TJMM
14 (2022), No. 1, 11-17

A NEW WAY OF FINDING TRAPEZIUM INEQUALITY INVOLVING HARMONIC CONVEX FUNCTIONS THROUGH GENERALIZED FRACTIONAL INTEGRALS

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Abstract. The main objective of this paper is to obtain some new trapezium type inequalities essentially involving the class of harmonic convex functions and generalized fractional integrals.

1. Introduction and Preliminaries

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if

$$f((1-t)x+ty) \leq (1-t)f(x)+tf(y), \quad \forall x,y \in I, t \in [0,1].$$

In recent years several new extensions of classical convex functions have been proposed in the literature. For example Iscan [2] introduced the notion of harmonic convex functions as:
A function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)f(x)+tf(y), \quad \forall x,y \in I, t \in [0,1].$$

Theory of convexity in relation with theory of inequalities has attracted many mathematicians due to their closed relationship. Convexity played pivotal role in the development of theory of inequalities. Several known results in theory of inequalities can be obtained using the convexity property of the functions. One of the most significant result in this regard is trapezium inequality which can be viewed as a necessary and sufficient condition for a function to be convex. This result reads as:
Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be convex functions, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

For some more and interesting details, see [1].
Sarikaya et al. [5] obtained the fractional analogue of trapezium inequality as:

Theorem 1. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L^1[a,b]$. If f is a convex function on I, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\right] \leq \frac{f(a)+f(b)}{2},$$

where $\alpha > 0$ and $\Gamma(.)$ is the gamma function. $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ are the left and right sided Riemann-Liouville fractional integrals of order $\alpha > 0$ respectively.

2010 *Mathematics Subject Classification.* Primary: 26A51; Secondary: 26A33, 26D10, 26D15, 26D20.
Key words and phrases. Convex, Harmonic, Fractional, Generalized, trapezium Inequality.

The aim of this paper is to obtain some new refinements of trapezium inequality involving harmonic convex functions and generalized fractional integrals. Before we move further let us recall some previously known concepts and results pertaining to fractional calculus which will be helpful in the study of this paper.
We first give the definition of classical Riemann-Liouville fractional integrals.

Definition 1. *Let $f \in L_1[a,b]$. Then Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}f(\tau)\mathrm{d}t, \quad x > a,$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (\tau-x)^{\alpha-1}f(\tau)\mathrm{d}t, \quad x < b,$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-\tau}x^{\alpha-1}\mathrm{d}x,$$

is the well known gamma function.

In recent years several new fractional extensions have been made using classical Riemann-Liouville fractional integrals and its generalizations, for example, see [6, 7, 8, 9]. Katugampola [4] gave a new generalization of Riemann-Liouville fractional integral as:

Definition 2 ([4])**.** *Let $\rho > 0$. The generalized left sided fractional integral ${}^{\rho}I^{\alpha}_{a+}f(x)$ of order $\alpha > 0$ is defined as:*

$${}^{\rho}I^{\alpha}_{a+}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{w^{\rho-1}}{(x^{\rho}-w^{\rho})^{1-\alpha}}f(w)\mathrm{d}w, \quad x > a.$$

Similarly, the generalized right sided fractional integral ${}^{\rho}I^{\alpha}_{b-}f(x)$ of order $\alpha > 0$ is defined as:

$${}^{\rho}I^{\alpha}_{b-}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{w^{\rho-1}}{(w^{\rho}-x^{\rho})^{1-\alpha}}f(w)\mathrm{d}w, \quad b > x.$$

2. Results and Discussions

In this section, we discuss our main results. First of all changing the variables in Definition 2, we have

Lemma 1. *For $x > a$, we have*

$${}^{\rho}I^{\alpha}_{a+}f(x) = \frac{\rho^{1-\alpha}ax(x-a)}{\Gamma(\alpha)}\int_0^1 \frac{\left(\frac{ax}{sx+(1-s)a}\right)^{\rho-1}}{(sx+(1-s)a)^2\left(x^{\rho}-\left(\frac{ax}{sx+(1-s)a}\right)^{\rho}\right)^{1-\alpha}}f\left(\frac{ax}{sx+(1-s)a}\right)\mathrm{d}s.$$

For $x < b$, we have

$${}^{\rho}I^{\alpha}_{b-}f(x) = \frac{\rho^{1-\alpha}bx(b-x)}{\Gamma(\alpha)}\int_0^1 \frac{\left(\frac{bx}{sb+(1-s)x}\right)^{\rho-1}}{(sb+(1-s)x)^2\left(\left(\frac{bx}{sb+(1-s)x}\right)^{\rho}-x^{\rho}\right)^{1-\alpha}}f\left(\frac{bx}{sb+(1-s)x}\right)\mathrm{d}s.'$$

Lemma 2. *Let $f : I = (0,\infty) \to \mathbb{R}$ be a given function, where $a, b \in I$ and $0 < a < b < \infty$. We suppose that $f \in L^\infty(a,b)$ in such a way that ${}^\rho I^\alpha_{a+} f(x)$ and ${}^\rho I^\alpha_{b-} f(x)$ are well defined. Then*

$$\tilde{f}(x) = f\left(\frac{1}{\frac{a+b}{ab} - \frac{1}{x}}\right), \quad x \in [a,b], \tag{1}$$

and

$$F(x) = f(x) + \tilde{f}(x), \quad x \in [a,b]. \tag{2}$$

We now derive trapezium inequality.

Theorem 2. *Let $\alpha > 0$ and $\rho > 0$. If f is harmonic convex function on $[a,b]$, then*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{\rho^\alpha \Gamma(\alpha+1)}{4(b^\rho - a^\rho)^\alpha}[{}^\rho I^\alpha_{a+} F(b) + {}^\rho I^\alpha_{b-} F(a)] \le \frac{f(a)+f(b)}{2}.$$

Proof. For $t \in [0,1]$, let $x = \frac{ab}{at+(1-t)b}$ and $y = \frac{ab}{(1-t)a+tb}$, then

$$f\left(\frac{2ab}{a+b}\right) = f\left(\frac{2xy}{x+y}\right) \le \frac{f(x)+f(y)}{2}.$$

This implies

$$f\left(\frac{2ab}{a+b}\right) \le \frac{1}{2} f\left(\frac{ab}{at+(1-t)b}\right) + \frac{1}{2} f\left(\frac{ab}{(1-t)a+tb}\right). \tag{3}$$

Multiplying both sides of (3) by $\frac{\rho^{1-\alpha} ab(b-a)}{\Gamma(\alpha)} \frac{\left(\frac{ab}{tb+(1-t)a}\right)^{\rho-1}}{(tb+(1-t)a)^2\left(b^\rho - \left(\frac{ab}{tb+(1-t)a}\right)^\rho\right)^{1-\alpha}}$ and integrating with respect to t on $(0,1)$, we have

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right) \frac{\rho^{1-\alpha} ab(b-a)}{\Gamma(\alpha)} \int_0^1 \frac{\left(\frac{ab}{tb+(1-t)a}\right)^{\rho-1}}{(tb+(1-t)a)^2\left(b^\rho - \left(\frac{ab}{tb+(1-t)a}\right)^\rho\right)^{1-\alpha}} \mathrm{d}t \\ & \le \frac{\rho^{1-\alpha} ab(b-a)}{2\Gamma(\alpha)} \int_0^1 \frac{\left(\frac{ab}{tb+(1-t)a}\right)^{\rho-1}}{(tb+(1-t)a)^2\left(b^\rho - \left(\frac{ab}{tb+(1-t)a}\right)^\rho\right)^{1-\alpha}} f\left(\frac{ab}{at+(1-t)b}\right) \mathrm{d}t \\ & \quad + \frac{\rho^{1-\alpha} ab(b-a)}{2\Gamma(\alpha)} \int_0^1 \frac{\left(\frac{ab}{tb+(1-t)a}\right)^{\rho-1}}{(tb+(1-t)a)^2\left(b^\rho - \left(\frac{ab}{tb+(1-t)a}\right)^\rho\right)^{1-\alpha}} f\left(\frac{ab}{(1-t)a+tb}\right) \mathrm{d}t. \end{aligned} \tag{4}$$

Note that

$$\frac{\rho^{1-\alpha} ab(b-a)}{\Gamma(\alpha)} \int_0^1 \frac{\left(\frac{ab}{tb+(1-s)a}\right)^{\rho-1}}{(tb+(1-t)a)^2\left(b^\rho - \left(\frac{ab}{tb+(1-t)a}\right)^\rho\right)^{1-\alpha}} \mathrm{d}t = \frac{(b^\rho - a^\rho)^\alpha}{\rho^\alpha \Gamma(\alpha+1)}$$

Using Lemma 2, we have

$$\tilde{f}\left(\frac{ab}{(1-t)a+tb}\right) = f\left(\frac{ab}{at+(1-t)b}\right) \tag{5}$$

And

$$\frac{\rho^{1-\alpha}ab(b-a)}{2\Gamma(\alpha)}\int_0^1 \frac{\left(\frac{ab}{tb+(1-t)a}\right)^{\rho-1}}{(tb+(1-t)a)^2\left(b^\rho-\left(\frac{ab}{tb+(1-t)a}\right)^\rho\right)^{1-\alpha}} f\left(\frac{ab}{at+(1-t)b}\right)\mathrm{d}t$$

$$=\frac{1}{2}{}^\rho I^\alpha_{a^+}\tilde{f}(b) \tag{6}$$

$$\frac{\rho^{1-\alpha}ab(b-a)}{2\Gamma(\alpha)}\int_0^1 \frac{\left(\frac{ab}{tb+(1-t)a}\right)^{\rho-1}}{(tb+(1-t)a)^2\left(b^\rho-\left(\frac{ab}{tb+(1-t)a}\right)^\rho\right)^{1-\alpha}} f\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t$$

$$=\frac{1}{2}{}^\rho I^\alpha_{a^+}f(b) \tag{7}$$

Resultantly, we have

$$\frac{(b^\rho-a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}f\left(\frac{2ab}{a+b}\right)\leq\frac{1}{2}{}^\rho I^\alpha_{a^+}F(b). \tag{8}$$

Similarly, multiplying both sides of (3) by $\frac{\rho^{1-\alpha}ab(b-a)}{\Gamma(\alpha)}\frac{\left(\frac{ab}{tb+(1-s)a}\right)^{\rho-1}}{(tb+(1-t)a)^2\left(\left(\frac{ab}{tb+(1-t)a}\right)^\rho-a^\rho\right)^{1-\alpha}}$ and integrating with respect to t over $(0,1)$, we have

$$\frac{(b^\rho-a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}f\left(\frac{2ab}{a+b}\right)\leq\frac{1}{2}{}^\rho I^\alpha_{b^-}F(a). \tag{9}$$

Adding (8) and (10), we obtain

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{\rho^\alpha\Gamma(\alpha+1)}{4(b^\rho-a^\rho)^\alpha}[{}^\rho I^\alpha_{a^+}F(b)+{}^\rho I^\alpha_{b^-}F(a)]. \tag{10}$$

Now we prove the right hand side of the inequality.

$$f\left(\frac{ab}{at+(1-t)b}\right)+f\left(\frac{ab}{(1-t)a+tb}\right)\leq f(a)+f(b). \tag{11}$$

Multiplying both sides of (11) by $\frac{\rho^{1-\alpha}ab(b-a)}{\Gamma(\alpha)}\frac{\left(\frac{ab}{tb+(1-t)a}\right)^{\rho-1}}{(tb+(1-t)a)^2\left(b^\rho-\left(\frac{ab}{tb+(1-t)a}\right)^\rho\right)^{1-\alpha}}$ and integrating with respect to t over $(0,1)$, we have

$$\frac{\rho^{1-\alpha}ab(b-a)}{\Gamma(\alpha)}\int_0^1 \frac{\left(\frac{ab}{tb+(1-t)a}\right)^{\rho-1}}{(tb+(1-t)a)^2\left(b^\rho-\left(\frac{ab}{tb+(1-t)a}\right)^\rho\right)^{1-\alpha}} f\left(\frac{ab}{at+(1-t)b}\right)\mathrm{d}t$$

$$+\frac{\rho^{1-\alpha}ab(b-a)}{\Gamma(\alpha)}\int_0^1 \frac{\left(\frac{ab}{tb+(1-t)a}\right)^{\rho-1}}{(tb+(1-t)a)^2\left(b^\rho-\left(\frac{ab}{tb+(1-t)a}\right)^\rho\right)^{1-\alpha}} f\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t$$

$$\leq\frac{\rho^{1-\alpha}ab(b-a)}{\Gamma(\alpha)}\int_0^1 \frac{\left(\frac{ab}{tb+(1-s)a}\right)^{\rho-1}}{(tb+(1-t)a)^2\left(b^\rho-\left(\frac{ab}{tb+(1-t)a}\right)^\rho\right)^{1-\alpha}}\mathrm{d}t[f(a)+f(b)]. \tag{12}$$

We obtain

$${}^\rho I^\alpha_{a^+}F(b)\leq\frac{(b^\rho-a^\rho)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}[f(a)+f(b)]. \tag{13}$$

Similarly, multiplying both sides of (11) by $\frac{\rho^{1-\alpha}ab(b-a)}{\Gamma(\alpha)}\frac{\left(\frac{ab}{tb+(1-s)a}\right)^{\rho-1}}{(tb+(1-t)a)^2\left(\left(\frac{ab}{tb+(1-t)a}\right)^{\rho}-a^{\rho}\right)^{1-\alpha}}$ and integrating with respect to t over $(0,1)$, we obtain

$$ {}^{\rho}I^{\alpha}_{b^-}F(a) \leq \frac{(b^{\rho}-a^{\rho})^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}[f(a)+f(b)]. \tag{14}$$

Adding (13) and (14), we have

$$\frac{\rho^{\alpha}\Gamma(\alpha+1)}{4(b^{\rho}-a^{\rho})^{\alpha}}[{}^{\rho}I^{\alpha}_{a^+}F(b)+{}^{\rho}I^{\alpha}_{b^-}F(a)] \leq \frac{f(a)+f(b)}{2}. \tag{15}$$

By comparing (10) and (15), we get the required result. □

Lemma 3. *Let $\alpha,\rho>0$. If $f:[a,b]\to\mathbb{R}$ is differentiable mapping on (a,b) with $a<b$, then*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{\rho^{\alpha}\Gamma(\alpha+1)}{4(b^{\rho}-a^{\rho})^{\alpha}}[{}^{\rho}I^{\alpha}_{a^+}F(b)+{}^{\rho}I^{\alpha}_{b^-}F(a)]\\ &=\frac{ab(b-a)}{4(b^{\rho}-a^{\rho})^{\alpha}}\int_0^1\Omega_{\alpha,\rho}(t)f'\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t,\end{aligned} \tag{16}$$

where

$$\begin{aligned}\Omega_{\alpha,\rho}(t)=&\frac{1}{(ta+(1-t)b)^2}\left[\left[\left(\frac{ab}{ta+(1-t)b}\right)^{\rho}-a^{\rho}\right]^{\alpha}-\left[\left(\frac{ab}{tb+(1-t)a}\right)^{\rho}-a^{\rho}\right]^{\alpha}\right.\\ &\left.+\left[b^{\rho}-\left(\frac{ab}{tb+(1-t)a}\right)^{\rho}\right]^{\alpha}-\left[b^{\rho}-\left(\frac{ab}{ta+(1-t)b}\right)^{\rho}\right]^{\alpha}\right].\end{aligned}$$

Proof. Since

$$ {}^{\rho}I^{\alpha}_{a^+}F(b) = \frac{\rho^{1-\alpha}ab(b-a)}{\Gamma(\alpha)}\int_0^1\frac{\left(\frac{ab}{tb+(1-t)a}\right)^{\rho-1}}{(tb+(1-t)a)^2\left(b^{\rho}-\left(\frac{ab}{tb+(1-t)a}\right)^{\rho}\right)^{1-\alpha}}F\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t.$$

Using integration by parts, we have

$$\begin{aligned}{}^{\rho}I^{\alpha}_{a^+}F(b)=&\frac{(b^{\rho}-a^{\rho})^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}F(a)+\frac{ab(b-a)}{\rho^{\alpha}\Gamma(\alpha+1)}\int_0^1\frac{1}{(tb+(1-t)a)^2}\left(b^{\rho}-\left(\frac{ab}{tb+(1-t)a}\right)^{\rho}\right)^{\alpha}\\ &\times F'\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t.\end{aligned} \tag{17}$$

Similarly, we have

$$\begin{aligned}{}^{\rho}I^{\alpha}_{b^-}F(a)=&\frac{(b^{\rho}-a^{\rho})^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}F(b)-\frac{ab(b-a)}{\rho^{\alpha}\Gamma(\alpha+1)}\int_0^1\frac{1}{(tb+(1-t)a)^2}\left(\left(\frac{ab}{tb+(1-t)a}\right)^{\rho}-a^{\rho}\right)^{\alpha}\\ &\times F'\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t.\end{aligned} \tag{18}$$

Using (17) and (18), we get

$$\begin{aligned}&\frac{4(b^{\rho}-a^{\rho})^{\alpha}}{ab(b-a)}\left(\frac{f(a)+f(b)}{2}-\frac{\rho^{\alpha}\Gamma(\alpha+1)}{4(b^{\rho}-a^{\rho})^{\alpha}}[{}^{\rho}I^{\alpha}_{a^+}F(b)+{}^{\rho}I^{\alpha}_{b^-}F(a)]\right)\\ &=\int_0^1\frac{1}{(tb+(1-t)a)^2}\left[\left(\left(\frac{ab}{tb+(1-t)a}\right)^{\rho}-a^{\rho}\right)^{\alpha}-\left(\left(b^{\rho}-\frac{ab}{tb+(1-t)a}\right)^{\rho}\right)^{\alpha}\right]\\ &\times F'\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t.\end{aligned} \tag{19}$$

We note that

$$F\left(\frac{ab}{(1-t)a+tb}\right)=f\left(\frac{ab}{(1-t)a+tb}\right)+\tilde{f}\left(\frac{ab}{(1-t)a+tb}\right),$$

and

$$F'\left(\frac{ab}{(1-t)a+tb}\right)=f'\left(\frac{ab}{(1-t)a+tb}\right)-\frac{(bt+(1-t)a)^2}{(ta+(1-t)b)^2}f'\left(\frac{ab}{(1-t)b+ta}\right).$$

Then

$$\begin{aligned}&\int_0^1\frac{1}{(tb+(1-t)a)^2}\left(\left(\frac{ab}{tb+(1-t)a}\right)^{\rho}-a^{\rho}\right)^{\alpha}F'\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t\\&=\int_0^1\frac{1}{(ta+(1-t)b)^2}\left(\left(\frac{ab}{ta+(1-t)b}\right)^{\rho}-a^{\rho}\right)^{\alpha}f'\left(\frac{ab}{(ta+(1-t)b}\right)\mathrm{d}t\\&\quad-\int_0^1\frac{1}{(ta+(1-t)b)^2}\left(\left(\frac{ab}{tb+(1-t)a}\right)^{\rho}-a^{\rho}\right)^{\alpha}f'\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t.\end{aligned}\tag{20}$$

$$\begin{aligned}&\int_0^1\frac{1}{(tb+(1-t)a)^2}\left(\left(b^{\rho}-\frac{ab}{tb+(1-t)a}\right)^{\rho}\right)^{\alpha}F'\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t\\&=\int_0^1\frac{1}{(ta+(1-t)b)^2}\left(\left(b^{\rho}-\frac{ab}{ta+(1-t)b}\right)^{\rho}\right)^{\alpha}f'\left(\frac{ab}{(ta+(1-t)b}\right)\mathrm{d}t\\&\quad-\int_0^1\frac{1}{(ta+(1-t)b)^2}\left(b^{\rho}-\left(\frac{ab}{tb+(1-t)a}\right)^{\rho}\right)^{\alpha}f'\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t.\end{aligned}\tag{21}$$

Using (19), (20) and (21), we get the required result. □

Theorem 3. *Let $\alpha>0$ and $\rho>0$. If $f\in C^1(a,b)$ and $|f'|$ is harmonically convex function, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\rho^{\alpha}\Gamma(\alpha+1)}{4(b^{\rho}-a^{\rho})^{\alpha}}[{}^{\rho}I^{\alpha}_{a^+}F(b)+{}^{\rho}I^{\alpha}_{b^-}F(a)]\right|\\&\leq\frac{1}{4(b^{\rho}-a^{\rho})^{\alpha}(b-a)}\left(|f'(a)|\int_a^b\triangle(x)\frac{a(b-x)}{x}\mathrm{d}x+|f'(b)|\int_a^b\triangle(x)\frac{b(x-a)}{x}\mathrm{d}x\right),\end{aligned}\tag{22}$$

Proof.

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\rho^{\alpha}\Gamma(\alpha+1)}{4(b^{\rho}-a^{\rho})^{\alpha}}[{}^{\rho}I^{\alpha}_{a^+}F(b)+{}^{\rho}I^{\alpha}_{b^-}F(a)]\right|\\&\leq\frac{ab(b-a)}{4(b^{\rho}-a^{\rho})^{\alpha}}\int_0^1|\Omega_{\alpha,\rho}(t)|\left|f'\left(\frac{ab}{ta+(1-t)b}\right)\right|\mathrm{d}t\\&\leq\frac{ab(b-a)}{4(b^{\rho}-a^{\rho})^{\alpha}}\left(\int_0^1(1-t)\,|\Omega_{\alpha,\rho}(t)|\,|f'(a)|\,\mathrm{d}t+\int_0^1t\,|\Omega_{\alpha,\rho}(t)|\,|f'(b)|\,\mathrm{d}t\right).\end{aligned}\tag{23}$$

Note that

$$\int_0^1t\,|\Omega_{\alpha,\rho}(t)|\,\mathrm{d}t=\frac{1}{a(b-a)^2}\int_a^b\triangle(x)\frac{(x-a)}{x}\mathrm{d}x,$$

and

$$\int_0^1(1-t)\,|\Omega_{\alpha,\rho}(t)|\,\mathrm{d}t=\frac{1}{b(b-a)^2}\int_a^b\triangle(x)\frac{(b-x)}{x}\mathrm{d}x,$$

where

$$\triangle(x) = \Bigg[(x^\rho - a^\rho)^\alpha - \left(\left(\frac{abx}{x(a+b)-ab}\right)^\rho - a^\rho\right)^\alpha + \left(b^\rho - \left(\frac{abx}{x(a+b)-ab}\right)^\rho\right)^\alpha - (b^\rho - x^\rho)^\alpha\Bigg]$$

Also we observe that $\triangle$ is non decreasing function on $[a, b]$. Also

$$\triangle\left(\frac{2ab}{a+b}\right) = 0, \tag{24}$$

and

$$\triangle(a) = -2(b^\rho - a^\rho),$$
$$\triangle(b) = 2(b^\rho - a^\rho).$$

This completes the proof. □

Acknowledgment. The authors are thankful to the editor and the reviewer for their valuable comments and suggestions. This research is supported by HEC NRPU project titled: "Inequalities via convex functions and its generalizations" and No: 8081/Punjab/NRPU/R&D/HEC/2017.

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