum_{i=1}^{m}\dot q_i\frac{\partial\alpha_\nu}{\partial\dot q_i}, \tag{22}$$

$$\begin{aligned}\overline{B}_\nu(q_1,\dots,q_n,\dot q_1,\dots,\dot q_m,t)&=\sum_{r=1}^{m}\dot q_r\left(\frac{\partial\overline{\alpha}_\nu}{\partial q_r}-\frac{\partial\alpha_\nu}{\partial q_r}\right)+\sum_{\mu=1}^{k}\left(\alpha_\mu\frac{\partial\overline{\alpha}_\nu}{\partial q_{m+\mu}}-\overline{\alpha}_\mu\frac{\partial\alpha_\nu}{\partial q_{m+\mu}}\right)\\&+\sum_{r=1}^{m}\ddot q_r\left(\frac{\partial\overline{\alpha}_\nu}{\partial\dot q_r}-\frac{\partial\alpha_\nu}{\partial\dot q_r}\right)+\frac{\partial\overline{\alpha}_\nu}{\partial t}\\&=\frac{d}{dt}(\overline{\alpha}_\nu-\alpha_\nu)-\sum_{\mu=1}^{k}\frac{\partial\alpha_\nu}{\partial q_{m+\mu}}(\overline{\alpha}_\mu-\alpha_\mu)+\frac{\partial\alpha_\nu}{\partial t}.\end{aligned} \tag{23}$$

Proof. Basing on the formula

$$\frac{d}{dt}F(q_1,\dots,q_n,\dot q_1,\dots,\dot q_m,t)=\sum_{i=1}^{m}\frac{\partial F}{\partial q_i}\dot q_i+\sum_{\nu=1}^{k}\frac{\partial F}{\partial q_{m+\nu}}\alpha_\nu+\sum_{i=1}^{m}\frac{\partial F}{\partial\dot q_i}\ddot q_i+\frac{\partial F}{\partial t} \tag{24}$$

implemented with $F=T^*$ one finds $\sum\limits_{i=1}^{m}\dot q_i\left(\frac{d}{dt}\frac{\partial T^*}{\partial\dot q_i}-\frac{\partial T^*}{\partial q_i}-\sum\limits_{\nu=1}^{k}\frac{\partial T^*}{\partial q_{m+\nu}}\frac{\partial\alpha_\nu}{\partial\dot q_i}\right)$
$=\frac{d}{dt}\left(\sum\limits_{i=1}^{m}\dot q_i\frac{\partial T^*}{\partial\dot q_i}-T^*\right)+\frac{\partial T^*}{\partial t}-\sum\limits_{\nu=1}^{k}\frac{\partial T^*}{\partial q_{m+\nu}}\left(\sum\limits_{i=1}^{m}\dot q_i\frac{\partial\alpha_\nu}{\partial\dot q_i}-\alpha_\nu\right)$ so that

$$\sum_{i=1}^{m}\dot q_i\left(\frac{d}{dt}\frac{\partial T^*}{\partial\dot q_i}-\frac{\partial T^*}{\partial q_i}-\sum_{\nu=1}^{k}\frac{\partial T^*}{\partial q_{m+\nu}}\frac{\partial\alpha_\nu}{\partial\dot q_i}\right)=\frac{d}{dt}\left(\sum_{i=1}^{m}\dot q_i\frac{\partial T^*}{\partial\dot q_i}-T^*\right)+\frac{\partial T^*}{\partial t}-\sum_{\nu=1}^{k}\frac{\partial T^*}{\partial q_{m+\nu}}(\overline{\alpha}_\nu-\alpha_\nu). \tag{25}$$

Furthermore, recalling (10) one has

$$\begin{aligned}\sum_{i=1}^{m}B_i^\nu\dot q_i&=\frac{d}{dt}\left(\sum_{i=1}^{m}\dot q_i\frac{\partial\alpha_\nu}{\partial\dot q_i}\right)-\sum_{i=1}^{m}\left(\dot q_i\frac{\partial\alpha_\nu}{\partial q_i}+\ddot q_i\frac{\partial\alpha_\nu}{\partial\dot q_i}\right)-\sum_{\mu=1}^{k}\sum_{i=1}^{m}\dot q_i\frac{\partial\alpha_\mu}{\partial\dot q_i}\frac{\partial\alpha_\nu}{\partial q_{m+\mu}}\\&=\frac{d\overline{\alpha}_\nu}{dt}-\sum_{i=1}^{m}\left(\dot q_i\frac{\partial\alpha_\nu}{\partial q_i}+\ddot q_i\frac{\partial\alpha_\nu}{\partial\dot q_i}\right)-\sum_{\mu=1}^{k}\overline{\alpha}_\mu\frac{\partial\alpha_\nu}{\partial q_{m+\mu}}\end{aligned}$$

$$=\sum_{r=1}^{m}\dot q_r\left(\frac{\partial\overline{\alpha}_\nu}{\partial q_r}-\frac{\partial\alpha_\nu}{\partial q_r}\right)+\sum_{\mu=1}^{k}\left(\alpha_\mu\frac{\partial\overline{\alpha}_\nu}{\partial q_{m+\mu}}-\overline{\alpha}_\mu\frac{\partial\alpha_\nu}{\partial q_{m+\mu}}\right)+\sum_{r=1}^{m}\ddot q_r\left(\frac{\partial\overline{\alpha}_\nu}{\partial\dot q_r}-\frac{\partial\alpha_\nu}{\partial\dot q_r}\right)+\frac{\partial\overline{\alpha}_\nu}{\partial t}. \tag{26}$$

Concerning the forces, under assumption (19) and having in mind $\frac{dU}{dt} = \sum_{i=1}^{m} \frac{\partial U}{\partial q_i}\dot{q}_i + \sum_{\nu=1}^{k} \frac{\partial U}{\partial q_{m+\nu}}\alpha_\nu + \frac{\partial U}{\partial t}$, we can write

$$\sum_{i=1}^{m} \dot{q}_i \left(\mathcal{F}^{(q_i)} + \sum_{\nu=1}^{k} \frac{\partial \alpha_\nu}{\partial \dot{q}_i} \mathcal{F}^{(q_{m+\nu})} \right) = \frac{dU}{dt} - \frac{\partial U}{\partial t} + \sum_{\nu=1}^{k} \frac{\partial U}{\partial q_{m+\nu}} \left(\overline{\alpha}_\nu - \alpha_\nu \right). \tag{27}$$

By virtue of (25), (26) and (27), multiplying each of (9) by $\dot{q}_i$ and summing up with respect to i one gets the statement (21). The second equality for $\overline{B}_\nu$ in (23) is obtained by applying (24) with $F = \overline{\alpha}_\nu - \alpha_\nu$. □

Corollary 1. *For a system verifying assumption (13) (nonlinear Čaplygin's systems) and assumption (19) for $i = 1, \dots, m$, equation (21) takes the simpler form*

$$\frac{d}{dt}\left(\sum_{i=1}^{m} \dot{q}_i \frac{\partial \mathcal{L}^*}{\partial \dot{q}_i} - \mathcal{L}^* \right) - \sum_{\nu=1}^{k} \left(\frac{d}{dt}(\overline{\alpha}_\nu - \alpha_\nu) + \frac{\partial \alpha_\nu}{\partial t} \right) \frac{\partial T}{\partial \dot{q}_{m+\nu}} = -\frac{\partial \mathcal{L}^*}{\partial t} \tag{28}$$

Indeed, the terms containing $\frac{\partial \mathcal{L}^*}{\partial q_{m+\nu}}$ in (22) cancel out for each $\nu = 1, \dots, k$. Whenever T^* is the function (17), the energy of the system is

$$\begin{aligned} \sum_{i=1}^{m} \dot{q}_i \frac{\partial \mathcal{L}^*}{\partial \dot{q}_i} - \mathcal{L}^* &= \frac{1}{2}\sum_{r,s}^{m} g_{r,s}\dot{q}_r\dot{q}_s + \sum_{\nu,\mu=1}^{k} g_{m+\nu,m+\mu}(\overline{\alpha}_\nu\alpha_\mu - \frac{1}{2}\alpha_\nu\alpha_\mu) \\ &+ \sum_{r=1}^{m}\sum_{\nu=1}^{k} g_{r,m+\nu}\overline{\alpha}_\nu\dot{q}_r + \sum_{\nu=1}^{k} b_{m+\nu}(\overline{\alpha}_\nu - \alpha_\nu) - c - U. \end{aligned} \tag{29}$$

We refer to (29) as the energy of the system.

4. Special classes of nonholonomic constraints

As it emerges from the Examples, the energy balance (21) deserves a distinctive treatment whenever the constraint functions (2) take a specific form. In particular, the circumstance $\overline{\alpha}_\nu = \alpha_\nu$ play the crucial role for the conservation of the energy of the system. Let us start from the following

Lemma 1. *For a fixed ν from 1 up to k, $\overline{\alpha}_\nu = \alpha_\nu$ if and only if α_ν is a homogeneous function of degree 1 w. r. t. $\dot{q}_1, \dots, \dot{q}_m$.*

Proof. We simply turn to the Euler's theorem: $\overline{\alpha}_\nu = \sum_{r=1}^{m} \dot{q}_r \frac{\partial \alpha_\nu}{\partial \dot{q}_r} = \alpha_\nu$ if and only if α_ν is a homogeneous function of degree 1 with respect to the variables $\dot{q}_1, \dots, \dot{q}_m$. □
Let us now assume that each of the nonholonomic constraints verifies

$$\overline{\alpha}_\nu = \alpha_\nu \quad \text{for any} \ \ \nu = 1, \dots, k. \tag{30}$$

Then, the energy balance simplifies according to the following statement.

Proposition 3. *If (30): holds, the energy balance (21) takes the simpler form*

$$\frac{d}{dt}\left(\sum_{i=1}^{m} \dot{q}_i \frac{\partial \mathcal{L}^*}{\partial \dot{q}_i} - \mathcal{L}^* \right) = -\frac{\partial \mathcal{L}^*}{\partial t} + \sum_{\nu=1}^{k} \frac{\partial T}{\partial \dot{q}_{m+\nu}} \frac{\partial \alpha_\nu}{\partial t}. \tag{31}$$

Proof. Owing to (30) the terms with $\frac{\partial \mathcal{L}^*}{\partial q_{m+\nu}}$ in (21) are null. Moreover, the definition (22) shows $\overline{B}_\nu = \frac{\partial \overline{\alpha}_\nu}{\partial t} = \frac{\partial \alpha_\nu}{\partial t}$, hence $\sum\limits_{\nu=1}^{k} \overline{B}_\nu \frac{\partial T}{\partial \dot{q}_{m+\nu}} = \sum\limits_{\nu=1}^{k} \frac{\partial \overline{\alpha}_\nu}{\partial t} \frac{\partial T}{\partial \dot{q}_{m+\nu}}$ and (31) is proved. □

Remark 2. *The presence of the rheonomic contributions on the right side of equality (32) is easily explainable:* $\frac{\partial \mathcal{L}^*}{\partial t}$ *can be not null either because of the non–stationarity of the holonomic constraints (hence* $\frac{\partial T^*}{\partial t} \neq 0$*) or because of the presence of* t *in the forces (then* $\frac{\partial U}{\partial t} \neq 0$*). On the other hand, the possible non–stationarity of the nonholonomic constraints (3) gives rise to the terms containing* $\frac{\partial \alpha_\nu}{\partial t}$.

4.1. **Linear nonholonomic constraints.** A significant circumstance of validity of assumption is the case of linear nonholonomic constraints of the form (3). Indeed, the linear functions $\sum\limits_{j=1}^{m} \alpha_{\nu,j}(q_1, \dots, q_n, t)\dot{q}_j$, $\nu = 1, \dots, k$, are homogeneous functions of degree 1 w. r. t. $\dot{q}_1, \dots, \dot{q}_m$ and Lemma 1 is applicable. By virtue of (30) the coefficients (23) are $\overline{B}_\nu = \sum\limits_{i=1}^{m} \dot{q}_i \frac{\partial \alpha_{\nu,i}}{\partial t}$, hence the energy balance (31) takes the form

$$\frac{d}{dt}\left(\sum_{i=1}^{m} \dot{q}_i \frac{\partial \mathcal{L}^*}{\partial \dot{q}_i} - \mathcal{L}^*\right) = -\frac{\partial \mathcal{L}^*}{\partial t} + \sum_{\nu=1}^{k}\sum_{i=1}^{m} \dot{q}_i \frac{\partial T}{\partial \dot{q}_{m+\nu}} \frac{\partial \alpha_{\nu,i}}{\partial t}. \tag{32}$$

Concerning the linear stationary case, the balance (32) assumes the form pertinent to holonomic systems:

Corollary 2. *For a system such that* $\frac{\partial \mathcal{L}^*}{\partial t} = 0$ *and submitted to stationary linear kinematic constraints (3) with* $\alpha_{\nu,j} = \alpha_{\nu,j}(q_1, \dots, q_n)$ *for each* $\nu = 1, \dots, k$ *and* $j = 1, \dots, m$, *the quantity* $I(q_1, \dots, q_m, \dot{q}_1, \dots, \dot{q}_m) = \sum\limits_{i=1}^{m} \dot{q}_i \frac{\partial \mathcal{L}^*}{\partial \dot{q}_i} - \mathcal{L}^*$ *is conserved. Assuming for* T^* *the form (17) with* $g_{i,j}$, b_i *and* c *not depending on* t *explicitly,* $i, j = 1, \dots, n$, *the constant of motion is*

$$I = \sum_{r,s}^{m}\left(\frac{1}{2} g_{r,s} + \frac{1}{2}\sum_{\nu,\mu=1}^{k} g_{m+\nu,m+\mu}\alpha_{\nu,r}\alpha_{\mu,s} + \sum_{\nu=1}^{k} g_{r,m+\nu}\alpha_{\nu,s}\right)\dot{q}_r\dot{q}_s - U - c. \tag{33}$$

The conserved quantity, examined in [9] as well as in other textbooks, is the generalized energy integral, or Jacobi integral, of the Lagrangian $\mathcal{L}^*$.

4.2. **Nonlinear constraints: homogeneous quadratic functions.** A frequently encountered subcategory of nonholonomic constraints (1) encompasses restrictions of the form

$$\sum_{i,j=1}^{n} \gamma_{i,j}^{\nu}(q_1, \dots, q_n, t)\dot{q}_i\dot{q}_j = 0, \quad \nu = 1, \dots, k. \tag{34}$$

Assuming that the explicit form which singles out $\dot{q}_{m+1}, \dots, \dot{q}_n$ can be achieved, (2) takes the form

$$\dot{q}_{m+\nu} = \frac{\sum\limits_{r,s=1}^{m} \gamma_{r,s}^{\nu}(q_1, \dots, q_n, t)\dot{q}_r\dot{q}_s}{\sum\limits_{i=1}^{m} \beta_i^{\nu}(q_1, \dots, q_n, t)\dot{q}_i} = \alpha_\nu(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_m, t), \qquad \nu = 1, \dots, k \tag{35}$$

for appropriate functions β_i^ν. The functions α_ν of (35) are homogeneous of degree 1 w. r. t. $\dot{q}_1, \dots, \dot{q}_m$, hence the energy balance which pertains to such systems is (31), where the functions $\gamma^{\nu}_{r,s}$ and β^{ν}_{i} will appear; in the stationary case the energy is conserved. In a natural way, constraints (34) appear when the restrictions concern parallelism or orthogonality of the velocities, or the assignment of equal intensity of the velocities. A typical and simple instance takes into consideration two points P_1, P_2, for which the three restrictions $\dot{P}_1 \wedge \dot{P}_2 = 0$, $\dot{P}_1 \cdot \dot{P}_2 = 0$, $|\dot{P}_1|^2 = |\dot{P}_2|^2$ respectively read, in cartesian coordinates,

$$\begin{array}{ll} \dot{x}_1\dot{y}_2 - \dot{y}_1\dot{x}_2, \;\; \dot{x}_1\dot{z}_2 - \dot{z}_1\dot{x}_2 = 0 & \text{parallelism} \\ \dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2 + \dot{z}_1\dot{z}_2 = 0 & \text{orthogonality} \\ \dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2 - \dot{x}_2^2 - \dot{y}_2^2 - \dot{z}_2^2 = 0 & \text{same norm of velocity} \end{array}$$

belonging to class (34). It is worthwhile to stress that adding further resctrictions or specifying mechanisms regarding the nonholonomic constraints (34) may modify totally the typology of the restrictions: in this sense, the equivalence in realizing phisically nonholonomic restrictions by means of either linear or nonlinear equations sometimes claimed in literature needs to be debated.

4.3. **Linear affine constraints.** Finally, a special situation concerns the nonholonomic systems with affine constraints, which can be assumed of the form (not encompassed by (3))

$$\alpha_\nu = \sum_{j=1}^{m} a_{\nu,j}(q_1,\dots,q_n,t)\dot{q}_j + c_\nu(t), \qquad \nu = 1,\dots,k \tag{36}$$

with c_ν non zero function. In that case $\bar{\alpha}_\nu = \alpha_\nu - c_\nu$, so that $\overline{B}_\nu = \frac{\partial \overline{\alpha}_\nu}{\partial t}$ and (21) is

$$\frac{d}{dt}\left(\sum_{i=1}^{m} \dot{q}_i \frac{\partial \mathcal{L}^*}{\partial \dot{q}_i} - \mathcal{L}^*\right) + c_\nu \frac{\partial \mathcal{L}^*}{\partial q_{m+\nu}} - \sum_{\nu=1}^{k}\left(\sum_{j=1}^{m} \frac{\partial \alpha_{\nu,j}}{\partial t}\dot{q}_j + \dot{c}_\nu\right)\frac{\partial T}{\partial \dot{q}_{m+\nu}} = -\frac{\partial \mathcal{L}^*}{\partial t}. \tag{37}$$

The stationary case $\alpha_{\nu,j}(_1,\dots,q_n,\dot{q}_1,\dots,\dot{q}_m)$, c_ν constant provides the conservation of the quantity in round brackets (energy), whenever $\mathcal{L}^*$ does not depend explicitly on t and the forces verify special properties, as it is described in the following
A necessary and sufficient condition (in terms of geometrical properties of the constraint manifold) in order that (21) provides the energy integral is discussed and proved in [5].

5. Conclusions

The equations of motions for nonholonomic nonlinear systems are presented in both versions (6) and (9), each of them showing advantageous points. The context of nonlinear restrictions leads us to make use of the generalized velocities as kinematic variables (instead of quasi–coordinates), thus favouring the extension of the Voronec's method for linear kinematic constraints to the nonlinear case. The calculation of the power of the forces, by way of rearranging the equations of motion, generates (21), showing in an unified and consistent way the rate of change in time of the energy espressed by the independent velocities (the function in round brackets in (21)) in terms of the contributions due to the constraints forces (by means of the coefficients $\overline{\alpha}_\nu$ and $\overline{B}_\nu$) and of the possible explicit dependence of the restriction or of the forces on time (term on the right side).
The arrangement of the energy balance equation is suitable in order to identify the type of nonholonomic constraints exhibiting $\overline{\alpha}_\nu = \alpha_\nu$, which is the key point for the purposes of inferring the first integral of energy. At the same time, in some special modellistic circumstances the setup (21) shows directly the appropriate simplifcations, as in the case (28) or in the linear case. Several examples of simple but meaningful systems have been performed. Condition (30) fits for kinematic constraints whose explicit formulation (2) is

a homogeneous function of degree 1: the models frequently adopted in literature and accessible for implementations concerning special restrictions on the velocities (parallelism, equal norms, orthogonality) fall in this category. Theoretically, it is easy to extend the category to restrictions (1) which are homogeneous functions of arbitrary degree w. r. t. the generalized velocities $\dot{q}_1, \ldots, \dot{q}_n$; however, from the experimental point of view this may produce a not concrete realisation. Beyond the mere aspect of the balance of energy, the subject nonlinear constraints presents interesting questions somehow unexplored in literature and sometimes misleading. These aspects, just mentioned in the paper, concern the equivalence of linear kinematics models with nonlinear restrictions, the correctness of mergering part of the constraints giving rise to new conditions (tipically, two linear constraints are joined to form a quadratic condition); the Hamel–Appell example of a system with nonlinear nonholonomic constraint obtained by linear kinematic condition is a point of reference in this sense. A sistematic procedure for readily comparing the equations of motion whenever different sets of independent velocities are selected is also not secondary in order to take into the right consideration the local use of (2). The just mentioned open points are the next purpose of the research on nonlinear kinematic constraints.

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