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ON THE ADDITION OF A DIRAC MASS TO A q-LAGUERRE-HAHN FORM

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Abstract. Our goal is to study the addition of a Dirac mass to a H_q-Laguerre-Hahn form where H_q be the q-derivative operator. The H_q-Laguerre-Hahn character and the class of the obtained form is discussed into detail. An example in connection with the first order associated of a H_q-classical form is highlighted.

1. Introduction and preliminaries

The addition of a Dirac mass to a regular and D-semiclassical form was studied by F. Marcellán and P. Maroni in [7] where D is the derivative operator. Later, F. Marcellán and E. Prianes have studied the addition problem of a D-Laguerre-Hahn form [8]. In [3], the basic theory of H_q-Laguerre-Hahn (q-Laguerre-Hahn in short) forms (linear functionals) and a few generic examples related to some standard transformations (association, co-recursion, inversion) of H_q-classical [4, 6] or more generally H_q-semiclassical q-polynomials [5] were studied, where H_q be the q-derivative operator (see also [2]).

So, the aim of this work is to construct some new q-Laguerre-Hahn forms of class greater to one from old one's by using the following standard perturbation

$$\breve{u} = u + \lambda\delta_c,$$

or equivalently,

$$(x-c)\breve{u} = \lambda(x-c)u,$$

where c is a complex number, δ_c be the Dirac measure at c ($\delta_0 := \delta$), λ a non null complex number and u be a q-Laguerre-Hahn form of class s. The q-Laguerre-Hahn character of $\breve{u}$ is studied for any complex c. The variation of the class is examined into detail for $c = 0$ in order to avoid long calculations and an example in connection with the first associated of the natural q-analogue of Hermite is emphasized and provides two q-Laguerre-Hahn new forms of class 1 and 2 depending on the value of the parameter λ.

We denote by $\mathcal{P}$ the vector space of the polynomials with coefficients in $\mathbb{C}$ and by $\mathcal{P}'$ its dual space whose elements are forms. The action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ is denoted as $\langle u, f\rangle$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$ the moments of u. For instance, for any form u, any polynomial g and any $(a, c) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$, we let $H_q u$, gu, $h_a u$, Du, $(x-c)^{-1}u$ and δ_c, be the forms defined as usually [9] and [4] for the results related to the operator H_q

$$\langle H_q u, f\rangle := -\langle u, H_q f\rangle \ , \ \langle gu, f\rangle := \langle u, gf\rangle \ , \ \langle h_a u, f\rangle := \langle u, h_a f\rangle \ ,$$

$$\langle Du, f\rangle := -\langle u, f'\rangle \quad , \quad \langle (x-c)^{-1}u, f\rangle := \langle u, \theta_c f\rangle \quad , \quad \langle \delta_c, f\rangle := f(c) \quad ,$$

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where for all $f \in \mathcal{P}$ and $q \in \widetilde{\mathbb{C}} := \Big\{ z \in \mathbb{C},\ z \neq 0, z^n \neq 1, n \geq 1 \Big\}$ [4]

$$(H_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}\ ,\ (h_a f)(x) = f(ax)\ ,\ (\theta_c f)(x) = \frac{f(x) - f(c)}{x - c}.$$

In particular, this yields

$$(H_q u)_n = -[n]_q (u)_{n-1}\ ,\ n \geq 0\ ,$$

where $(u)_{-1} = 0$ and $[n]_q := \dfrac{q^n - 1}{q - 1}\ ,\ n \geq 0$ [4]. It is obvious that when $q \to 1$, we meet again the derivative D.
For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the product uf is the polynomial [9]

$$(uf)(x) := \langle u, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta} \rangle = \sum_{i=0}^{n} \Big(\sum_{j=i}^{n} (u)_{j-i}\, f_j \Big) x^i,$$

where $f(x) = \displaystyle\sum_{i=0}^{n} f_i x^i$. This allows us to define the Cauchy's product of two forms:

$$\langle uv, f \rangle := \langle u, vf \rangle\ ,\ f \in \mathcal{P}.$$

The Stieltjes formal series of $u \in \mathcal{P}'$ is defined by [9]

$$S(u)(z) := -\sum_{n \geq 0} \frac{(u)_n}{z^{n+1}}.$$

A form u is said to be regular whenever there is a sequence of monic polynomials $\{P_n\}_{n\geq 0}$, $\deg P_n = n$, $n \geq 0$ MPS such that $\langle u, P_n P_m \rangle = r_n \delta_{n,m}$ with $r_n \neq 0$ for any $n, m \geq 0$. In this case, $\{P_n\}_{n\geq 0}$ is called a monic orthogonal polynomials sequence MOPS and it is characterized by the following three-term recurrence relation (Favard's theorem) (TTRR in short) [1, 9]

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ &P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x), \quad n \geq 0, \end{aligned} \tag{1}$$

where $\beta_n = \dfrac{\langle u, xP_n^2 \rangle}{r_n} \in \mathbb{C}$, $\gamma_{n+1} = \dfrac{r_{n+1}}{r_n} \in \mathbb{C} \setminus \{0\},\ n \geq 0.$

The shifted MOPS $\{\widehat{P}_n := a^{-n}\big(h_a P_n\big)\}_{n\geq 0}$ is then orthogonal with respect to $\widehat{u} = h_{a^{-1}} u$ and satisfies (1.1) with [9]

$$\widehat{\beta}_n = \frac{\beta_n}{a} \quad ,\quad \widehat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2} \quad ,\quad n \geq 0.$$

Moreover, the form u is said to be normalized if $(u)_0 = 1$. In this paper, we suppose that any regular form will be normalized. The form u is said to be positive definite if and only if $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} > 0$ for all $n \geq 0$. When u is regular, $\{P_n\}_{n\geq 0}$ is a symmetrical MOPS if and only if $\beta_n = 0,\ n \geq 0$ or equivalently $(u)_{2n+1} = 0,\ n \geq 0$ [1].

Given a regular form u and the corresponding MOPS $\{P_n\}_{n\geq 0}$, we define the associated sequence of the first kind $\{P_n^{(1)}\}_{n\geq 0}$ of $\{P_n\}_{n\geq 0}$ by [9]

$$P_n^{(1)}(x) = \Big\langle u, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x - \xi} \Big\rangle = (u\theta_0 P_{n+1})(x),\ n \geq 0.$$

Proposition 1. [9] *Let $\{P_n\}_{n\geq 0}$ be a MOPS satisfying the TTRR (1.1), then its associated sequence $\{P_n^{(1)}\}_{n\geq 0}$ satisfies the TTRR*

$$P_0^{(1)}(x)=1,\quad P_1^{(1)}(x)=x-\beta_1,$$

$$P_{n+2}^{(1)}(x)=(x-\beta_{n+2})P_{n+1}^{(1)}(x)-\gamma_{n+2}P_n^{(1)}(x),\quad n\geq 0,$$

Lemma 1. [4, 5, 9] *For $f,g\in\mathcal{P}$, $u,v\in\mathcal{P}'$, $a\in\mathbb{C}\setminus\{0\}$ and $c\in\mathbb{C}$, we have*

$$h_a(gu)=(h_{a^{-1}}g)(h_au)\,,\ h_a(uv)=(h_au)(h_av)\,,\ h_a\delta_c=\delta_{ac}\,,\ f(x)\delta_c=f(c)\delta_c, \tag{2}$$

$$h_{q^{-1}}\circ H_q=H_{q^{-1}}\quad,\quad H_q\circ h_{q^{-1}}=q^{-1}H_{q^{-1}}\ ,\ in\ \mathcal{P}, \tag{3}$$

$$h_{q^{-1}}\circ H_q=q^{-1}H_{q^{-1}}\quad,\quad H_q\circ h_{q^{-1}}=H_{q^{-1}}\ ,\ in\ \mathcal{P}', \tag{4}$$

$$H_q(fg)(x)=(h_qf)(x)(H_qg)(x)+g(x)(H_qf)(x), \tag{5}$$

$$H_q(gu)=(h_{q^{-1}}g)H_qu+q^{-1}(H_{q^{-1}}g)u, \tag{6}$$

$$S(H_qu)(z)=q^{-1}(H_{q^{-1}}(S(u)))(z)\quad,\quad (h_{q^{-1}}S(u))(z)=qS(h_qu)(z), \tag{7}$$

$$H_q(gu)=gH_qu+(H_{q^{-1}}g)h_qu, \tag{8}$$

$$(x-\tau)^{-1}\big((x-\tau)u\big)=u-(u)_0\delta_\tau,\quad (x-\tau)\big((x-\tau)^{-1}u\big)=u, \tag{9}$$

$$f(x^{-1}u)=x^{-1}(fu)+\langle u,\theta_0f\rangle\delta, \tag{10}$$

$$f(uv)=(fu)v+x(u\theta_0f)(x)v, \tag{11}$$

$$S(fu)(z)=f(z)S(u)(z)+(u\theta_0f)(z). \tag{12}$$

We will give now some future about the q-Laguerre-Hahn character.

Definition 1. [3] *A form u is called q-Laguerre-Hahn when it is regular and satisfies the q-difference equation*

$$H_q(\Phi u)+\Psi u+B\big(x^{-1}u(h_qu)\big)=0, \tag{13}$$

where Φ,Ψ,B are polynomials, with Φ monic. The corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ is called q-Laguerre-Hahn MOPS.

Remark 1.

1. *When $B=0$ and the form u is regular then u is H_q-semiclassical* [5].
2. *When u satisfies* (13), *then $\widehat{u}=h_{a^{-1}}u$ fulfills the q-difference equation* [3]
$$H_q(a^{-\deg\Phi}\Phi(ax)\widehat{u})+a^{1-\deg\Phi}\Psi(ax)\widehat{u}+a^{-\deg\Phi}B(ax)\big(x^{-1}\widehat{u}(h_q\widehat{u})\big)=0. \tag{14}$$
3. *Put $t=\deg\Phi$, $p=\deg\Psi$, $r=\deg B$ and $d=\max(t,r)$, we define the class of u the nonnegative integer s* [3]
$$s=\min\max(p-1,d-2)$$
where the minimum is taken over all triplets (Φ,Ψ,B) satisfying (13). Moreover, the q-Laguerre-Hahn form u satisfying (13) *is of class $s=\max(p-1,d-2)$ if and only if*

$$\prod_{c\in\mathcal{Z}_\Phi}\left\{\big|q(h_q\Psi)(c)+(H_q\Phi)(c)\big|+\big|q(h_qB)(c)\big|+\left|\left\langle u,q(\theta_{cq}\Psi)+(\theta_{cq}\circ\theta_c\Phi)+q\big(h_qu(\theta_0\circ\theta_{cq}B)\big)\right\rangle\right|\right\}>0, \tag{15}$$

where $\mathcal{Z}_\Phi$ is the set of roots of Φ [3]. *When $c\in\mathcal{Z}_\Phi$ and (13) may be simplified by $x-c$, then (13) becomes*

$$H_q((\theta_c\Phi)u)+(q\theta_{cq}\Psi+\theta_{cq}\circ\theta_c\Phi)u+q(\theta_{cq}B)\big(x^{-1}u(h_qu)\big)=0. \tag{16}$$

Proposition 2. [3] *Let u be a regular form. the following statement are equivalents:*

(a) *u belongs to the q-Laguerre-Hahn class, satisfying* (13).

(b) *The Stieljes formal series $S(u)$ satisfies the q-Riccati equation*

$$(h_{q^{-1}}\Phi)(z)H_{q^{-1}}(S(u))(z) = B(z)S(u)(z)(h_{q^{-1}}S(u))(z) + C(z)S(u)(z) + D(z), \tag{17}$$

where Φ and B are polynomials defined in (13) *and*

$$\begin{cases} C(z) = -(H_{q^{-1}}\Phi)(z) - q\Psi(z) \\ D(z) = -\big\{H_{q^{-1}}(u\theta_0\Phi)(z) + q(u\theta_0\Psi)(z) + q(uh_q u)(\theta_0^2 B)(z)\big\}. \end{cases} \tag{18}$$

Moreover, u is of class s if and only if

$$\prod_{c\in Z_\Phi}\Big\{|B(cq)| + |C(cq)| + |D(cq)|\Big\} > 0, \tag{19}$$

and one may write

$$s = \max\big(\deg B - 2, \deg C - 1, \deg D\big). \tag{20}$$

Lastly, the following results and notations will be needed in the sequel.

Lemma 2. [3] *If u be a q-Laguerre-Hahn form of class s fulfilling (13) such that its Stieltjes formal series $S\big(u\big)$ satisfies (17), then the associated form $u^{(1)}$ is q-Laguerre-Hahn of the same class s fulfilling (21) and its Stieltjes formal series $S\big(u^{(1)}\big)$ satisfying (22) where*

$$H_q(\Phi^{(1)}u^{(1)}) + \Psi^{(1)}u^{(1)} + B^{(1)}\big(x^{-1}u^{(1)}(h_q u^{(1)})\big) = 0, \tag{21}$$

$$(h_{q^{-1}}\Phi^{(1)})(z)H_{q^{-1}}(S(u^{(1)}))(z) = \\ B^{(1)}(z)S(u^{(1)})(z)(h_{q^{-1}}S(u^{(1)}))(z) + C^{(1)}(z)S(u^{(1)})(z) + D^{(1)}(z), \tag{22}$$

with

$$\begin{cases} K\Phi^{(1)}(x) = \Phi(x) + (q-1)x\{(qx-\beta_0)(h_q D)(x) - (h_q C)(x)\}\ , \\ K\Psi^{(1)}(x) = -\big\{q^{-1}(H_{q^{-1}}\Phi)(x) + q^{-1}(q^{-1}x-\beta_0)D(x) + (qx-\beta_0)(h_q D)(x) - (h_q C)(x)\big\}, \\ KB^{(1)}(x) = \gamma_1 D(x)\ , \\ KC^{(1)}(x) = ((q^{-1}+1)x - 2\beta_0)D(x) - C(x)\ , \\ KD^{(1)}(x) = \gamma_1^{-1}\big\{B(x) + (q^{-1}x-\beta_0)(x-\beta_0)D(x) - (q^{-1}x-\beta_0)C(x) - (h_{q^{-1}}\Phi)(x)\big\}, \end{cases} \tag{23}$$

and K is a normalization constant.

The quantum factorial symbol is defined by [4]

$$(x;q)_0 = 1,\ (x;q)_n = \prod_{k=0}^{n-1}(1-xq^k),\ x,q\in\mathbb{C}. \tag{24}$$

2. The addition of a Dirac mass problem: the q-Laguerre-Hahn case

Let u be a regular form and $\{P_n\}_{n\geq 0}$ its MOPS satisfying the TTRR (1). Let $\check{u}\in\mathcal{P}'$ such that

$$\check{u} = u + \lambda\delta_c, \quad \lambda, c\in\mathbb{C}, \tag{25}$$

or equivalently,

$$(x-c)\check{u} = \lambda(x-c)u. \tag{26}$$

It is seen in [7] that $\check{u}$ is regular, if and only if, $\lambda\neq\lambda_n,\ n\geq 0$ where

$$\lambda_n = -\left(\sum_{\nu=0}^{n}\frac{P_\nu^2(c)}{\tau_\nu}\right)^{-1}, \quad \tau_n = \prod_{\nu=0}^{n}\gamma_\nu,\ n\geq 0,\ \gamma_0 := 1. \tag{27}$$

In this case, denoting $\{\check{P}_n\}_{n\geq 0}$ its MOPS and $\check{\beta}_n,\ \check{\gamma}_{n+1},\ n\geq 0$ the recurrence elements of its TTRR, we have [7]

$$\begin{cases} (x-c)\check{P}_{n+1}(x) = P_{n+2}(x) + b_{n+1}P_{n+1}(x) + \dfrac{d_{n+1}}{d_n}P_n(x),\ n\geq 0, \\ b_0 = \beta_0 - c, \quad b_{n+1} = \beta_{n+1} - c - \lambda\dfrac{P_n(c)P_{n+1}(c)}{d_n},\ n\geq 0, \\ \check{\beta}_n = \beta_{n+1} + b_n - b_{n+1},\ n\geq 0, \\ \check{\gamma}_{n+1} = \gamma_n\,\dfrac{d_{n+1}d_{n-1}}{d_n^2},\ n\geq 0, \qquad d_0 = 1+\lambda, \quad d_{-1} := 1,\ \check{\gamma}_0 = 1+\lambda. \end{cases} \tag{28}$$

with

$$d_n = \left(\prod_{\nu=0}^{n}\gamma_\nu\right)\left(1+\lambda\sum_{\nu=0}^{n}\frac{P_\nu^2(c)}{\tau_\nu}\right),\ n\geq 0. \tag{29}$$

Moreover [7],

- ▷ when u is positive definite and $c\in\mathbb{R}$, then the form $\check{u}$ is positive definite for any $\lambda>0$ and regular for any $\lambda\in\mathbb{C}\backslash]-\infty,0[$.
- ▷ When u is symmetric regular and real, then for any c,λ such that $\Re(c) = 0,\ \Im(\lambda)\neq 0$, the form $\check{u}$ is regular.

2.1. **The q-Laguerre-Hahn character of $\check{u}$.** From now on, let u be a q-Laguerre-Hahn form of class s satisfying (13) and its corresponding MOPS $\{P_n\}_{n\geq 0}$ fulfilling the TTRR (1). We suppose that $\lambda\neq\lambda_n,\ n\geq 0$. Consequently, the form $\check{u}$ defined by (25) is regular. We are going to study the q-Laguerre-Hahn character of $\check{u}$ and the variation of its class $\check{s}$ according to that of u.

Proposition 3. *The regular form $\check{u}$ is q-Laguerre-Hahn of class $\check{s}$ such that $s-2\leq\check{s}\leq s+2$ and satisfying the q-difference equation*

$$H_q\big(\check{\Phi}(x)\check{u}\big) + \check{\Psi}(x)\check{u} + \check{B}(x)\big(x^{-1}\check{u}(h_q\check{u})\big) = 0, \tag{30}$$

with

$$\begin{cases} K\check{\Phi}(x) = q(x-c)\{(qx-c)\Phi(x) + \lambda(q-1)xB(qx)\}, \\ K\check{\Psi}(x) = (x-c)\{(x-cq)\Psi(x) - (1+q)\Phi(x) - \lambda(B(x)+qB(qx))\}, \\ K\check{B}(x) = (x-c)(x-cq)B(x). \end{cases} \tag{31}$$

In addition, its Stieltjes formal series $S(\check{u})$ satisfies the q-Riccati equation

$$(h_{q^{-1}}\check{\Phi})(z)H_{q^{-1}}(S(\check{u}))(z) = \check{B}(z)S(\check{u})(z)(h_{q^{-1}}S(\check{u}))(z) + \check{C}(z)S(\check{u})(z) + \check{D}(z), \tag{32}$$

where

$$\begin{cases} K\check{\Phi}(z) = q(z-c)\{(qz-c)\Phi(z) + \lambda(q-1)zB(qz)\}, \\ K\check{B}(z) = (z-c)(z-cq)B(z), \\ K\check{C}(z) = \{(z-c)(z-cq)C(z) + \lambda\big((1+q)z-2cq\big)B(z)\}, \\ K\check{D}(z) = \{(z-c)(z-cq)D(z) + q\lambda^2B(z) + \lambda(z-cq)C(z) + \lambda(h_{q^{-1}}\Phi)(z)\}. \end{cases} \tag{33}$$

Proof. Accordingly to (25) and the third formula in (2), the q-difference equation (13) is equivalent to

$$H_q\big(\Phi(x)(\check{u}-\lambda\delta_c)\big) + \Psi(x)(\check{u}-\lambda\delta_c) + B(x)\big(x^{-1}(\check{u}-\lambda\delta_c)((h_q\check{u})-\lambda\delta_{cq})\big) = 0. \tag{34}$$

Multiplying (34) by $(x-c)(x-cq)$, we have for the second term and the first one of the obtained q-difference equation,

$$(x-c)(x-cq)\Psi(x)(\check{u}-\lambda\delta_c) = (x-c)(x-cq)\Psi(x)\check{u},$$

$$(x-c)(x-cq)H_q\big(\Phi(x)(\check{u}-\lambda\delta_c)\big) = H_q\big(q(x-c)(qx-c)\Phi(x)(\check{u}-\lambda\delta_c)\big) -(1+q)(x-c)\Phi(x)(\check{u}-\lambda\delta_c)$$
$$= H_q\big(q(x-c)(qx-c)\Phi(x)\check{u}\big)-(1+q)(x-c)\Phi(x)\check{u},$$

thanks to (6) and the fact that $(x-c)\delta_c = 0$.

To simplify the third term $(x-c)(x-cq)B(x)\big(x^{-1}(\check{u}-\lambda\delta_c)((h_q\check{u})-\lambda\delta_{cq})\big)$, we have

$$(x-c)(x-cq)\big(x^{-1}(\check{u}-\lambda\delta_c)((h_q\check{u})-\lambda\delta_{cq})\big) = (x-c)(x-cq)\Big(x^{-1}\check{u}h_q\check{u}-\lambda x^{-1}(\delta_c h_q\check{u}+\check{u}\delta_{cq})+\lambda^2 x^{-1}\delta_c\delta_{cq}\Big)$$

with

$$(x-c)(x-cq)\Big(x^{-1}\delta_c h_q\check{u}\Big) = x^{-1}\Big((x-c)(x-cq)(\delta_c h_q\check{u})\Big)+\langle\delta_c h_q\check{u}, x-c(1+q)\rangle\delta$$
$$= x^{-1}\Big(x(x-cq)h_q\check{u}\Big)+\langle h_q\check{u}, x-cq\rangle\delta$$
$$= (x-cq)h_q\check{u}-((x-cq)h_q\check{u})_0\delta+\langle h_q\check{u}, x-cq\rangle\delta = (x-cq)h_q\check{u},$$

$$(x-c)(x-cq)\Big(x^{-1}\check{u}\delta_{cq}\Big) = x^{-1}\Big((x-c)(x-cq)(\delta_{cq}\check{u})\Big)+\langle\delta_{cq}\check{u}, x-c(1+q)\rangle\delta$$
$$= x^{-1}\Big(x(x-c)\check{u}\Big)+\langle\check{u}, x-c\rangle\delta$$
$$= (x-c)\check{u}-((x-c)\check{u})_0\delta+\langle\check{u}, x-c\rangle\delta = (x-c)\check{u},$$

and

$$(x-c)(x-cq)\Big(x^{-1}\delta_c\delta_{cq}\Big) = x^{-1}\Big((x-c)(x-cq)(\delta_c\delta_{cq})\Big)+\langle\delta_c\delta_{cq}, x-c(1+q)\rangle\delta$$
$$= x^{-1}\Big(x(x-cq)\delta_{cq}\Big)+\langle\delta_{cq}, x-cq\rangle\delta = 0,$$

since (2), (9)-(11) and definitions. Therefore, the third term of the obtained q-difference equation becomes

$$(x-c)(x-cq)B(x)\big(x^{-1}(\check{u}-\lambda\delta_c)((h_q\check{u})-\lambda\delta_{cq})\big) =$$
$$(x-c)(x-cq)B(x)\big(x^{-1}\check{u}h_q\check{u}\big)-\lambda(x-cq)B(x)h_q\check{u}-\lambda(x-c)B(x)\check{u}. \qquad (35)$$

Now, combining (6) and (8) this allows to deduce that

$$h_q\check{u} = (1-q)H_q(x\check{u})+\check{u}. \qquad (36)$$

Injecting (36) in (35) and thanks to (6) another time leads to

$$(x-c)(x-cq)B(x)\big(x^{-1}(\check{u}-\lambda\delta_c)((h_q\check{u})-\lambda\delta_{cq})\big) = (x-c)(x-cq)B(x)\big(x^{-1}\check{u}h_q\check{u}\big)$$
$$-\lambda\Big\{q(1-q)H_q\big(x(x-c)B(qx)\check{u}\big)+(x-c)(B(x)+qB(qx))\check{u}\Big\}.$$

Consequently, we obtain (30)-(31). Moreover, on account of (31) and the definition of the class, it is easy to see that $\check{s}\le s+2$. The multiplication of (17) by $(z-c)(z-cq)$, while using (5) and the formula (see (12) for $f(z)=z-c$) $S((z-c)u) = (z-c)S(u)+1$, give

$$h_{q^{-1}}(q^2(z-c)(z-cq^{-1})\Phi)(z)H_{q^{-1}}(S(\check{u}))(z) = (z-c)(z-cq)B(z)S(\check{u})(z)(h_{q^{-1}}S(\check{u}))(z)$$
$$+S(\check{u})(z)\{(z-c)(z-cq)C(z)q\lambda(z-c)B(z)\}+\lambda(z-cq)B(z)\big(h_{q^{-1}}S(\check{u})\big)(z)$$
$$+\{q\lambda^2B(z)+\lambda(z-cq)C(z)+(z-c)(z-cq)D(z)+\lambda(h_{q^{-1}}\Phi)(z)\}. \quad (37)$$

But, from (7)-(8) we have

$$\big(h_{q^{-1}}S(\check{u})\big)(z) = qS(h_q\check{u})(z) = S(\check{u})(z)-(1-q^{-1})z\big(H_{q^{-1}}S(\check{u})\big)(z). \qquad (38)$$

By replacing (38) in (37), we obtain the desired result (32)-(33). Lastly, we have from definitions $\check{t} = \deg\check{\Phi}\le\max(\deg\Phi+2, \deg B+2)\le s+4$, $\check{p} = \deg\check{\Psi}\le\max(\deg\Psi+2, \deg\Phi+1, \deg B+1)\le s+3$, $\check{r} = \deg\check{B} = \deg B+2\le s+4$. Then, $\check{d} = \max(\check{t},\check{r})\le s+4$, $\check{s} = \max(\check{p}-1, \check{d}-2)\le s+2$. On the other hand, since $u = \check{u}-\lambda\delta_c$ is a regular form we deduce from the above result that $s\le\check{s}+2$. Therefore, $s-2\le\check{s}$. □

Lemma 3. *Let $\check{u}$ the q-Laguerre-Hahn form which its Stieltjes formal series satisfies (32)-(33).*

(1) *For any root τ of $\check{\Phi}$ such that $\tau \neq c$ and $\tau \neq cq^{-1}$, the equation (32) with (33) cannot be simplified by $z - \tau q$.*

(2) *If τ is a root of $\check{\Phi}$ such that $\tau = c$, the equation (32) with (33) cannot be simplified by $z - cq$ if and only if*

$$| cq(q-1)B(qc) | + | q\lambda B(qc) + \Phi(c) | \neq 0. \tag{39}$$

(3) *If τ is a root of $\check{\Phi}$ such that $\tau = cq^{-1}$, the equation (32) with (33) cannot be simplified by $z - c$ if and only if*

$$| c(1-q)B(c) | + | q\lambda B(c) + \Phi(cq^{-1}) + c(1-q)C(c) | \neq 0. \tag{40}$$

Proof. (1) Let τ be a root of the polynomial $\check{\Phi}$, such that $\tau \neq c$ and $\tau \neq cq^{-1}$. If $\check{B}(q\tau) = \check{C}(q\tau) = \check{D}(q\tau) = 0$, then $B(q\tau) = 0$, which gives from the expression of $\check{\Phi}$, $\Phi(\tau) = 0$. Therefore $\Phi(\tau) = 0$ since $\tau \neq c$ and $\tau \neq cq^{-1}$. Now, since $B(q\tau) = \big(h_{q^{-1}}\Phi\big)(q\tau) = \Phi(\tau) = 0$ we have $C(q\tau) = 0$. Next, from the expression of $\check{D}$, we see that $D(q\tau) = 0$ since $\check{D}(q\tau) = B(q\tau) = C(q\tau) = 0$ and $\Phi(\tau) = 0$. Thus, $B(q\tau) = C(q\tau) = D(q\tau) = 0$. From (17), we deduce then a contradiction since u is a q-Laguerre-Hahn form of class s.
(2)-(3) If c is a root of $\check{\Phi}$, then $\check{B}(qc) = 0$, $K\check{C}(qc) = \lambda cq(q-1)B(qc)$ and $\check{D}(qc) = q\lambda^2 B(qc) + \lambda\Phi(c)$ and if cq^{-1} is a root of $\check{\Phi}$, then $\check{B}(c) = 0$, $K\check{C}(c) = \lambda c(q-1)B(c)$ and $K\check{D}(c) = q\lambda^2 B(c) + \lambda c(1-q)C(c) + \lambda\Phi(cq^{-1})$. Thus, the results in (39)-(40) are consequences of (17). □

2.2. **The class of $\check{u}$ when $c = 0$.** Now, $\check{u} = u + \lambda\delta$ with $\lambda \neq \lambda_n$, $n \geq 0$. From Proposition 3. and Lemma 3., we have

($c_{2,1}$) The regular form $\check{u}$ is q-Laguerre-Hahn form of class $\check{s}$ such that $s-2 \leq \check{s} \leq s+2$ and satisfying the q-difference equation

$$H_q\big(\check{\Phi}(x)\check{u}\big) + \check{\Psi}(x)\check{u} + \check{B}(x)\big(x^{-1}\check{u}(h_q\check{u})\big) = 0, \tag{41}$$

with

$$\begin{cases} K\check{\Phi}(x) = qx^2\{q\Phi(x) + \lambda(q-1)B(qx)\}, \quad K\check{B}(x) = x^2 B(x), \\ K\check{\Psi}(x) = x\{x\Psi(x) - (1+q)\Phi(x) - \lambda(B(x) + qB(qx))\}. \end{cases} \tag{42}$$

In addition, its Stieltjes formal series $S(\check{u})$ satisfies the q-Riccati equation

$$(h_{q^{-1}}\check{\Phi})(z)H_{q^{-1}}(S(\check{u}))(z) = \check{B}(z)S(\check{u})(z)(h_{q^{-1}}S(\check{u}))(z) + \check{C}(z)S(\check{u})(z) + \check{D}(z), \tag{43}$$

where

$$\begin{cases} K\check{\Phi}(z) = qz^2\{q\Phi(z) + \lambda(q-1)B(qz)\}, \quad K\check{B}(z) = z^2 B(z), \\ K\check{C}(z) = z\{zC(z) + \lambda(1+q)B(z)\}, \\ K\check{D}(z) = z^2 D(z) + q\lambda^2 B(z) + \lambda z C(z) + \lambda(h_{q^{-1}}\Phi)(z). \end{cases} \tag{44}$$

($c_{2,2}$) For any root τ of $\check{\Phi}$ such that $\tau \neq 0$, the equation (43) with (44) cannot be simplified by $z - \tau$.

($c_{2,3}$) If τ is a root of $\check{\Phi}$ such that $\tau = 0$, the equation (43 with (44) cannot be simplified by z if and only if

$$| q\lambda B(0) + \Phi(0) | \neq 0. \tag{45}$$

Theorem 1. *The form $\check{u} = u + \lambda\delta$ when it is regular is q-Laguerre-Hahn of class $\check{s}$ fulfilling (41)-(42) and its Stieltjes formal series $S(\check{u})$ satisfies the q-Riccati equation (43) with (44). Moreover,*

(1) *If $q\lambda B(0) + \Phi(0) \neq 0$, then $\check{s} = s+2$. In this case, $S(\check{u})$ satisfies the q-Riccati equation*

(43) with (44).

(2) *If* $\big(q\lambda B(0)+\Phi(0)=0,\ \ B(0)\neq 0\big)$ *or* $\big(q\lambda B(0)+\Phi(0)=0,\ \ C(0)+q\lambda B^{'}(0)+q^{-1}\Phi(0)\neq 0\big)$*, then* $\check{s}=s+1$*. In this case,* $S(\check{u})$ *satisfies the q-Riccati equation*

$$(h_{q^{-1}}\check{\Phi}_{0,1})(z)H_{q^{-1}}(S(\check{u}))(z)=\check{B}_{0,1}(z)S(\check{u})(z)(h_{q^{-1}}S(\check{u}))(z)+\check{C}_{0,1}(z)S(\check{u})(z)+\check{D}_{0,1}(z),\quad (46)$$

where

$$\begin{cases} K\check{\Phi}_{0,1}(z)=z\{q^2\Phi(z)+\lambda q(q-1)B(qz)\}, \quad K\check{B}_{0,1}(z)=zB(z),\\ K\check{C}_{0,1}(z)=zC(z)+\lambda(1+q)B(z),\\ K\check{D}_{0,1}(z)=zD(z)+q\lambda^2B_{0,1}(z)+\lambda C(z)+\lambda q^{-1}(h_{q^{-1}}\Phi_{0,1})(z). \end{cases}\quad (47)$$

(3) *If* $\big((B(0),\Phi(0))=(0,0),\ \ C(0)+q\lambda B^{'}(0)=0,\ \ B^{'}(0)\neq 0\big)$ *or* $\big((B(0),\Phi(0))=(0,0),$ $C(0)+q\lambda B^{'}(0)=0,\ \ D(0)+\lambda C^{'}(0)+\frac{q\lambda^2}{2}B^{''}(0)+\frac{q^{-2}\lambda}{2}\Phi^{''}(0)\neq 0\ \big)$*, then* $\check{s}=s$*. In this case,* $S(\check{u})$ *satisfies the q-Riccati equation*

$$(h_{q^{-1}}\check{\Phi}_{0,2})(z)H_{q^{-1}}(S(\check{u}))(z)=\check{B}_{0,2}(z)S(\check{u})(z)(h_{q^{-1}}S(\check{u}))(z)+\check{C}_{0,2}(z)S(\check{u})(z)+\check{D}_{0,2}(z),\quad (48)$$

where

$$\begin{cases} K\check{\Phi}_{0,2}(z)=q^2\Phi(z)+\lambda q(q-1)B(qz), \quad K\check{B}_{0,2}(z)=B(z),\\ K\check{C}_{0,2}(z)=C(z)+\lambda(1+q)B_{0,1}(z),\\ K\check{D}_{0,2}(z)=D(z)+q\lambda^2B_{0,2}(z)+\lambda C_{0,1}(z)+\lambda q^{-2}(h_{q^{-1}}\Phi_{0,2})(z). \end{cases}\quad (49)$$

(4) *If* $\big((B(0),\Phi(0))=(C(0),B^{'}(0))=(0,0),\ \ D(0)+\lambda C^{'}(0)+\frac{q\lambda^2}{2}B^{''}(0)+\frac{q^{-2}\lambda}{2}\Phi^{''}(0)=0,$ $\frac{\lambda(1+q)}{2}B^{''}(0)+C^{'}(0)\neq 0\ \big)$ *or* $\big((B(0),\Phi(0))=(C(0),B^{'}(0))=(0,0),\ \ D(0)+\lambda C^{'}(0)+$ $\frac{q\lambda^2}{2}B^{''}(0)+\frac{q^{-2}\lambda}{2}\Phi^{''}(0)=0,\ \ D^{'}(0)+\frac{\lambda}{2}C^{''}(0)+\frac{q\lambda^2}{3!}B^{'''}(0)+\frac{q^{-3}\lambda}{3!}\Phi^{'''}(0)\neq 0\big)$*, then* $\check{s}=s-1$*. In this case,* $S(\check{u})$ *satisfies the q-Riccati equation*

$$(h_{q^{-1}}\check{\Phi}_{0,3})(z)H_{q^{-1}}(S(\check{u}))(z)=\check{B}_{0,3}(z)S(\check{u})(z)(h_{q^{-1}}S(\check{u}))(z)+\check{C}_{0,3}(z)S(\check{u})(z)+\check{D}_{0,3}(z),\quad (50)$$

where

$$\begin{cases} K^{'}\check{\Phi}_{0,3}(z)=\{q^2\Phi_{0,1}(z)+\lambda q^2(q-1)B(qz)\}, \quad K^{'}\check{B}_{0,3}(z)=B_{0,1}(z),\\ K^{'}\check{C}_{0,3}(z)=C_{0,1}(z)+\lambda(1+q)B_{0,1}(z),\\ K^{'}\check{D}_{0,3}(z)=D_{0,1}(z)+q\lambda^2B_{0,3}(z)+\lambda C_{0,2}(z)+\lambda q^{-3}(h_{q^{-1}}\Phi_{0,1})(z). \end{cases}\quad (51)$$

(5) *If* $\big((B(0),\Phi(0))=(C(0),B^{'}(0))=(0,0),\ \ D(0)-\frac{\lambda^2}{2}B^{''}(0)+\frac{q^{-2}\lambda}{2}\Phi^{''}(0)=0,\ \ D^{'}(0)+$ $\frac{\lambda}{2}C^{''}(0)+\frac{q\lambda^2}{3!}B^{'''}(0)+\frac{q^{-3}\lambda}{3!}\Phi^{'''}(0)=0\ \big)$*, then* $\check{s}=s-2$*. In this case,* $S(\check{u})$ *satisfies the q-Riccati equation*

$$(h_{q^{-1}}\check{\Phi}_{0,4})(z)H_{q^{-1}}(S(\check{u}))(z)=\check{B}_{0,4}(z)S(\check{u})(z)(h_{q^{-1}}S(\check{u}))(z)+\check{C}_{0,4}(z)S(\check{u})(z)+\check{D}_{0,4}(z),\quad (52)$$

where

$$\begin{cases} K^{''}\check{\Phi}_{0,4}(z)=q^2\Phi_{0,2}(z)+\lambda q^3(q-1)B_{0,2}(qz), \quad K^{''}\check{B}_{0,4}(z)=B_{0,2}(z),\\ K^{''}\check{C}_{0,4}(z)=C_{0,2}(z)+\lambda(1+q)B_{0,3}(z),\\ K^{''}\check{D}_{0,4}(z)=D_{0,2}(z)+q\lambda^2B_{0,4}(z)+\lambda C_{0,3}(z)+\lambda q^{-4}(h_{q^{-1}}\Phi_{0,4})(z), \end{cases}\quad (53)$$

and for $0\leq i\leq 3$,

$$\begin{cases} \Phi_{0,i}(z)=z\Phi_{0,i+1}(z)+t_{0,i+1},\ \Phi_{0,0}=\Phi,\ B_{0,i}(z)=zB_{0,i+1}(z)+s_{0,i+1},\ B_{0,0}=B,\\ C_{0,i}(z)=zC_{0,i+1}(z)+q_{0,i+1},\ C_{0,0}=C,\ D_{0,i}(z)=zD_{0,i+1}(z)+p_{0,i+1},\ D_{0,0}=D. \end{cases}$$

Proof. On account of $(c_{2,1})$ we have $s-2\leq\check{s}\leq s+2$ and $S(\check{u})$ satisfies the q-Riccati equation (43) with (44). Therefore,
(1) if $q\lambda B(0)+\Phi(0)\neq 0$, then $\check{s}=s+2$ since $(c_{2,2})-(c_{2,3})$ and $S(\check{u})$ satisfies the q-Riccati equation (43)-(44).
(2) If $q\lambda B(0)+\Phi(0)=0$, we have 0 is a root of $\check{\Phi}$, so the equation (43) with (44) is

simplified by z and $S(\check{u})$ satisfies the Ricatti equation (46)-(47). In this case, according to (19), the equation (46) with (47) cannot be simplified, if and only if,

$$| B(0) | + | C(0) + q\lambda B^{'}(0) + q^{-1}\Phi(0) | \neq 0.$$

(2.1) If $B(0) \neq 0$ or $C(0) + q\lambda B^{'}(0) + q^{-1}\Phi(0) \neq 0$, then $\check{s} = s + 1$.

(2.2) If $B(0) = 0$ and $C(0) + q\lambda B^{'}(0) + q^{-1}\Phi(0) = 0$, it is easy to see that 0 is a root of $\check{\Phi}_{0,1}$, so the equation (46)-(47) is simplified by z and $S(\check{u})$ satisfies the q-Riccati equation (48)-(49). In this case, according to (19), the equation (48) with (49) cannot be simplified, if and only if,

$$| \lambda(1 + q)B^{'}(0) + C(0) | + | D(0) + \lambda C^{'}(0) + \tfrac{q\lambda^2}{2}B^{''}(0) + \tfrac{q^{-2}\lambda}{2}\Phi^{''}(0) | \neq 0.$$

(2.2.1) If $\lambda(1 + q)B^{'}(0) + C(0) \neq 0$ or $D(0) + \lambda C^{'}(0) + \frac{q\lambda^2}{2}B^{''}(0) + \frac{q^{-2}\lambda}{2}\Phi^{''}(0) \neq 0$, then $\check{s} = s$.

(2.2.2) If $\lambda(1 + q)B^{'}(0) + C(0) = 0$ and $D(0) + \lambda C^{'}(0) + \frac{q\lambda^2}{2}B^{''}(0) + \frac{q^{-2}\lambda}{2}\Phi^{''}(0) = 0$, we have $\Phi(0) = B(0) = 0$, then 0 is a root of $\check{\Phi}_{0,2}$. Consequently, the equation (48) with (49) is simplified by z and $S(\check{u})$ satisfies the q-Riccati equation (50) with (51). In this case, according to (19), the equation (50) with (51) cannot be simplified, if and only if,

$$| B^{'}(0) | + | \tfrac{\lambda(1+q)}{2}B^{''}(0) + C^{'}(0) | + | D^{'}(0) + \tfrac{\lambda}{2}C^{''}(0) + \tfrac{q\lambda^2}{3!}B^{'''}(0) + \tfrac{q^{-3}\lambda}{3!}\Phi^{'''}(0) | \neq 0.$$

There are two subcases:

(2.2.2.1) If $B^{'}(0) \neq 0$ or $\frac{\lambda(1+q)}{2}B^{''}(0) + C^{'}(0) \neq 0$ or $D^{'}(0) + \frac{\lambda}{2}C^{''}(0) + \frac{q\lambda^2}{3!}B^{'''}(0) + \frac{q^{-3}\lambda}{3!}\Phi^{'''}(0) \neq 0$, then $\check{s} = s - 1$.

(2.2.2.2) If $B^{'}(0) = 0$, $\frac{\lambda(1+q)}{2}B^{''}(0)+C^{'}(0) = 0$, $D^{'}(0)+\frac{\lambda}{2}C^{''}(0)+\frac{q\lambda^2}{3!}B^{'''}(0)+\frac{q^{-3}\lambda}{3!}\Phi^{'''}(0) = 0$, then $\check{s} = s - 2$ and $S(\check{u})$ satisfies the q-Riccati equation (52) with (53). The theorem is then proved. □

Remark 2. *When $q \to 1$, we recover the results established in* [8].

Example 1. *First of all, let us recall $\mathbf{H}(q)$ the natural q-analoque of the Hermite form which is symmetric H_q-classical and $\{H_n(.;q)\}_{n\geq 0}$ be its MOPS. We have* [4, 6]

$$\gamma_{n+1} = \frac{1}{2}\, q^n\, [n+1]_q,\ H_q\big(\mathbf{H}(q)\big) + 2x\mathbf{H}(q) = 0,\ H_{q^{-1}}(S(\mathbf{H}(q)))(z) = -2qzS(\mathbf{H}(q))(z) - 2q. \quad (54)$$

From (54), for $0 < q < 1$, the form $\mathbf{H}(q)$ is positive definite. Denoting $\mathbf{H}^{(1)}(q)$ its first associated form and $\{H_n^{(1)}(.;q)\}_{n\geq 0}$ its MOPS. On account of Proposition 1., Proposition 2. and Lemma 2., the symmetric form $\mathbf{H}^{(1)}(q)$ is q-Laguerre-Hahn of class $s = 0$ fulfilling

$$\begin{cases} \beta_n^{(1)} = 0, \quad \gamma_{n+1}^{(1)} = \frac{1}{2}\, q^{n+1}\, [n+2]_q,\ n \geq 0, \\ H_q\big(\mathbf{H}^{(1)}(q)\big) + 2q^{-1}x\mathbf{H}^{(1)}(q) - q\big(x^{-1}\mathbf{H}^{(1)}(q)(h_q\mathbf{H}^{(1)}(q))\big) = 0, \\ H_{q^{-1}}(S(\mathbf{H}^{(1)}(q)))(z) = -qS(\mathbf{H}^{(1)}(q))(z)(h_{q^{-1}}S(\mathbf{H}^{(1)}(q))(z) - 2zS(\mathbf{H}^{(1)}(q))(z) - 2. \end{cases} \quad (55)$$

Morover, from (55) and (20) we have$H_0^{(1)}(0;q) = 1$, $H_{2n+1}^{(1)}(0;q) = 0$, $n \geq 0$, and

$$H_{2n+2}^{(1)}(0;q) = (-1)^{n+1} \prod_{k=0}^{n} \gamma_{2k+1}^{(1)} = \frac{(-1)^{n+1}\, q^{(n+1)^2}\, (q^2;q^2)_{n+1}}{2^{n+1}\, (1-q)^{n+1}},\ n \geq 0. \quad (56)$$

Now, from the fact that $\mathbf{H}^{(1)}(q)$ is also positive definite for $0 < q < 1$, then $\check{u} = \mathbf{H}^{(1)}(q) + \lambda\delta$ is positive definite for $0 < q < 1$ and $\lambda > 0$. Moreover, thanks to the coefficients of the q-Riccati equation in (55), we have $B^{(1)}(0) = -q$, $\Phi^{(1)}(0) = 1$ and $q\lambda B^{(1)}(0) + \Phi^{(1)}(0) = -q^2\lambda + 1$. Two cases arise.

▷ *$q\lambda B^{(1)}(0) + \Phi^{(1)}(0) \neq 0$, equivalently $\lambda \neq q^{-2}$. From the point (1) of Theorem 1., we deduce that for $0 < q < 1$ and for all $\lambda > 0$, $\lambda \neq q^{-2}$, the positive definite form $\check{u} = \mathbf{H}^{(1)}(q) + \lambda\delta$ is q-Laguerre-Hahn of class $\check{s} = 2$. On the other hand, thanks to (56)* *and put* $\varpi_n(\lambda,q) := \left(1+\lambda\sum_{k=0}^{n}\frac{(q^2;q^2)_k}{(q^3;q^2)_k}\,q^{-k}\right)$, $n\geq 0$, $\lambda>0$, $0<q<1$, *we get for (27)-(29) and (43)-(44),*

$$\begin{cases}\tau_n=\frac{q^{\frac{n(n+1)}{2}}(q^2;q)_n}{2^n\,(1-q)^n}, \quad b_n=0, \quad \check{\beta}_n=0,\ n\geq 0,\\ d_{2n}=\frac{q^{n(2n+1)}(q^2;q)_{2n}}{2^{2n}(1-q)^{2n}}\,\varpi_n(\lambda,q), \quad d_{2n+1}=\frac{q^{(n+1)(2n+1)}(q^2;q)_{2n+1}}{2^{2n+1}(1-q)^{2n+1}}\,\varpi_n(\lambda,q),\ n\geq 0,\\ \check{\gamma}_0=1+\lambda, \quad \check{\gamma}_{2n+2}=\frac{1}{2}\,q^{2n+2}\,[2n+3]_q\,\frac{\varpi_{n+1}(\lambda,q)}{\varpi_n(\lambda,q)},\ n\geq 0,\\ \check{\gamma}_1=\frac{q(1+q)}{2(1+\lambda)}, \quad \check{\gamma}_{2n+1}=\frac{1}{2}\,q^{2n+1}\,[2n+2]_q\,\frac{\varpi_{n-1}(\lambda,q)}{\varpi_n(\lambda,q)},\ n\geq 1,\\ q^{-2}z^2H_{q^{-1}}(S(\check{u}))(z)=-q^{-1}(1-\lambda(q-1))^{-1}z^2S(\check{u})(z)(h_{q^{-1}}S(\check{u}))(z)\\ \qquad -q^{-2}(1-\lambda(q-1))^{-1}z(2z^2+\lambda q(1+q))S(\check{u})(z)\\ \qquad +q^{-2}(1-\lambda(q-1))^{-1}(-2(1+\lambda)z^2+\lambda(1-q^2\lambda)).\end{cases}$$

$\triangleright$ $q\lambda B^{(1)}(0)+\Phi^{(1)}(0)=0$, *equivalently* $\lambda=q^{-2}$. *Moreover, it is seen that* $B^{(1)}(0)=-q\neq 0$. *Consequently, the first condition in (2) of Theorem 1. is valid and for* $0<q<1$, $\lambda=q^{-2}$, *the positive definite form* $\check{u}=\mathbf{H}^{(1)}(q)+q^{-2}\delta$ *is q-Laguerre-Hahn of class* $\check{s}=1$ *satisfying*

$$q^{-1}zH_{q^{-1}}(S(\check{u}))(z)=-q(q^2-q+1)^{-1}zS(\check{u})(z)(h_{q^{-1}}S(\check{u}))(z)$$
$$-q^{-1}(q^2-q+1)^{-1}(2qz^2+1+q)S(\check{u})(z)-2q^{-2}(1+q^2)(q^2-q+1)^{-1}z.$$

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