TJMM
14 (2022), No. 1, 19-34

ON SOME REFINEMENTS OF JENSEN AND RELATED INEQUALITIES WITH APPLICATIONS

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Abstract. The aim of this article is to give the cyclic refinement of the Jensen's inequality, its variant and extension by considering real weights.Applications to Ky Fan inequality and cyclic mixed symmetric means are also given.

1. Introduction and Preliminaries

Jensen's inequality for convex functions is one of the most celebrated inequality in Mathematics and other fields of science. Due to fact that many other reowned inequalities such as Arithmetic-Geometric, Ky Fan inequality, Minkoski inequality and Holder inequality etc, can be obtained as its particular case. Due to its highly importance there are given numerous variants, generalizations and refinements of Jensen's inequalities (for reference see [6, 8, 12, 13, 14, 15, 28, 29, 19, 20, 3, 21, 22, 34, 35, 36, 37, 38]). We also adduce to [31] and [33] for detailed discussion on Jensen's inequality and for some remarks on literature and history of the topic.

The well-known Jensen's inequality given in [33] as follow:

Proposition 1. *Let $\phi : I \to \mathbb{R}$ be a convex function and I be an interval in $\mathbb{R}$. If $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ be a n-tuple and $\mathbf{w}$ be a non-negative n-tuple such that $\sum_{i=1}^{n} w_i > 0$. Then following inequality*

$$\phi\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i \phi(x_i) \tag{1}$$

holds, where $W_n = \sum_{i=1}^{n} w_i, i \in \{1, \dots, n\}$.

The more generalized form of Jensn's inequality is established by Steffsen, which is referred as Jensen-Steffsen's inequality given in [33, pg. 38] as follow:

Proposition 2. *Let $\mathbf{x} = (x_1, \dots, x_n) \in I^n$, be a real monotonic n-tuple and $\mathbf{w}$ be a real n-tuple such that*

$$0 \leq W_i \leq W_n, \quad W_n > 0, i \in \{1, \dots, n\}. \tag{2}$$

If ϕ is convex on I, then (1) is valid.

In [33, p. 83] reverse-Jensen's Inequality is give as follow:

2010 *Mathematics Subject Classification.* 26D99, 26A51.
Key words and phrases. Convex function, Ky Fan Inequality, Jensen-Steffsen's inequality.

Proposition 3. *Let $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$, be a real monotonic n-tuple and $\mathbf{w}$ be a real n-tuple with $\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \in I^n$, where $w_1 > 0, w_i \leq 0$ for $i \in \{2, \ldots, n\}$ and $W_n > 0$. If ϕ is convex function on I, then reverse inequality in (1) is valid.*

A variant of Jensen-Steffsense's inequality is given in [1] as follows:

Proposition 4. *Let $\phi : I \to \mathbb{R}$ be a convex function and I be an interval in $\mathbb{R}$ and $[a, b] \in I^n, a < b$ and also $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ be real monotonic n-tuple and $\mathbf{w}$ be a real n-tuple and conditions stated in (2) be valid. Then following inequality holds:*

$$\phi\left(a + b - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \phi(a) + \phi(b) - \frac{1}{W_n}\sum_{i=1}^{n} w_i \phi(x_i). \tag{3}$$

The cyclic refinement of the Jensen's inequality in paper [7] is given as follows:

Theorem 1. *Let $\phi : I \to \mathbb{R}$ be a convex function and I be an interval in $\mathbb{R}, \mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a nonnegative n-tuple such that $\sum_{i=1}^{k} \lambda_i = 1$ for some k, $2 \leq k \leq n$. Then*

$$\phi\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} \phi\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \leq \frac{1}{n}\sum_{i=1}^{n} \phi(x_i), \tag{4}$$

where $i + j$ means $i + j - n$ in case of $i + j > n$.

In this article we are going to use some of the following assumptions:

(A_1) : Let I be a real interval, $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ be a real nondecreasing n-tuple such that $x_{i+n} = x_i$ for $i \in \{1, 2, \ldots, n\}$.

(A_2) : Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a non-negative n-tuple such that $\displaystyle\sum_{i=1}^{k} \lambda_i = 1$ for some k, $2 \leq k \leq n$.

(A_3) : Let $a = \min\limits_{1 \leq i \leq n} \{x_i\}$ and $b = \max\limits_{1 \leq i \leq n} \{x_i\}$.

(A_4) : Let $\phi : I \to \mathbb{R}$ be a convex function.

(A_5) : Let $\phi, \psi : I \to \mathbb{R}$ be continuous and strictly monotone functions.

Under the assumptions stated above it should be noted that:

$$\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}$$

and

$$a + b - \sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j} \in I$$

The main theme of this article is to refine Jensen's inequality, its variant and extension by considering real weights and satisfying the assumptions as stated in (2). The first section deals with introduction and preliminaries. The second section is devoted to the refinement of Jensen's inequality for real weights. In third section we present cyclic refinement of the variant of Jensen's inequality for real weights referred as Jensen-Mercer inequality. In application section, with the help of obtained results we will give refinements of Ky Fan, arithmetic-geometric means inequalities and their related results. We also define cyclic mixed symmetric means, power mean and generalized quasi-arithmetic means and study their properties. We follow the techniques given in [7].

2. Refinement of Jensen's inequality for real weights

Now we establish refinement of (1) under assumptions stated in (2) as follows:

Theorem 2. *Let the assumptions stated in* $(A_1),(A_2)$ *and* (A_4) *be true. Then the following inequalities hold*

$$\phi\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i \phi\left(\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i \phi\left(x_i\right), \tag{5}$$

for all $x_i \in I$ *for* $1 \leq i \leq n$.

Proof. We prove first inequality of (5) since, ϕ is convex function and $\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j} \in I$. Therefore by Jensen-Steffensen's inequality we have,

$$\begin{aligned} &\frac{1}{W_n}\sum_{i=1}^{n} w_i \phi\left(\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right) \\ &\geq \phi\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \sum_{j=0}^{k-1}\lambda_{j+1}w_i x_{i+j}\right) \\ &= \phi\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \sum_{j=0}^{k-1}\lambda_{j+1}w_i x_i\right) \\ &= \phi\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right). \end{aligned}$$

To prove second inequality, for each $i \in \{1,2,\ldots,n\}$ we consider

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i \phi\left(\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right).$$

Since for each $i \in \{1,2,\ldots,n\}$, $\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j} \in I$, therefore by Jensen-Steffensen's inequality we have:

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i \phi\left(\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i \sum_{j=0}^{k-1}\lambda_{j+1}\phi\left(x_{i+j}\right)$$

where,

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i \sum_{j=0}^{k-1}\lambda_{j+1}\phi\left(x_{i+j}\right) = \frac{1}{W_n}\sum_{i=1}^{n} w_i \sum_{j=1}^{k}\lambda_{j}\phi\left(x_i\right) = \frac{1}{W_n}\sum_{i=1}^{n} w_i \phi\left(x_i\right),$$

which concludes our proof. □

3. Refinement of Jensen-Mercer inequality for real weights

Before we further proceed we recall here a lemma from [30] stated as under:

Lemma 1. *Let the assumptions stated in* (A_1) *and* (A_2) *be true. Then*

$$\phi(a+b-x_i) \leq \phi(a)+\phi(b)-\phi(x_i), \quad 1 \leq i \leq n. \tag{6}$$

Theorem 3. *Let the assumptions stated in* $(A_1), (A_2), (A_3)$ *and* (A_4) *and* (2) *be true. Then the following inequalities hold*

$$\phi\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i \phi\left(a+b-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right) \leq \phi(a)+\phi(b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i\phi(x_i), \tag{7}$$

for all $x_i \in I$ *for* $1 \leq i \leq n$.

Proof. To prove first inequality, since ϕ is convex function and $a+b-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j} \in I$, therefore by reverse Jensen's inequality we have,

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i \phi\left(a+b-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right) \geq \phi\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a + \frac{1}{W_n}\sum_{i=1}^{n} w_i b - \frac{1}{W_n}\sum_{i=1}^{n}\sum_{j=0}^{k-1}\lambda_{j+1}w_i x_{i+j}\right)$$

Equivalent to

$$\phi\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i a + \frac{1}{W_n}\sum_{i=1}^{n} w_i b - \frac{1}{W_n}\sum_{i=1}^{n}\sum_{j=0}^{k-1}\lambda_{j+1}w_i x_{i+j}\right) = \phi\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \sum_{j=1}^{k}\lambda_j\right) = \phi\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right).$$

To prove second inequality, for each $i \in \{1, 2, \ldots, n\}$ we consider

$$\phi\left(a+b-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right).$$

Since for each $i \in \{1, 2, \ldots, n\}$, $\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j} \in I$, therefore by Lemma 1 we have:

$$\begin{aligned}
&\phi\left(a+b-\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \\
&\leq \phi(a)+\phi(b)-\phi\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \\
&=\phi(a)+\phi(b)-\phi\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \\
&=\phi(a)+\phi(b)-\phi\left(\sum_{j=1}^{k} \lambda_{j} x_{i}\right) \\
&=\phi(a)+\phi(b)-\phi(x_i),
\end{aligned}$$

or we can write it as for each $i \in \{1, 2, \ldots, n\}$,

$$\phi\left(a+b-\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \leq \phi(a)+\phi(b)-\phi(x_i), \tag{8}$$

Now multiplying inequality (8) with $\frac{1}{W_n} \sum_{i=1}^{n} w_i$, we get our required result. □

4. Applications

4.1. **For cyclic refinement of Jensen's inequality via real weights.** Throughout this section, let the assumptions stated in Theorem 2 be valid. We define generalized (or modified) arithmetic, geometric and harmonic mean respectively as follows:

$$\begin{aligned}
A_n &= \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i, \\
G_n &= \left(\prod_{i=1}^{n} x_i^{w_i}\right)^{\frac{1}{W_n}}, \\
H_n &= \left(\frac{1}{W_n} \sum_{i=1}^{n} w_i \frac{1}{x_i}\right)^{-1}.
\end{aligned}$$

Further for $x_i \in (0, \frac{1}{2}]$ we have

$$A'_n = \frac{1}{W_n}\sum_{i=1}^{n} w_i\,(1-x_i)\,,$$

$$G'_n = \left(\prod_{i=1}^{n}(1-x_i)^{w_i}\right)^{\dfrac{1}{W_n}},$$

$$H'_n = \left(\frac{1}{W_n}\sum_{i=1}^{n} w_i\left(\frac{1}{1-x_i}\right)\right)^{-1}$$

We also define new notations $A(\lambda, \mathbf{x})$ and $G(\lambda, \mathbf{x})$ as under:

$$A(\lambda, \mathbf{x}) = \sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j},$$

$$G(\lambda, \mathbf{x}) = \prod_{j=0}^{k-1}(x_{i+j})^{\lambda_{j+1}},$$

Here also $x_{i+j} \in (0, \frac{1}{2}]$

$$A'(\lambda, \mathbf{x}) = \sum_{j=0}^{k-1}\lambda_{j+1}\,(1-x_{i+j})\,,$$

$$G'(\lambda, \mathbf{x}) = \prod_{j=0}^{k-1}(1-x_{i+j})^{\lambda_{j+1}}.$$

Now, we present the refinement of the Ky Fan type inequality. For Ky Fan inequality and related results see [2] and [9] and references given therein.

Theorem 4. *Under the assumptions stated in Theorem 2 ,We have following inequality:*

$$\frac{A_n}{A'_n} \le \left(\prod_{i=1}^{n}\left(\frac{A(\lambda, \mathbf{x})}{A'(\lambda, \mathbf{x})}\right)^{w_i}\right)^{\frac{1}{W_n}} \le \frac{G_n}{G'_n}.$$

Proof. By applying the convex function $\phi(x) = ln\left(\frac{x}{1-x}\right)$ for all $x \in (0, \frac{1}{2}]$, to the inequality (1), we get,

$$\ln\left(\frac{\dfrac{1}{W_n}\displaystyle\sum_{i=1}^{n} w_i x_i}{1-\dfrac{1}{W_n}\displaystyle\sum_{i=1}^{n} w_i x_i}\right)$$

$$\le \frac{1}{W_n}\sum_{i=1}^{n} w_i \ln\left(\frac{\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}}{1-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}}\right)$$

$$\le \frac{1}{W_n}\sum_{i=1}^{n} w_i \ln\left(\frac{x_i}{1-x_i}\right)$$

consequently,

$$\ln\left(\frac{A_n}{A'_n}\right) \leq \frac{1}{W_n} \ln \prod_{i=1}^{n} \left(\frac{\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}}{1-\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}}\right)^{w_i} \leq \ln\left(\frac{G_n}{G'_n}\right)$$

Finally, we obtain

$$\left(\frac{A_n}{A'_n}\right) \leq \left(\prod_{i=1}^{n} \left(\frac{A(\lambda,\mathbf{x})}{A'(\lambda,\mathbf{x})}\right)^{w_i}\right)^{\frac{1}{W_n}} \leq \left(\frac{G_n}{G'_n}\right),$$

which completes the proof. □

Now, we present refinement of arithmetic-geometric mean type inequality as follow:

Corollary 1. *Let the assumptions stated in Theorem 2 be true. Then following inequality holds:*

$$A_n \geq \left(\prod_{i=1}^{n} (A(\lambda,\mathbf{x}))^{w_i}\right)^{\frac{1}{W_n}} \geq G'_n.$$

Proof. By applying the convex function $\phi(x) = -\ln(x), x \in (0, \frac{1}{2}]$ to Theorem 5 we obtain required result. □

Now, we present refinement of harmonic and geometric means inequality as follow:

Corollary 2. *Let the assumptions stated in Theorem 2 be true be true. Then following inequalities hold:*

$$(G'_n)^{-1} \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i (G'(\lambda,\mathbf{x}))^{-1} \leq (H'_n)^{-1}.$$

Proof. By applying the convex function $\phi(x) = exp(x), x \in (0, \frac{1}{2}]$ to Theorem 5 and by replacing x_i by $\ln\left(\frac{1}{1-x_i}\right)$ respectively, we get required result. □

We would also establish refinements related to arithmetic-harmonic means inequalities as follow:

Corollary 3. *Let the assumptions stated in Theorem 2 be true. Then following inequalities hold:*

$$\frac{1}{A_n} \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \left(\frac{1}{A(\lambda,\mathbf{x})}\right) \leq \frac{1}{H_n}, \tag{9}$$

$$\frac{1}{A'_n} \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \left(\frac{1}{A'(\lambda,\mathbf{x})}\right) \leq \frac{1}{H'_n}. \tag{10}$$

Proof. By applying convex function $f(x) = \frac{1}{x}, x \in (0, \frac{1}{2}]$ to Theorem 2 we get inequality (9). Similarly, by using convex function $f(x) = \frac{1}{1-x}, x \in (0, \frac{1}{2}]$ to Theorem 2 we get inequality (10). □

We establish a refinement of the difference of the arithmetic and harmonic mean.

Corollary 4. *Let the assumptions* $(A_1), (A_3)$ *and* (A_5) *be true. Then following inequalities hold:*

$$\frac{1}{A_n} - \frac{1}{A'_n} \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \left(\frac{1}{A(\lambda, \mathbf{x})} - \frac{1}{A'(\lambda, \mathbf{x})} \right) \leq \frac{1}{H_n} - \frac{1}{H'_n}.$$

Proof. By applying convex function $f(x) = \frac{1}{x} - \frac{1}{1-x}, x \in (0, \frac{1}{2}]$ to Theorem 2 we obtain required result. □

The Jensen's inequality and Jensen-Mercer inequality are much fertile to study about mixed means (see[27] and [17]). Let the assumptions stated in $(A_1), (A_2)$ and (A_5) be true. Then we define power mean of the order $r \in \mathbb{R}$, for positive n-tuple $\mathbf{x}$ as follow:

$$M_r\left(x_i, \ldots, x_{i+k-1}; \lambda_1, \ldots, \lambda_k\right) = \begin{cases} \left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}^r \right)^{\frac{1}{r}}, & r \neq 0, \\ \prod_{j=0}^{k-1} \left(x_{i+j}\right)^{\lambda_{j+1}}, & r = 0, \end{cases}$$

and cyclic mixed symmetric means corresponding to (5) is given as:

$$M_{r,s}(\mathbf{x}, \lambda) = \begin{cases} \left(\frac{1}{W_n} \sum_{i=1}^{n} w_i M_r^s\left(x_i, \cdots, x_{i+k-1}, \lambda_1, \cdots, \lambda_k\right) \right)^{\frac{1}{s}}, & s \neq 0, \\ \left(\prod_{i=1}^{n} M_r\left(x_i, \cdots, x_{i+k-1}, \lambda_1, \cdots, \lambda_k\right)^{w_i} \right)^{\frac{1}{W_n}}, & s = 0. \end{cases}$$

The standard power mean of order $r \in \mathbb{R}$ for the positive n-tuple $\mathbf{x}$ are define as follow:

$$M_r\left(\mathbf{x}\right) = \begin{cases} \left(\frac{1}{W_n} \sum_{i=1}^{n} w_i x_i^r \right)^{\frac{1}{r}}, & r \neq 0, \\ \frac{cd}{\left(\prod_{i=1}^{n} \left(x_i\right)^{w_i} \right)^{\frac{1}{W_n}}}, & r = 0. \end{cases}$$

Corollary 5. *For* $r \leq 1$ *and by considering the assumptions state in* (A_1) *for positive* n*-tuple* $\mathbf{x}$*, then the following inequality hold.*

$$M_r\left(\mathbf{x}\right) \leq M_r\left(\mathbf{x}, \lambda\right) \leq A_n. \tag{11}$$

For $r \geq 0$*, the inequality* (11) *is reversed.*

Proof. For For $r \leq 1, r \neq 0$, by applying the convex function $\phi(x) = x^{\frac{1}{r}}$ to the Theorem 2 and replacing x_i with x_i^r respectively and for $r = 0$ applying convex function $\phi(x) = \exp(x)$ to the Theorem 2, replacing xi with $\ln xi$ respectively, we obtain 11. If $r \geq 1$, then the function $\phi(x) = x^{\frac{1}{r}}$ is concave, so the inequality in (11) is reversed. □

Now, we define the bounds for power mean and cyclic mixed symmetric means as follow:

Corollary 6. *Let $r, s \in \mathbb{R}$ such that $r \leq s$ and considering the assumption stated in A_1 for positive n-tuple $\mathbf{x}$, then following inequalities holds.*

$$M_r(\mathbf{x}) \leq M_{r,s}(\mathbf{x}, \lambda) \leq M_s(\mathbf{x}),$$
$$M_r(\mathbf{x}) \leq M_{r,s}(\mathbf{x}, \lambda) \leq A_n. \tag{12}$$

Proof. Let $r, s \neq 0$. By applying the Theorem 2 for convex function $\phi(x) = x^{\frac{s}{r}}$, $x > 0$ and by replacing positive n-tuple $\mathbf{x}$ by $(x_1^r, \ldots, x_n^r)$ and then raising the power $\frac{1}{s}$ we get,

$$M_r(\mathbf{x}) \leq M_{r,s}(\mathbf{x}, \lambda) \leq M_s(\mathbf{x}).$$

For $s = 0$ or $r = 0$, we obtain the required result by taking limit. □

Let $\phi, \psi : I \to \mathbb{R}$ be a strictly monotonic and continuous functions and under the assumptions stated in $(A_1), (A_2)$ and (A_5), we define generalized means with respect to (2) as follow:

$$M_{\phi,\psi}(\mathbf{x}, \lambda) = \phi^{-1}\left[\frac{1}{W_n}\sum_{i=1}^{n} w_i \left(\phi \circ \psi^{-1}\right)\left(\sum_{j=0}^{k-1} \lambda_{j+1}\psi(x_{i+j})\right)\right]. \tag{13}$$

Now, we establish the relation among generalized means and quasi-arithmetic means as follow:

Corollary 7. *Let assumptions (A_1) and (A_3) be true. Then*

$$\tilde{M}_\psi(\mathbf{x}) \leq \tilde{M}_{\phi,\psi}(\mathbf{x}, \lambda) \leq \tilde{M}_\phi(\mathbf{x}),$$

if either $\phi\ \psi^{-1}$ is convex and ψ is strictly increasing or $\phi \circ \psi^{-1}$ is concave and ψ is strictly decreasing.

Proof. By applying Theorem 2 to the convex function $\phi \circ \psi^{-1}$ and replacing n-tuples $\mathbf{x}$ by $(\psi(x_1), \ldots, \psi(x_n))$, we get

$$\phi \circ \psi^{-1}\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i\psi(x_i)\right)$$
$$\leq \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi \circ \psi^{-1}\left(\sum_{j=0}^{k-1}\lambda_{j+1}\psi(x_{i+j})\right)$$
$$\leq \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi \circ \psi^{-1}\left(\psi(x_i)\right),$$

consequently,

$$\phi \circ \psi^{-1}\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i\psi(x_i)\right)$$
$$\leq \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi \circ \psi^{-1}\left(\sum_{j=0}^{k-1}\lambda_{j+1}\psi(x_{i+j})\right)$$
$$\leq \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi(x_i),$$

by applying ϕ^{-1}, we get

$$\phi^{-1}\phi \circ \psi^{-1}\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i\psi(x_i)\right) \leq \phi^{-1}\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i\phi \circ \psi^{-1}\left(\sum_{j=0}^{k-1}\lambda_{j+1}\psi(x_{i+j})\right)\right) \leq \phi^{-1}\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i\phi(x_i)\right),$$

and after some simplification we obtained required result. □

4.2. **For refinement of Jesen-Seffensen's inequality of Jensen-Mercer Type.** Throughout this section, let the assumptions stated in $(A_1), (A_2), (A_3)$ and (A_5) be valid with $0 < a < b$. We define generalized (or modified) arithmetic, geometric and harmonic mean respectively as follows:

$$A_n = a + b - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i,$$

$$G_n = \frac{ab}{\left(\prod_{i=1}^{n} x_i^{w_i}\right)^{\frac{1}{W_n}}},$$

$$H_n = \left(a^{-1} + b^{-1} - \frac{1}{W_n}\sum_{i=1}^{n} w_i \frac{1}{x_i}\right)^{-1}.$$

Further for $x_i \in (0, \frac{1}{2}]$ we have

$$A_n' = (1-a) + (1-b) - \frac{1}{W_n}\sum_{i=1}^{n} w_i\left(1 - x_i\right),$$

$$G_n' = \frac{(1-a)(1-b)}{\left(\prod_{i=1}^{n}(1-x_i)^{w_i}\right)^{\dfrac{1}{W_n}}},$$

$$H_n' = \left((1-a)^{-1} + (1-b)^{-1} - \frac{1}{W_n}\sum_{i=1}^{n} w_i\left(\frac{1}{1-x_i}\right)\right)^{-1}.$$

We also define new notations $A(\lambda, \mathbf{x})$ and $G(\lambda, \mathbf{x})$ as under:

$$A(\lambda, \mathbf{x}) = a + b - \sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j},$$
$$G(\lambda, \mathbf{x}) = \frac{ab}{\prod_{j=0}^{k-1} (x_{i+j})^{\lambda_{j+1}}}.$$

Here also $x_{i+j} \in (0, \frac{1}{2}]$

$$A'(\lambda, \mathbf{x}) = (1-c) + (1-d) - \sum_{j=0}^{k-1} \lambda_{j+1} (1 - x_{i+j}),$$
$$G'(\lambda, \mathbf{x}) = \frac{(1-c)(1-d)}{\prod_{j=0}^{k-1} (1 - x_{i+j})^{\lambda_{j+1}}}.$$

Now, we present the refinement of the Ky Fan type inequality.

Theorem 5. *Let assumptions stated in Theorem 3 be true. Then following inequality holds:*

$$\frac{A_n}{A'_n} \leq \left(\prod_{i=1}^{n} \left(\frac{A(\lambda, \mathbf{x})}{A'(\lambda, \mathbf{x})} \right)^{w_i} \right)^{\frac{1}{W_n}} \leq \frac{G_n}{G'_n}.$$

Proof. By applying the convex function $\phi(x) = ln\left(\frac{x}{1-x}\right)$ for all $x \in (0, \frac{1}{2}]$, to the inequality (7), we obtained required result. □

Now, we present refinement of arithmetic-geometric mean type inequality as follow:

Corollary 8. *Let assumptions stated in Theorem 3 be true. Then following inequality holds:*

$$A_n \geq \left(\prod_{i=1}^{n} (A(\lambda, \mathbf{x}))^{w_i} \right)^{\frac{1}{W_n}} \geq G'_n.$$

Proof. By applying the convex function $\phi(x) = -\ln(x), x \in (0, \frac{1}{2}]$ to Theorem 5 we obtain required result. □

Now, we present refinement of harmonic and geometric means inequality as follow:

Corollary 9. *Let assumptions stated in Theorem 3 be true. Then following inequalities hold:*

$$(G'_n)^{-1} \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i (G'(\lambda, \mathbf{x}))^{-1} \leq (H'_n)^{-1}.$$

Proof. By applying the convex function $\phi(x) = exp(x), x \in (0, \frac{1}{2}]$ to Theorem 5 and by replacing a, b and x_i by $\ln\left(\frac{1}{1-a}\right), \ln\left(\frac{1}{1-b}\right)$ and $\ln\left(\frac{1}{1-x_i}\right)$ respectively , we obtain required results after some simplifications. □

We would also establish refinements related to arithmetic-harmonic means inequalities as follow:

Corollary 10. *Let assumptions stated in Theorem* 3 *be true.Then following inequalities hold:*

$$\frac{1}{A_n} \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \left(\frac{1}{A(\lambda, \mathbf{x})} \right) \leq \frac{1}{H_n}, \tag{14}$$

$$\frac{1}{A'_n} \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \left(\frac{1}{A'(\lambda, \mathbf{x})} \right) \leq \frac{1}{H'_n}. \tag{15}$$

Proof. By applying convex function $f(x) = \frac{1}{x}, x \in (0, \frac{1}{2}]$ to Theorem 4 we get inequality (14). Similarly, by using convex function $f(x) = \frac{1}{1-x}, x \in (0, \frac{1}{2}]$ to Theorem 4 we get inequality (15). □

We establish a refinement of the difference of the arithmetic and harmonic mean.

Corollary 11. *Let assumptions stated in Theorem* 3 *be true. Then following inequalities hold:*

$$\frac{1}{A_n} - \frac{1}{A'_n} \leq \frac{1}{W_n} \sum_{i=1}^{n} w_i \left(\frac{1}{A(\lambda, \mathbf{x})} - \frac{1}{A'(\lambda, \mathbf{x})} \right) \leq \frac{1}{H_n} - \frac{1}{H'_n}.$$

Proof. By applying convex function $f(x) = \frac{1}{x} - \frac{1}{1-x}, x \in (0, \frac{1}{2}]$ to Theorem 4 we obtain required result. □

The Jensen's inequality and Jensen-Mercer inequality are much fertile to study about mixed means (see[27] and [17]). Let the assumptions stated in $(A_1), (A_2), (A_3)$ and (A_5) be true. Then we define power mean of the order $r \in \mathbb{R}$, for positive n-tuple $\mathbf{x}$ as follow:

$$M_r\left(x_i, \ldots, x_{i+k-1}; \lambda_1, \ldots, \lambda_k\right) = \begin{cases} \left(a^r + b^r - \sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}^r \right)^{\frac{1}{r}}, & r \neq 0, \\ \dfrac{ab}{\prod_{j=0}^{k-1} \left(x_{i+j}\right)^{\lambda_{j+1}}}, & r = 0, \end{cases}$$

and cyclic mixed symmetric means corresponding to (1) is given as:

$$M_{r,s}(\mathbf{x}, \lambda) = \begin{cases} \left(\frac{1}{W_n} \sum_{i=1}^{n} w_i M_r^s\left(x_i, \cdots, x_{i+k-1}, \lambda_1, \cdots, \lambda_k\right) \right)^{\frac{1}{s}}, & s \neq 0, \\ \left(\prod_{i=1}^{n} M_r\left(x_i, \cdots, x_{i+k-1}, \lambda_1, \cdots, \lambda_k\right)^{w_i} \right)^{\frac{1}{W_n}}, & s = 0. \end{cases}$$

The standard power mean of order $r \in \mathbb{R}$ for the positive n-tuple $\mathbf{x}$ are define as follow:

$$M_r(\mathbf{x}) = \begin{cases} \left(a^r + b^r - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i^r\right)^{\frac{1}{r}}, & r \neq 0, \\ \dfrac{ab}{\left(\prod_{i=1}^{n}(x_i)^{w_i}\right)^{\frac{1}{W_n}}}, & r = 0. \end{cases}$$

Corollary 12. *For $r \leq 1$ and by considering the assumptions state in (A_1) for positive n-tuple $\mathbf{x}$, then the following inequality hold.*

$$M_r(\mathbf{x}) \leq M_r(\mathbf{x}, \lambda) \leq A_n. \tag{16}$$

For $r \geq 0$, the inequality (16) is reversed.

Proof. For For $r \leq 1, r \neq 0$, by applying the convex function $\phi(x) = x^{\frac{1}{r}}$ to the Theorem 5 and replacing a, b, and x_i with a^r, b^r and x_i^r respectively and for $r = 0$ applying convex function $\phi(x) = \exp(x)$ to the Theorem 5, replacing a, b, and xi with $\ln a, \ln b$, and $\ln xi$ respectively, we obtain 16.

If $r \geq 1$, then the function $\phi(x) = x^{\frac{1}{r}}$ is concave, so the inequality in (16) is reversed. □

Now, we define the bounds for power mean and cyclic mixed symmetric means as follow:

Corollary 13. *Let $r, s \in \mathbb{R}$ such that $r \leq s$ and considering the assumption stated in (A_1) for positive n-tuple $\mathbf{x}$, then following inequalities holds.*

$$M_r(\mathbf{x}) \leq M_{r,s}(\mathbf{x}, \lambda) \leq M_s(\mathbf{x}), \tag{17}$$

Proof. Let $r, s \neq 0$. By applying the Theorem 5 for convex function $\phi(x) = x^{\frac{s}{r}}$, $x > 0$ and by replacing a, b and positive n-tuple $\mathbf{x}$ by a^r, b^r and $(x_1^r, \ldots, x_n^r)$ respectively, and then raising the power $\frac{1}{s}$ we get,

$$M_r(\mathbf{x}) \leq M_{r,s}(\mathbf{x}, \lambda) \leq M_s(\mathbf{x}).$$

For $s = 0$ or $r = 0$, we obtain the required result by taking limit. □

Let $\phi : [a, b] \to \mathbb{R}$ be a continuous and strictly monotone function then cyclic quasi-arithmetic means are define as

$$M_\phi(\mathbf{x}) := \phi^{-1}\left(\phi(a) + \phi(b) - \frac{1}{W_n}\sum_{i=1}^{n} w_i \phi(x_i)\right) \tag{18}$$

.

Let $\phi, \psi : [a, b] \to \mathbb{R}$ be a strictly monotonic and continuous functions and under the assumptions stated in (A_1) and (A_2), we define generalized means with respect to (1) as follow:

$$M_{\phi,\psi}(\mathbf{x},\lambda) = \phi^{-1}\Big[\phi\circ\psi^{-1}(a)+\psi\circ\psi^{-1}(b) - \frac{1}{W_n}\sum_{i=1}^{n} w_i\left(\phi\circ\psi^{-1}\right)\left(\sum_{j=0}^{k-1}\lambda_{j+1}\psi(x_{i+j})\right)\Big]. \tag{19}$$

Now, we establish the relation among generalized means and quasi-arithmetic means as follow:

Corollary 14. *Let assumptions (A_1) and (A_3) be true. Then*

$$M_{\psi}(\mathbf{x}) \le M_{\phi,\psi}(\mathbf{x},\lambda) \le M_{\phi}(\mathbf{x}),$$

if either $\phi\circ\psi^{-1}$ is convex and ψ is strictly increasing or $\phi\circ\psi^{-1}$ is concave and ψ is strictly decreasing.

Proof. By applying Theorem 5 to the convex function $\phi\circ\psi^{-1}$ and replacing a by $\psi(a)$, b by $\psi(b)$ and n-tuples $\mathbf{x}$ by $(\psi(x_1),\ldots,\psi(x_n))$, and after some simplification we obtained required result. □

5. Further Results and Future Work

Under the assumptions of Theorem 5 and Theorem 3, we define positive linear functionals as

$$\varphi_1(\mathbf{x},\lambda,\phi) = \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi(x_i) - \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi\left(\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right),$$

$$\varphi_2(\mathbf{x},\lambda,\phi) = \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi\left(\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right) - \phi\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)$$

$$\varphi_3(\mathbf{x},\lambda,\phi) = \phi(a)+\phi(b) - \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi(x_i) - \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi\left(a+b-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right),$$

$$\varphi_4(\mathbf{x},\lambda,\phi) = \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi\left(a+b-\sum_{j=0}^{k-1}\lambda_{j+1}x_{i+j}\right) - \phi\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)$$

We can state number of different results for these functionals defined above which may be listed as follows:

(1) We can state Lagrange type and Cauchy type mean value theorems and results related to $n-$exponential and logarithmic convexity by using similar techniques as stated in [7] and [26].
(2) We can also state number of applications by using method of article [18].
(3) We can state further results using technique of index set function with series of refinements and plenty of applications including Rado and Popovicu series of inequality by using method of [23] and [25].

Here we state some future ideas for interested readers:

(1) One can state similar results as stated in this article for Niezgoda's inequality [32].
(2) One can also work on similar results as stated in this article for generalized convex functions including functions with nondecreasing increments and functions with non-decreasing increments of convex type see for example [10], [4] and [24].
(3) One can also try its integral version as well.

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