TJMM
14 (2022), No. 1, 63-71

A NEW EFFICIENT STRATEGY FOR SOLVING THE SYSTEM OF CAUCHY INTEGRAL EQUATIONS VIA TWO PROJECTION METHODS

ABDELAZIZ MENNOUNI

Abstract. It is interesting to solve a system of singular integral equations with bounded non-compact operators by projection methods in Hilbert space. The theory of projection methods for solving a system of Cauchy integral equations in $L^2([0,1],\mathbb{C})\times L^2([0,1],\mathbb{C})$ is developed and extended in this work. We look at two situations: Galerkin approximations and Kulkarni approximations. This is accomplished through the use of a sequence of orthogonal finite rank projections. A new efficient technique is used to obtain an equivalent system of two separable equations. The existence of the approach solution, as well as the error analysis, are established. A numerical example illustrates theoretical results.

1. Introduction

Various models of science and engineering problems may require solving the operator equations, which involve the Cauchy integral equations (CIEs). Different approximation schemes are developed in the research literature to solve these equations numerically, (see [4, 5, 7, 8, 9]). Projection methods are very popular methods for this topic, such as the classical Galerkin method and its variants. Several papers on numerically solving compact operator equations via Galerkin and other approaches have been published.

The author of [1] showed how the differential transform approach can be applied to solve CIEs over a finite interval. Approximate solutions to CIEs are obtained using two kinds of kernels: convolution and degenerate. Analytically, it is possible to solve the obtained system for the characteristic equation. A new approach to solving CIEs of the first kind is presented in [9]. The process aims at examining a regularized integral equation and then converting it into a canonical form suitable for application of the Adomian decomposition method.

To solve other classes of CIEs, the authors of [12] used an approximate methodology with high precision. They employed an iterative method to derive both the integral limits fulfill the physical argument and the solution of the CIEs consisting of multiple parameter functions via the quadrature approximation.

According to [3], Kulkarni developed a new and more precise approximation method for compact operator equations of the second kind in a complex Banach space via orthogonal projections. Kulkarni's process improves both the Sloan and Galerkin numerical techniques.

For the first time, we introduced the Kulkarni approach and a Galerkin one to approximate the solution of second kind noncompact bounded operator equations. In [6], we investigated some bounded finite rank projection approximations for solving CIEs of the second kind of Fredholm type. Then, for a class of generalized CIEs of the second kind

2010 *Mathematics Subject Classification.* 45A05, 45F15.
Key words and phrases. Linear integral equations, system of integral equations, Cauchy kernel, projection methods.

with constant coefficients in $L^2([0,1],\mathbb{C})$, an extended version of the piecewise constant Galerkin method is presented. Following that, we extended and improved on previous work using the Kulkarni method for an integro-differential equation, (see [11]).

The main idea of this new work is to extend the application of Galerkin and Kulkarni methods for solving a system of operator equations with bounded non-compact operators in Hilber space by using a new efficient technique. The important thing to remember here is that we obtained a system of two separable equations, which we then solved using the Galerkin and Kulkarni methods. A Cauchy type system of integral equations in $L^2([0,1],\mathbb{C}) \times L^2([0,1],\mathbb{C})$. is used to implement the application.

2. System of operator equations with bounded non-compact operators in Hilber space

Assume $\mathcal{H}$ is a Hilbert space, and $\mathcal{A}$ is a bounded linear operator from $\mathcal{H}$ into itself. For a given function $(f,g) \in \mathcal{H}\times\mathcal{H}$, we regard the problem of finding a function $(\varphi,\psi) \in \mathcal{H}\times\mathcal{H}$ such that:

$$\varphi - \mathcal{A}\psi = f \tag{1}$$

$$\psi - \mathcal{A}\varphi = g \tag{2}$$

Letting

$$u := \varphi - \psi, \quad \xi := f - g;$$
$$v := \varphi + \psi, \quad \zeta := f + g.$$

Assume that $\mathcal{A}$ is skew-hermitian: that is $\mathcal{A}^* = -\mathcal{A}$, where $\mathcal{A}^*$ denotes the adjoint of $\mathcal{A}$.

Theorem 1. *The problem (1), (2) can be represented as follows*

$$v - \mathcal{A}v = \zeta, \tag{3}$$

$$u - \mathcal{A}^*u = \xi. \tag{4}$$

Proof. It is obvious that

$$\varphi = \frac{v+u}{2}, \quad f = \frac{\zeta+\xi}{2};$$
$$\psi = \frac{v-u}{2}, \quad g = \frac{\zeta-\xi}{2}.$$

Substituting this into (3) and (4) respectively, yields

$$\frac{v+u}{2} - \mathcal{A}\left(\frac{v-u}{2}\right) = \frac{\zeta+\xi}{2},$$
$$\frac{v-u}{2} - \mathcal{A}\left(\frac{v+u}{2}\right) = \frac{\zeta-\xi}{2}.$$

That is

$$v + u - (\mathcal{A}v - \mathcal{A}u) = \zeta + \xi, \tag{5}$$

$$v - u - (\mathcal{A}v + \mathcal{A}u) = \zeta - \xi. \tag{6}$$

By adding the two equations (5) and (6) together, we get (3). Again, by subtracting (6) from (5) we obtain (4). □

Following [6], the above problem has a unique solution $(\varphi,\psi) \in \mathcal{H}\times\mathcal{H}$. Let us consider a sequence $(\mathcal{A}_n)_{n\geq 1}$ of skew-Hermitian operators from $\mathcal{H}$ into itself.

Theorem 2. *For all n, the operators $I - \mathcal{A}_n$ and $I - \mathcal{A}_n^*$ are invertible, moreover,*

$$\|(I-\mathcal{A}_n)^{-1}\| \leq 1, \text{ and } \|(I-\mathcal{A}_n^*)^{-1}\| \leq 1.$$

Proof. The proof of this theorem is analogous to that of [6, Theorem 1]. □

3. Second-kind Cauchy integral equations system

Setting $\mathcal{H} := L^2([0,1],\mathbb{C})$. In this segment, we will look at the following system of second-kind Cauchy integral equations:

$$\varphi(\sigma) - \fint_0^1 \frac{\psi(\rho)}{\rho-\sigma} d\rho = f(\sigma), \tag{7}$$

$$\psi(\sigma) - \fint_0^1 \frac{\varphi(\rho)}{\rho-\sigma} d\rho = g(\sigma), \tag{8}$$

here f and g are functions that are well-known. The integral displayed above is taken to be the Cauchy principal value:

$$\fint_0^1 \frac{\varphi(\rho)}{\rho-\sigma} d\rho = \lim_{\epsilon\to 0}\left[\int_0^{\sigma-\epsilon} \frac{\varphi(\rho)}{\rho-\sigma} d\rho + \int_{\sigma+\epsilon}^1 \frac{\varphi(\rho)}{\rho-\sigma} d\rho\right].$$

Letting

$$\mathcal{A}\varphi(\sigma) := \fint_0^1 \frac{\varphi(\rho)}{\rho-\sigma} d\rho, \quad 0<\sigma<1.$$

Problem (7), (8) is equivalent to the problem (1), (2). We are reminded of the fact that for each $(f,g)\in\mathcal{H}\times\mathcal{H}$, problem (7), (8) has a unique solution $(\varphi,\psi)\in\mathcal{H}\times\mathcal{H}$, and the singular integral operator $\mathcal{A}$ is bounded from $\mathcal{H}$ into itself and thus further $\mathcal{A}^* = -\mathcal{A}$.

We construct a sequence $(\pi_n)_{n\geq 1}$ of bounded projections, each of which has a finite rank, by considering a grid $(\sigma_{n,j})_{j=0}^n$ on $[0,1]$ as well as

$$0<\sigma_{n,0}<\sigma_{n,1}<\ldots<\sigma_{n,n}<1.$$

Letting

$$d_{n,i} := \sigma_{n,i}-\sigma_{n,i-1}, \quad i\in[\![1,n]\!], \quad d_n := (d_{n,1}, d_{n,2},\ldots,d_{n,n}),$$

and

$$\pi_n x := \sum_{k=1}^n \langle x, e_{n,k}\rangle\, e_{n,k},$$

with

$$e_{n,k} := \frac{\phi_{n,k}}{\sqrt{d_{n,k}}}, \quad \phi_{n,k}(\sigma) := \begin{cases} 1 \text{ for } \sigma\in]\sigma_{n,k-1},\sigma_{n,k}[, \\ 0 \text{ otherwise.}\end{cases}$$

The modulus of continuity of the function $\psi\in\mathcal{H}$ relative to d_n is defined as follows:

$$\omega_2(\vartheta,\delta_n) := \sup_{0\leq\delta\leq d_n}\left(\int_0^1 |\vartheta(\tau+\delta)-\vartheta(\tau)|^2 d\tau\right)^{\frac{1}{2}}, \text{ where } \delta_n := \{\sigma_{n,j}, \quad j\in[\![0,n]\!]\}.$$

Outside of $[0,1]$, all functions are extended by 0. As stated in [6],

$$\omega_2(\vartheta,\delta_n)\to 0 \text{ as } n\to\infty, \quad \text{for all } \vartheta\in\mathcal{H}.$$

Moreover,

$$\|(I-\pi_n)\vartheta\|_2 \leq \omega_2(\vartheta,\delta_n), \quad \text{for all } \vartheta\in\mathcal{H}. \tag{9}$$

3.1. **Galerkin approximation.** We have recently introduced the Galerkin method to approximate the solution of the bounded integral equation, (see [6]). We extend the results of this work concerning the application of the Galerkin approximation to a system of Cauchy integral equations of the second kind with bounded non-compact operators in this section.

As mentioned in [6], first observe that $\pi_n^* = \pi_n$, so $(\mathcal{A}_n^G)^* = -\mathcal{A}_n^G$ and $(\mathcal{A}_n^{*G})^* = -\mathcal{A}_n^{*G}$, with $\mathcal{A}_n^G := \pi_n \mathcal{A} \pi_n$ and $\mathcal{A}_n^{*G} := \pi_n \mathcal{A}^* \pi_n$. It follows that the the solution $\left(u_n^G, v_n^G\right)$ of the following Galerkin system

$$u_n^G - \mathcal{A}_n^{*G} u_n^G = \pi_n \xi \tag{10}$$

$$v_n^G - \mathcal{A}_n^G v_n^G = \pi_n \zeta \tag{11}$$

is unique and it is addicted by

$$u_n^G = \sum_{p=1}^{n} x_{n,p} e_{n,p},$$

$$v_n^G = \sum_{p=1}^{n} y_{n,p} e_{n,p}$$

for some scalars $x_{n,p}, y_{n,p}$, respectively. Problem (10), (11) reads as

$$\sum_{p=1}^{n} x_{n,p} \left[e_{n,p} - \pi_n \mathcal{A}^* e_{n,p}\right] = \pi_n \xi,$$

$$\sum_{j=1}^{n} y_{n,p} \left[e_{n,p} - \pi_n \mathcal{A} e_{n,p}\right] = \pi_n \zeta,$$

so that

$$\sum_{p=1}^{n} x_{n,p} \left[e_{n,p} - \sum_{i=1}^{n} \langle \mathcal{A}^* e_{n,p}, e_{n,i} \rangle e_{n,i}\right] = \sum_{i=1}^{n} \langle \xi, e_{n,i} \rangle e_{n,i},$$

$$\sum_{p=1}^{n} y_{n,p} \left[e_{n,p} - \sum_{i=1}^{n} \langle \mathcal{A} e_{n,p}, e_{n,i} \rangle e_{n,i}\right] = \sum_{i=1}^{n} \langle \zeta, e_{n,i} \rangle e_{n,i}.$$

In other words, the coefficients $x_{n,p}$, $y_{n,p}$ are obtained by solving the two linear systems given below:

$$(I - G_n^*) x_n = b_n,$$

$$(I - G_n) y_n = c_n,$$

where

$$\begin{aligned}
G_n(k,j) &:= \frac{1}{\sqrt{d_{n,j} d_{n,k}}} \int_{\sigma_{n,k-1}}^{\sigma_{n,k}} \oint_{\sigma_{n,j-1}}^{n,\sigma_j} \frac{d\rho}{\rho - \sigma} d\sigma, \\
G_n^*(k,j) &:= \frac{1}{\sqrt{d_{n,j} d_{n,k}}} \int_{\sigma_{n,k-1}}^{\sigma_{n,k}} \oint_{\sigma_{n,j-1}}^{n,\sigma_j} \frac{d\rho}{\sigma - \rho} d\sigma, \\
b_n(k) &:= \frac{1}{\sqrt{d_{n,k}}} \int_{\sigma_{k-1}}^{\sigma_k} \xi(\sigma) d\sigma, \\
c_n(k) &:= \frac{1}{\sqrt{d_{n,k}}} \int_{\sigma_{k-1}}^{\sigma_k} \zeta(\sigma) d\sigma.
\end{aligned}$$

Theorem 3. *The following estimates hold:*

$$\begin{aligned}\left\|\varphi^G-\varphi_n^G\right\|_2 &\leq \max\Big\{\omega_2(\xi,\delta_n)+\omega_2(\mathcal{A}^*u^G,\delta_n)+\pi\omega_2(u^G,\delta_n),\omega_2(\zeta,\delta_n)+\omega_2(\mathcal{A}v^G,\delta_n)+\pi\omega_2(v^G,\delta_n)\Big\},\\ \left\|\psi^G-\psi_n^G\right\|_2 &\leq \max\Big\{\omega_2(\zeta,\delta_n)+\omega_2(\mathcal{A}v^G,\delta_n)+\pi\omega_2(v^G,\delta_n),\omega_2(\zeta,\delta_n)+\omega_2(\mathcal{A}v^G,\delta_n)+\pi\omega_2(v^G,\delta_n)\Big\}.\end{aligned}$$

Proof. We have

$$\begin{aligned}\varphi^G-\varphi_n^G &= \frac{\left(v^G-v_n^G\right)+\left(u^G-u_n^G\right)}{2},\\ \psi^G-\psi_n^G &= \frac{\left(v^G-v_n^G\right)-\left(u^G-u_n^G\right)}{2}.\end{aligned}$$

Hence

$$\begin{aligned}\left\|\varphi^G-\varphi_n^G\right\|_2 &\leq \frac{\left\|u^G-u_n^G\right\|_2+\left\|v^G-v_n^G\right\|_2}{2},\\ \left\|\psi^G-\psi_n^G\right\|_2 &\leq \frac{\left\|u^G-u_n^G\right\|_2+\left\|v^G-v_n^G\right\|_2}{2}.\end{aligned}$$

We can now proceed analogously to the proof of Theorem 2 in [6]. It follows immediately that

$$\begin{aligned}u_n^G-u &= (I-\mathcal{A}_n^{*G})^{-1}\pi_n\xi-(I-\mathcal{A}^*)^{-1}\xi\\ &= (I-\mathcal{A}_n^{*G})^{-1}\pi_n\xi-(I-\mathcal{A}_n^{*G})^{-1}\xi+(I-\mathcal{A}_n^{*G})^{-1}\xi-(I-\mathcal{A}^*)^{-1}\xi\\ &= (I-\mathcal{A}_n^{*G})^{-1}(\pi_n-I)\xi+(I-\mathcal{A}_n^{*G})^{-1}\left[(I-\mathcal{A}^*)-(I-\mathcal{A}_n^{*G})\right](I-\mathcal{A}^*)^{-1}\xi\\ &= (I-\mathcal{A}_n^{*G})^{-1}\left[(\pi_n-I)\xi+(\mathcal{A}_n^{*G}-\mathcal{A}^*)u^G\right].\end{aligned}$$

Of course $\left\|(I-\mathcal{A}_n^{*G})^{-1}\right\|\leq 1$, $\|\pi_n\|=1$ and $\|\mathcal{A}^*\|\leq\pi$. The equality

$$(\mathcal{A}_n^{*G}-\mathcal{A}^*)u^G=(\pi_n-I)\mathcal{A}^*u^G+\pi_n\mathcal{A}^*(\pi_n-I)u^G$$

implies that

$$\|u_n^G-u^G\|_2 \leq \omega_2(\xi,\delta_n)+\omega_2(\mathcal{A}^*u^G,\delta_n)+\pi\omega_2(u^G,\delta_n).$$

We apply the same argument again, with u replaced by v and ξ replaced by ζ to obtain

$$\|v_n^G-v^G\|_2 \leq \omega_2(\zeta,\delta_n)+\omega_2(\mathcal{A}v^G,\delta_n)+\pi\omega_2(v^G,\delta_n).$$

Since

$$\begin{aligned}\left\|\varphi^G-\varphi_n^G\right\|_2 &\leq \max\left\{\left\|u^G-u_n^G\right\|_2,\left\|v^G-v_n^G\right\|_2\right\},\\ \left\|\psi^G-\psi_n^G\right\|_2 &\leq \max\left\{\left\|u^G-u_n^G\right\|_2,\left\|v^G-v_n^G\right\|_2\right\},\end{aligned}$$

we get the aims results. □

3.2. **Kulkarni approximation.** Once-more, we have proved the application of the Kulkarni method for bounded noncompact integral equation for the first time, (see [6]). We recall that this method this method has been proposed by Kulkarni (cf. [3]) to approach a compact linear operator $\mathcal{A}$ by the finite rank operator $\mathcal{A}_n^K$ which is given by

$$\mathcal{A}_n^K:=\pi_n\mathcal{A}+\mathcal{A}\pi_n-\pi_n\mathcal{A}\pi_n.$$

In this section, we come up with the approximation of solution of Cauchy integral system with noncompact bounded operator $\mathcal{A}$ by this finite rank operator. Let (φ_n^K,ψ_n^K) be the approximate solution of the system (10), (11) via $\mathcal{A}_n^K$. For this purpose, letting

$$\begin{aligned}\eta_n &:= \pi_n u_n^K\\ \nu_n &:= \pi_n v_n^K\end{aligned}$$

Since $\pi_n\eta_n = \eta_n$ and $\pi_n\nu_n = \nu_n$ there exist scalars $\alpha_{n,j}$, $\beta_{n,j}$ aiming to

$$\begin{aligned}\eta_n &= \sum_{p=1}^{n} \alpha_{n,p}\gamma_{n,p},\\ \nu_n &= \sum_{p=1}^{n} \beta_{n,p}\gamma_{n,p}.\end{aligned}$$

We follow [3, 6] in concluding that

$$\begin{aligned}\eta_n - [\pi_n\mathcal{A}^*\pi_n + \pi_n\mathcal{A}^*(I-\pi_n)\mathcal{A}^*\pi_n]\,\eta_n &= \pi_n\xi + \pi_n\mathcal{A}^*(I-\pi_n)\xi,\\ \nu_n - [\pi_n\mathcal{A}\pi_n + \pi_n\mathcal{A}(I-\pi_n)\mathcal{A}\pi_n]\,\nu_n &= \pi_n\zeta + \pi_n\mathcal{A}(I-\pi_n)\zeta,\end{aligned}$$

which leads to

$$\begin{aligned}\sum_{j=1}^{n} \alpha_{n,j}\,[\gamma_{n,j} - (\pi_n\mathcal{A}^*\gamma_{n,j} + \pi_n\mathcal{A}^*(I-\pi_n)\mathcal{A}^*\gamma_{n,j})] &= \pi_n\xi + \pi_n\mathcal{A}^*(I-\pi_n)\xi,\\ \sum_{j=1}^{n} \beta_{n,j}\,[\gamma_{n,j} - (\pi_n\mathcal{A}\gamma_{n,j} + \pi_n\mathcal{A}(I-\pi_n)\mathcal{A}\gamma_{n,j})] &= \pi_n\zeta + \pi_n\mathcal{A}(I-\pi_n)\zeta.\end{aligned}$$

Therefore,

$$\sum_{j=1}^{n} \alpha_{n,j}\left[\gamma_{n,j} - \sum_{k=1}^{n}(\langle\mathcal{A}^*\gamma_{n,j},\gamma_{n,k}\rangle + \langle\mathcal{A}^*(I-\pi_n)\mathcal{A}^*\gamma_{n,j},\gamma_{n,k}\rangle)\gamma_{n,k}\right] = \sum_{k=1}^{n}\langle\xi,\gamma_{n,k}\rangle\gamma_{n,k} + \sum_{k=1}^{n}\langle\mathcal{A}^*(I-\pi_n)\xi,\gamma_{n,k}\rangle\gamma_{n,k}.$$

$$\sum_{j=1}^{n} \beta_{n,j}\left[\gamma_{n,j} - \sum_{k=1}^{n}(\langle\mathcal{A}\gamma_{n,j},\gamma_{n,k}\rangle + \langle\mathcal{A}(I-\pi_n)\mathcal{A}\gamma_{n,j},\gamma_{n,k}\rangle)\gamma_{n,k}\right] = \sum_{k=1}^{n}\langle\zeta,\gamma_{n,k}\rangle\gamma_{n,k} + \sum_{k=1}^{n}\langle\mathcal{A}(I-\pi_n)\zeta,\gamma_{n,k}\rangle\gamma_{n,k}.$$

We obtain the two linear systems by accomplishing the inner product with $\gamma_{n,i}$

$$\begin{aligned}\alpha_{n,i} - \sum_{j=1}^{n}\alpha_{n,j}\,[\langle\mathcal{A}^*\gamma_{n,j},\gamma_{n,i}\rangle + \langle\mathcal{A}^*(I-\pi_n)\mathcal{A}^*\gamma_{n,j},\gamma_{n,i}\rangle] &= \langle\xi,\gamma_{n,i}\rangle + \langle\mathcal{A}^*(I-\pi_n)\xi,\gamma_{n,i}\rangle, \quad i\in[\![1,n]\!],\\ \beta_{n,i} - \sum_{j=1}^{n}\beta_{n,j}\,[\langle\mathcal{A}^*\gamma_{n,j},\gamma_{n,i}\rangle + \langle\mathcal{A}^*(I-\pi_n)\mathcal{A}^*\gamma_{n,j},\gamma_{n,i}\rangle] &= \langle\zeta,\gamma_{n,i}\rangle + \langle\mathcal{A}^*(I-\pi_n)\zeta,\gamma_{n,i}\rangle, \quad i\in[\![1,n]\!].\end{aligned}$$

Thus,

$$\begin{aligned}&\alpha_{n,i} - \sum_{j=1}^{n}\left[\langle\mathcal{A}^*\gamma_{n,j},\gamma_{n,i}\rangle + \left\langle\mathcal{A}^2\gamma_{n,j},\gamma_{n,i}\right\rangle - \sum_{k=1}^{n}\langle\mathcal{A}^*\gamma_{n,j},\gamma_{n,k}\rangle\langle\mathcal{A}^*\gamma_{n,k},\gamma_{n,i}\rangle\right]\alpha_{n,j}\\ &\qquad = \langle\xi,\gamma_{n,i}\rangle + \langle\mathcal{A}^*\xi,\gamma_{n,i}\rangle - \sum_{k=1}^{n}\langle\xi,\gamma_{n,k}\rangle\langle A^*\gamma_{n,k},\gamma_{n,i}\rangle, \quad i\in[\![1,n]\!]. \quad (12)\end{aligned}$$

$$\beta_{n,i} - \sum_{j=1}^{n} \left[\langle \mathcal{A}\gamma_{n,j}, \gamma_{n,i} \rangle + \left\langle \mathcal{A}^2\gamma_{n,j}, \gamma_{n,i} \right\rangle - \sum_{k=1}^{n} \langle \mathcal{A}\gamma_{n,j}, \gamma_{n,k} \rangle \langle \mathcal{A}\gamma_{n,k}, \gamma_{n,i} \rangle \right] \beta_{n,j}$$
$$= \langle \zeta, \gamma_{n,i} \rangle + \langle \mathcal{A}\zeta, \gamma_{n,i} \rangle - \sum_{k=1}^{n} \langle \zeta, \gamma_{n,k} \rangle \langle \mathcal{A}\gamma_{n,k}, \gamma_{n,i} \rangle, \quad i \in [\![1, n]\!]. \tag{13}$$

The succeeding computations are required:

$$\begin{aligned}
\langle \mathcal{A}\gamma_{n,j}, \gamma_{n,i} \rangle &= \frac{1}{\sqrt{d_{n,j}d_{n,i}}} \int_{\sigma_{i-1}}^{\sigma_i} \oint_{\sigma_{j-1}}^{\sigma_j} \frac{d\rho}{\rho - \sigma} d\sigma, \\
\langle \mathcal{A}^*\gamma_{n,j}, \gamma_{n,i} \rangle &= \frac{1}{\sqrt{d_{n,j}d_{n,i}}} \int_{\sigma_{i-1}}^{\sigma_i} \oint_{\sigma_{j-1}}^{\sigma_j} \frac{d\rho}{\sigma - \rho} d\sigma, \\
\left\langle \mathcal{A}^2\gamma_{n,j}, \gamma_{n,i} \right\rangle &= \frac{1}{\sqrt{d_{n,j}d_{n,i}}} \int_{\sigma_{i-1}}^{\sigma_i} \oint_0^1 \frac{1}{\rho - \sigma} \oint_{\sigma_{j-1}}^{\sigma_j} \frac{d\tau}{\tau - \rho} d\rho d\sigma, \\
\langle \mathcal{A}\xi, \gamma_{n,i} \rangle &= \frac{1}{\sqrt{d_{n,i}}} \int_{\sigma_{i-1}}^{\sigma_i} \oint_0^1 \frac{\xi(\rho)}{\rho - \sigma} d\rho d\sigma, \\
\langle \mathcal{A}^*\xi, \gamma_{n,i} \rangle &= \frac{1}{\sqrt{d_{n,i}}} \int_{\sigma_{i-1}}^{\sigma_i} \oint_0^1 \frac{\xi(\rho)}{\sigma - \rho} d\rho d\sigma, \\
\langle \mathcal{A}\zeta, \gamma_{n,i} \rangle &= \frac{1}{\sqrt{d_{n,i}}} \int_{\sigma_{i-1}}^{\sigma_i} \oint_0^1 \frac{\zeta(\rho)}{\rho - \sigma} d\rho d\sigma, \\
\langle \mathcal{A}^*\zeta, \gamma_{n,i} \rangle &= \frac{1}{\sqrt{d_{n,i}}} \int_{\sigma_{i-1}}^{\sigma_i} \oint_0^1 \frac{\zeta(\rho)}{\sigma - \rho} d\rho d\sigma, \\
\langle \xi, \gamma_{n,i} \rangle &= \frac{1}{\sqrt{d_{n,i}}} \int_{\sigma_{i-1}}^{\sigma_i} \xi(\rho) d\rho, \\
\langle \zeta, \gamma_{n,i} \rangle &= \frac{1}{\sqrt{d_{n,i}}} \int_{\sigma_{i-1}}^{\sigma_i} \zeta(\rho) d\rho.
\end{aligned}$$

Once the systems (12) and (13) are solved, the solution (u_n^K, u_n^K) are built through

$$\begin{aligned}
u_n^K &= \eta_n + (I - \pi_n)\mathcal{A}^*\eta_n + (I - \pi_n)\xi, \\
v_n^K &= \nu_n + (I - \pi_n)\mathcal{A}\nu_n + (I - \pi_n)\zeta.
\end{aligned}$$

Hence

$$\begin{aligned}
u_n^K(\sigma) &= \eta_n(\sigma) + \oint_0^1 \frac{\eta_n(\rho)}{\rho - \sigma} d\rho - \sum_{k=1}^{n} \frac{\phi_{n,k}(\sigma)}{d_{n,k}} \int_{\sigma_{k-1}}^{\sigma_k} \oint_0^1 \frac{\eta_n(\tau)}{\tau - \rho} d\tau d\rho + \xi(\sigma) - \sum_{k=1}^{n} \frac{\phi_{n,k}(\sigma)}{d_{n,k}} \int_{\sigma_{k-1}}^{\sigma_k} \xi(\rho) d\rho, \\
v_n^K(\sigma) &= \nu_n(\sigma) + \oint_0^1 \frac{\nu_n(\rho)}{\rho - \sigma} d\rho - \sum_{k=1}^{n} \frac{\phi_{n,k}(\sigma)}{d_{n,k}} \int_{\sigma_{k-1}}^{\sigma_k} \oint_0^1 \frac{\nu_n(\tau)}{\tau - \rho} d\tau d\rho + \zeta(\sigma) - \sum_{k=1}^{n} \frac{\phi_{n,k}(\sigma)}{d_{n,k}} \int_{\sigma_{k-1}}^{\sigma_k} \zeta(\rho) d\rho.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\varphi_n^K &= \frac{1}{2} \left[\eta_n + \nu_n + (I - \pi_n)\mathcal{A}^*(\eta_n + \nu_n) + (I - \pi_n)(\xi + \zeta) \right], \\
\psi_n^K &= \frac{1}{2} \left[\eta_n - \nu_n + (I - \pi_n)\mathcal{A}(\eta_n - \nu_n) + (I - \pi_n)(\xi - \zeta) \right].
\end{aligned}$$

This gives

$$\begin{aligned}\varphi_n^K(\sigma) &= \frac{1}{2}\left[\eta_n(\sigma)+\nu_n(\sigma)+\oint_0^1 \frac{\eta_n(\rho)+\nu_n(\rho)}{\sigma-\rho}d\rho - \sum_{k=1}^{n}\frac{\phi_{n,k}(\sigma)}{d_{n,k}}\int_{\sigma_{k-1}}^{\sigma_k}\oint_0^1 \frac{\eta_n(\tau)+\nu_n(\tau)}{\rho-\tau}d\tau d\rho+\xi(\sigma)\right] \\ &- \frac{1}{2}\sum_{k=1}^{n}\frac{\phi_{n,k}(\sigma)}{d_{n,k}}\int_{\sigma_{k-1}}^{\sigma_k}\xi(\rho)d\rho, \\ \psi_n^K(\sigma) &= \frac{1}{2}\left[\nu_n(\sigma)-\nu_n(\sigma)+\oint_0^1 \frac{\eta_n(\rho)-\nu_n(\rho)}{\rho-\sigma}d\rho - \sum_{k=1}^{n}\frac{\phi_{n,k}(\sigma)}{d_{n,k}}\int_{\sigma_{k-1}}^{\sigma_k}\oint_0^1 \frac{\eta_n(\tau)-\nu_n(\tau)}{\tau-\rho}d\tau d\rho+\zeta(\sigma)\right] \\ &- \frac{1}{2}\sum_{k=1}^{n}\frac{\phi_{n,k}(\sigma)}{d_{n,k}}\int_{\sigma_{k-1}}^{\sigma_k}\zeta(\rho)d\rho.\end{aligned}$$

Theorem 4. *The following estimates are available:*

$$\|\varphi_n^K-\varphi\|_2 \le \max\left\{\left[2\omega_2(u,\delta_n)\left\|(I-\pi_n)\mathcal{A}^2\,(I-\pi_n)\,u\right\|_2\right]^{\frac{1}{2}}, \left[2\omega_2(v,\delta_n)\left\|(I-\pi_n)\mathcal{A}^2\,(I-\pi_n)\,v\right\|_2\right]^{\frac{1}{2}}\right\},$$

$$\|\psi_n^K-\psi\|_2 \le \max\left\{\left[2\omega_2(u,\delta_n)\left\|(I-\pi_n)\mathcal{A}^2\,(I-\pi_n)\,u\right\|_2\right]^{\frac{1}{2}}, \left[2\omega_2(v,\delta_n)\left\|(I-\pi_n)\mathcal{A}^2\,(I-\pi_n)\,v\right\|_2\right]^{\frac{1}{2}}\right\}.$$

Proof. An analysis similar to that used in Theorem 3 in [6] reveals that

$$\|u_n^K-u\|_2 \le \left[2\omega_2(u,\delta_n)\left\|(I-\pi_n)\mathcal{A}^2\,(I-\pi_n)\,u\right\|_2\right]^{\frac{1}{2}},$$

$$\|v_n^K-v\|_2 \le \left[2\omega_2(v,\delta_n)\left\|(I-\pi_n)\mathcal{A}^2\,(I-\pi_n)\,v\right\|_2\right]^{\frac{1}{2}}.$$

Using the main formulates

$$\begin{aligned}\varphi^K-\varphi_n^K &= \frac{\left(v^K-v_n^K\right)+\left(u^K-u_n^K\right)}{2}, \\ \psi^K-\psi_n^K &= \frac{\left(v^K-v_n^K\right)-\left(u^K-u_n^K\right)}{2}\end{aligned}$$

we obtain the desired result. □

4. Numerical example

Consider the system of second-kind Cauchy integral equations (7), (8), which has the following exact solution

$$\varphi(\sigma)=\sigma^2,\ \ \psi(\sigma)=e^{-\sigma}.$$

The way of connecting errors for this example are shown in table (1).

n	$\left\|\varphi-\varphi_n^G\right\|_2\le$	$\left\|\psi-\psi_n^G\right\|_2\le$	$\left\|\varphi-\varphi_n^K\right\|_2\le$	$\left\|\psi-\psi_n^K\right\|_2\le$
3	4.10e-2	0.020e-2	3.81e-3	0.017e-3
5	2.72e-2	0.015e-2	2.93e-3	0.013e-3
7	1.63e-2	0.010e-2	1.77e-3	0.009e-3
15	1.29e-2	0.005e-2	1.13e-4	0.003e-4
25	7.66e-3	0.005e-3	8.89e-5	0.007e-5

Table 1. *Example 1*

5. Conclusions

The purpose of this paper is to enlarge the application of projection methods to a system of skew-hermitian operator equations, more accurately to a system of singular integral equations with Cauchy kernels. The foremost advantage of the new method is that it generates a system of two separable equations. As stated previously in [3, 6], the Kulkarni method produces more precise results than the traditional Galerkin method. Due to the fact that each coefficient of the matrix associated with the auxiliary linear system requires an additional evaluation of the integral operator, the Kulkarni method doubles the computation cost of the Galerkin method.

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University of Batna 2, Mostefa Ben Boulaïd
Department of Mathematics, LTM,
Fesdis, Batna 05078, Algeria
E-mail address: a.mennouni@univ-batna2.dz