TJMM
14 (2022), No. 1, 01-09

FIXED POINT THEOREM FOR A MEIR-KEELER TYPE MAPPING IN A METRIC SPACE WITH A TRANSITIVE RELATION

KOJI AOYAMA AND MASASHI TOYODA

Abstract. The aim of this paper is to provide characterizations of a Meir-Keeler type mapping and a fixed point theorem for the mapping in a metric space endowed with a transitive relation.

1. Introduction

Let X be a metric space with metric d, R a subset of $X \times X$, and $T\colon X \to X$ a mapping. We say that T is a *Meir-Keeler type* mapping on R if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$(x, y) \in R \text{ and } \epsilon \leq d(x, y) < \epsilon + \delta \text{ imply } d(Tx, Ty) < \epsilon.$$

This notion of mappings is related to the notion introduced in Meir and Keeler [4]. Indeed, a Meir-Keeler type mapping T on $X \times X$ is a *weakly uniformly strict contraction* in the sense of [4], which is often called a *Meir-Keeler contraction*.

In Section 3, we provide some characterizations of a Meir-Keeler type mapping (Theorem 1). The result includes characterizations of a Meir-Keeler contraction by Wong [9], Lim [3], and Gavruta et al. [2].

In Section 4, we establish a fixed point theorem for a Meir-Keeler type mapping (Theorem 3) in a metric space endowed with a transitive relation. The result is related to the study of Ben-El-Mechaiekh [1] and fixed point theorems in a metric space with a partial order proved in Ran and Reurings [6], Nieto and Rodríguez-López [5], and Reich and Zaslavski [7].

2. Preliminaries

Throughout the present paper, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_+$ the set of nonnegative real numbers.

A function $l\colon \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of *type (L)* if for any $s > 0$ there exists $\delta > 0$ such that $l(t) \leq s$ for all $t \in [s, s + \delta]$. It is clear that if a function $l\colon \mathbb{R}_+ \to \mathbb{R}_+$ is of type (L), then $l(t) \leq t$ for all $t > 0$.

Remark 1. *The notion of a function of type (L) as above is related to the notion of an L-function introduced in* [3]*. We say that a function $l\colon \mathbb{R}_+ \to \mathbb{R}_+$ is an* L-function [3] *if $l(0) = 0$, $l(s) > 0$ for all $s > 0$, and l is of type (L).*

We say that a function $w\colon \mathbb{R}_+ \to \mathbb{R}$ is *right lower semicontinuous* at $t_0 \in \mathbb{R}_+$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that $w(t_0) - \epsilon < w(s)$ for all $s \in [t_0, t_0 + \delta)$; a function $\psi\colon \mathbb{R}_+ \to \mathbb{R}$ is *right upper semicontinuous* at $t_0 \in \mathbb{R}_+$ if $-\psi$ is right lower semicontinuous at t_0. It is clear that if $w\colon \mathbb{R}_+ \to \mathbb{R}$ is a nondecreasing function, then w is right lower semicontinuous at any $t \in \mathbb{R}_+$. It is known that a function $w\colon \mathbb{R}_+ \to \mathbb{R}$ is right lower

2010 *Mathematics Subject Classification.* 47H09.
Key words and phrases. Meir-Keeler type mapping, fixed point, transitive relation.

semicontinuous at $t_0 \in \mathbb{R}_+$ if and only if $w(t_0) \leq \liminf_n w(s_n)$ whenever $\{s_n\}$ is a sequence in $[t_0, \infty)$ such that $s_n \to t_0$.

3. Characterizations of a Meir-Keeler type mapping

The aim of this section is to prove the following theorem, which provides characterizations of a Meir-Keeler type mapping defined on a metric space endowed with a relation.

Theorem 1. *Let X be a metric space with metric d, $T\colon X \to X$ a mapping, and R a nonempty subset of $X \times X$. Then the following are equivalent:*

1. *T is a Meir-Keeler type mapping on R, that is, for any $\epsilon > 0$ there exists $\delta > 0$ such that $(x, y) \in R$ and $\epsilon \leq d(x, y) < \epsilon + \delta$ imply $d(Tx, Ty) < \epsilon$;*
2. *for any $\epsilon > 0$ there exists $\delta > 0$ such that $(x, y) \in R$ and $d(x, y) < \epsilon + \delta$ imply $d(Tx, Ty) < \epsilon$;*
3. *there exists a nondecreasing function $\gamma\colon \mathbb{R}_+ \to [0, \infty]$ such that $\gamma(s) > s$ for all $s > 0$ and $\gamma\big(d(Tx, Ty)\big) \leq d(x, y)$ for all $(x, y) \in R$;*
4. *there exists a function $w\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $w(s) > s$ for all $s > 0$, w is right lower semicontinuous on $(0, \infty)$, and $w\big(d(Tx, Ty)\big) \leq d(x, y)$ for all $(x, y) \in R$;*
5. *there exists a function $l\colon (0, \infty) \to \mathbb{R}_+$ of type (L) such that $d(Tx, Ty) < l\big(d(x, y)\big)$ for all $(x, y) \in R$ with $x \neq y$;*
6. *there exist a nondecreasing function $\phi\colon \mathbb{R}_+ \to [0, \infty]$ and a function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that ψ is right upper semicontinuous on $(0, \infty)$, $\phi(t) > \psi(t)$ for all $t > 0$, and $\phi\big(d(Tx, Ty)\big) \leq \psi\big(d(x, y)\big)$ for all $(x, y) \in R$.*

Moreover, in (5)*, one can choose l to be a right continuous and nondecreasing function such that $l(s) > 0$ for all $s > 0$.*

Obviously, Theorem 1 is valid in case of $R = X \times X$. Therefore Theorem 1 provides characterizations of a Meir-Keeler contraction [4] on a metric space.

Remark 2. *The condition (3) is related to the modulus of uniform continuity of T; see Lim* [3]. *The conditions (4) and (5) are based on* [3, Theorem 1]*; see also Wong* [9] *for (4). The condition (6) comes from a* weak type contraction *introduced in* [2].

Theorem 1 above is a direct consequence of Theorem 2 below. We first prove it by using lemmas in Section 5.

Theorem 2. *Let K be a nonempty set and let $f\colon K \to \mathbb{R}_+$ and $g\colon K \to \mathbb{R}_+$ be functions. Suppose that $g^{-1}(0) \subset f^{-1}(0)$. Then the following are equivalent:*

1. *For any $\epsilon > 0$ there exists $\delta > 0$ such that $x \in K$ and $\epsilon \leq g(x) < \epsilon + \delta$ imply $f(x) < \epsilon$;*
2. *for any $\epsilon > 0$ there exists $\delta > 0$ such that $x \in K$ and $g(x) < \epsilon + \delta$ imply $f(x) < \epsilon$;*
3. *there exists a nondecreasing function $\gamma\colon \mathbb{R}_+ \to [0, \infty]$ such that $\gamma(s) > s$ for all $s > 0$ and $\gamma\big(f(x)\big) \leq g(x)$ for all $x \in K$;*
4. *there exists a function $w\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $w(s) > s$ for all $s > 0$, w is right lower semicontinuous on $(0, \infty)$, and $w\big(f(x)\big) \leq g(x)$ for all $x \in K$;*
5. *there exists a function $l\colon (0, \infty) \to \mathbb{R}_+$ of type (L) such that $f(x) < l\big(g(x)\big)$ for all $x \in K$ with $g(x) \neq 0$.*
6. *there exist a nondecreasing function $\phi\colon \mathbb{R}_+ \to [0, \infty]$ and a function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi(t) > \psi(t)$ for all $t > 0$, ψ is right upper semicontinuous on $(0, \infty)$, and $\phi\big(f(x)\big) \leq \psi\big(g(x)\big)$ for all $x \in K$.*

Moreover, in (5)*, one can choose l to be a right continuous and nondecreasing function such that $l(s) > 0$ for all $s > 0$.*

Proof. The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (6) are clear. Lemma 4 shows that (1) and (5) are equivalent, and that l in (5) can be chosen to be a right continuous and nondecreasing function such that $l(s) > 0$ for all $s > 0$. Lemmas 5, 6, and 7 show the implications (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (2), respectively. Moreover, the implication (1) $\Rightarrow$ (2) and (6) $\Rightarrow$ (1) follow from Lemmas 8 and 9, respectively. This completes the proof. □

The following example shows that the implication (1) $\Rightarrow$ (2) in Theorem 2 does not hold without the assumption $g^{-1}(0) \subset f^{-1}(0)$.

Example 1. *Let $K = \{x\}$ be a singleton and let $f\colon K \to \mathbb{R}_+$ and $g\colon K \to \mathbb{R}_+$ be functions defined by $f(x) = 1$ and $g(x) = 0$. Then (1) in Theorem 2 holds, but (2) in Theorem 2 does not hold.*

Proof. Let $\epsilon = 1$. Then $0 = g(x) < \epsilon + \delta$ and $f(x) \geq \epsilon$ for all $\delta > 0$. Thus (2) does not hold. On the other hand, let $\epsilon > 0$ and $\delta = 1$. Then $\{y \in K : \epsilon \leq g(y) < \epsilon + \delta\} = \emptyset$. Therefore, (1) does hold. □

Remark 3. *Let K, f, and g be the same as in Example 1 and let $\phi\colon \mathbb{R}_+ \to [0, \infty]$ and $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be functions defined by $\phi(t) \equiv 1/2$ and*

$$\psi(t) = \begin{cases} 1 & \text{if } t = 0; \\ 1/4 & \text{otherwise.} \end{cases}$$

Then ϕ is nondecreasing, ψ is right upper semicontinuous on $(0, \infty)$, and $\phi(t) > \psi(t)$ for all $t > 0$. Since

$$\phi\big(f(x)\big) = \phi(1) = 1/2 \leq 1 = \psi(0) = \psi\big(g(x)\big),$$

it follows that $\phi\big(f(y)\big) \leq \big(g(y)\big)$ for all $y \in K$. Therefore Example 1 also shows that the implication (6) $\Rightarrow$ (2) in Theorem 2 does not hold without the assumption $g^{-1}(0) \subset f^{-1}(0)$.

Using Theorem 2, we can easily obtain Theorem 1.

Proof of Theorem 1. Let $f\colon R \to \mathbb{R}_+$ and $g\colon R \to \mathbb{R}_+$ be functions defined by $f(x, y) = d(Tx, Ty)$ and $g(x, y) = d(x, y)$ for $(x, y) \in R$. Then it is clear that $g^{-1}(0) \subset f^{-1}(0)$. Therefore Theorem 2 implies the conclusion. □

4. Fixed point theorems

The aim of this section is to establish fixed point theorems for a Meir-Keeler type mapping defined on a complete metric space endowed with a transitive relation or a partial order.

Theorem 3. *Let X be a complete metric space with metric d, $T\colon X \to X$ a mapping, and R a nonempty subset of $X \times X$. Suppose that*

(1) $(u, v) \in R$ and $(v, w) \in R$ imply $(u, w) \in R$;
(2) there exists $x \in X$ such that $(x, Tx) \in R$;
(3) $(Tu, Tv) \in R$ for all $(u, v) \in R$;
(4) for any $\epsilon > 0$ there exists $\delta > 0$ such that $(u, v) \in R$ and $\epsilon \leq d(u, v) < \epsilon + \delta$ imply $d(Tu, Tv) < \epsilon$;
(5) if $\{x_n\}$ is a sequence in X such that $x_n \to y$ and $(x_n, x_{n+1}) \in R$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k} \to Ty$ as $k \to \infty$.

Then $\{T^n x\}$ converges to a fixed point of T, that is, T has a fixed point. Moreover, suppose that

(6) $(x, y) \in R$ *for all* $y \in X$*;*
(7) R *is closed in* $X \times X$*.*

Then T *has a unique fixed point.*

Remark 4. *The assumptions* (6) *and* (7) *in Theorem 3 can be replaced by the following condition:*

> *If* y *is a fixed point of* T*, and* $\{x_n\}$ *is a sequence in* X *such that* $x_n \to z \in X$ *and* $(x_n, y) \in R$ *for all* $n \in \mathbb{N}$*, then* $(z, y) \in R$*.*

To prove Theorem 3, we need lemmas below, which are based on the results in [4, §2].

Lemma 1. *Let* X *be a metric space with metric* d*,* $T \colon X \to X$ *a mapping,* $x \in X$*, and* $\{x_n\}$ *a sequence in* X *defined by* $x_n = T^n x$ *for* $n \in \mathbb{N}$*. Suppose that for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$n \in \mathbb{N},\ \epsilon \leq d(x_n, x_{n+1}) < \epsilon + \delta \Rightarrow d(x_{n+1}, x_{n+2}) < \epsilon. \tag{1}$$

Then $\{d(x_n, x_{n+1})\}$ *is nonincreasing and* $\lim_n d(x_n, x_{n+1}) = 0$*.*

Proof. Suppose that $d(x_m, x_{m+1}) = 0$. Then $x_m = x_{m+1}$. Thus we have $x_{m+1} = T^{m+1}x = Tx_m = Tx_{m+1} = x_{m+2}$, and hence $d(x_{m+1}, x_{m+2}) = 0$. On the other hand, suppose that $\epsilon = d(x_m, x_{m+1}) > 0$. Then there exists $\delta > 0$ such that (1) holds. Thus we have $d(x_{m+1}, x_{m+2}) < \epsilon = d(x_m, x_{m+1})$. Consequently, we know that $\{d(x_n, x_{n+1})\}$ is nonincreasing, and hence $\lim_n d(x_n, x_{n+1})$ exists. Suppose that $\epsilon = \lim_n d(x_n, x_{n+1}) > 0$. Then there exists $\delta > 0$ such that (1) holds. Since $d(x_n, x_{n+1}) \searrow \epsilon$, there exists $k \in \mathbb{N}$ such that $\epsilon \leq d(x_k, x_{k+1}) < \epsilon + \delta$. Thus we have $\epsilon \leq d(x_{k+1}, x_{k+2}) < \epsilon$, which is a contradiction. Therefore, $\lim_n d(x_n, x_{n+1}) = \epsilon = 0$. □

Lemma 2. *Let* X *be a metric space with metric* d*,* $\{x_n\}$ *a sequence in* X*,* l, m *positive integers, and* ϵ, η *positive real numbers. Suppose that* $l < m$*,* $\eta \leq \epsilon$*,* $d(x_l, x_m) \geq 2\epsilon$*, and* $d(x_i, x_{i+1}) < \eta/3$ *for all* $i \in \mathbb{N}$ *with* $l \leq i \leq m$*. Then there exists* $j \in \mathbb{N}$ *such that* $l < j < m$ *and* $\epsilon + 2\eta/3 \leq d(x_l, x_j) < \epsilon + \eta$*.*

Proof. Set $A = \{i \in \mathbb{N} \colon l < i < m,\ \epsilon + 2\eta/3 \leq d(x_l, x_i)\}$. We first show that $m - 1 \in A$. Suppose that $m - 1 \leq l$. Then $m = l + 1$, and we have

$$2\epsilon \leq d(x_l, x_m) = d(x_l, x_{l+1}) < \eta/3 \leq \epsilon/3,$$

which is a contradiction. Thus $l < m - 1$. Moreover, we have

$$d(x_l, x_{m-1}) \geq d(x_l, x_m) - d(x_m, x_{m-1}) \geq 2\epsilon - \eta/3 \geq \epsilon + 2\eta/3.$$

Therefore, $m - 1 \in A$, and hence A is nonempty.

Set $j = \min A$. Suppose that $l \geq j - 1$. Then $j = l + 1$. Thus we have $\epsilon + 2\eta/3 \leq d(x_l, x_j) = d(x_l, x_{l+1}) < \eta/3$, which is a contradiction. Therefore, $l < j - 1 < j < m$. Since $j - 1 \notin A$, we have $d(x_l, x_{j-1}) < \epsilon + 2\eta/3$, and hence

$$d(x_l, x_j) \leq d(x_l, x_{j-1}) + d(x_{j-1}, x_j) < \epsilon + 2\eta/3 + \eta/3 = \epsilon + \eta.$$

As a result, we conclude that $l < j < m$ and $\epsilon + 2\eta/3 \leq d(x_l, x_j) < \epsilon + \eta$. □

Lemma 3. *Let* X*,* T*,* x*, and* $\{x_n\}$ *be the same as in Lemma 1. Suppose that for any* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$i, j \in \mathbb{N},\ \epsilon \leq d(x_i, x_j) < \epsilon + \delta \Rightarrow d(x_{i+1}, x_{j+1}) < \epsilon. \tag{2}$$

Then $\{x_n\}$ *is a Cauchy sequence.*

Proof. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that for each $i \in \mathbb{N}$ there exist $m_i, n_i \in \mathbb{N}$ such that

$$i \leq m_i < n_i \text{ and } d(x_{m_i}, x_{n_i}) \geq 2\epsilon. \tag{3}$$

By assumption, we know that there exists $\delta > 0$ such that (2) holds. Set $\eta = \min\{\delta, \epsilon\}$. Since $d(x_n, x_{n+1}) \searrow 0$ by Lemma 1, it follows from (3) that there exist $m, n \in \mathbb{N}$ with $m < n$ such that $d(x_m, x_n) \geq 2\epsilon$ and

$$d(x_i, x_{i+1}) < \eta/3 \tag{4}$$

for all $i \in \mathbb{N}$ with $i \geq m$. Thus Lemma 2 shows that there exists $j \in \mathbb{N}$ such that $m < j < n$ and

$$\epsilon + 2\eta/3 \leq d(x_m, x_j) < \epsilon + \eta.$$

As a result, we see that $\epsilon \leq d(x_m, x_j) < \epsilon + \delta$. Taking into account (4) and (2), we have

$$\begin{aligned}\epsilon + 2\eta/3 \leq d(x_m, x_j) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{j+1}) + d(x_{j+1}, x_j) \\ &< \eta/3 + \epsilon + \eta/3 = \epsilon + 2\eta/3,\end{aligned}$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. □

Now we prove Theorem 3.

Proof of Theorem 3. Let $\{x_n\}$ be a sequence in X defined by $x_n = T^n x$ for $n \in \mathbb{N}$. Then, by the assumptions (1), (2), and (3), we see that $(x_m, x_n) \in R$ for all $m, n \in \mathbb{N}$ with $m < n$. Thus it follows from the assumption (4) that for any $\epsilon > 0$ there exists $\delta > 0$ such that (2) holds. Since X is complete, Lemma 3 shows that $\{x_n\}$ converges to some point $z \in X$. We show that z is a fixed point of T. By virtue of the assumption (5), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k} \to Tz$ as $k \to \infty$. Taking into account $x_{n_k+1} \to z$, we conclude that

$$d(Tz, z) \leq d(Tz, x_{n_k+1}) + d(x_{n_k+1}, z) = d(Tz, Tx_{n_k}) + d(x_{n_k+1}, z) \to 0$$

as $k \to \infty$. Therefore, $Tz = z$, and hence z is a fixed point of T.

We next show that z is the unique fixed point of T under the assumptions (6) and (7). Let y be a fixed point of T. Since $(x, y) \in R$ by (6), it follows from (3) that $(Tx, y) = (Tx, Ty) \in R$. Therefore, $(T^n x, y) \in R$ for all $n \in \mathbb{N}$. Since $T^n x \to z$ and R is closed by (7), we conclude that $(z, y) \in R$. Using Theorem 1 and the function γ in Theorem 1 (3), we have

$$\gamma\bigl(d(z, y)\bigr) = \gamma\bigl(d(Tz, Ty)\bigr) \leq d(z, y),$$

and hence $z = y$. □

Using Theorem 3, we obtain the following:

Corollary 1 (Nieto & Rodríguez-López [5, Theorem 2.2])**.** *Let X be a complete metric space with metric d, $T\colon X \to X$ a mapping, and $\preceq$ a partial order in X. Suppose that*

- *(NR1) there exists $x \in X$ such that $x \preceq Tx$;*
- *(NR2) $Tu \preceq Tv$ for all $u, v \in X$ with $u \preceq v$;*
- *(NR3) there exists $\theta \in [0, 1)$ such that $d(Tu, Tv) \leq \theta d(u, v)$ for all $u, v \in X$ with $u \preceq v$;*
- *(NR4) if $\{x_n\}$ is a sequence in X such that $x_n \to y$ and $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, then then $x_n \preceq y$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. Set $R = \{(u, v) \in X \times X : u \preceq v\}$. Since $(x, Tx) \in R$ by (NR1), we know that R is a nonempty subset of $X \times X$ and the assumption (2) in Theorem 3 holds. The assumption (1) in Theorem 3 is valid clearly. The assumptions (3) and (4) in Theorem 3 follow from (NR2) and (NR3), respectively. We must check the assumption (5) in Theorem 3. Let $\{x_n\}$ be a sequence in X such that $x_n \to y$ and $(x_n, x_{n+1}) \in R$ for all $n \in \mathbb{N}$. Taking into account (NR3) and (NR4), we see that

$$d(Tx_n, Ty) \leq \theta d(x_n, y) \to 0$$

as $n \to \infty$. Therefore Theorem 3 implies the conclusion. □

Using Theorem 3, we also deduce the following fixed point theorem, which is similar to [7, Theorem 1.2].

Theorem 4. *Let Y be a complete metric space with metric d, $\preceq$ a partial order in Y, X a nonempty closed subset of Y, and $T: X \to X$ a mapping. Suppose that*

- *(RZ0) $\{(u, v) \in Y \times Y : u \preceq v\}$ is closed in $Y \times Y$;*
- *(RZ1) the graph of T is closed in $Y \times Y$;*
- *(RZ2) $Tu \preceq Tv$ for all $u, v \in X$ with $u \preceq v$;*
- *(RZ3) there exists a right upper semicontinuous function $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ such that $t > \psi(t)$ for all $t > 0$ and $d(Tu, Tv) \leq \psi\big(d(u, v)\big)$ for all $u, v \in X$ with $u \preceq v$;*
- *(RZ4) there exists $x \in X$ such that $x \preceq y$ for all $y \in X$.*

Then $\{T^n x\}$ converges to a unique fixed point of T.

Proof. By assumption, it is clear that X is complete. Set $R = \{(u, v) \in X \times X : u \preceq v\}$. By virtue of (RZ4), $(x, x) \in R$, and hence R is nonempty. Moreover, since $\preceq$ is a partial order, the assumption (1) in Theorem 3 holds. The assumptions (2) and (6) in Theorem 3 follow from (RZ4); the assumption (3) in Theorem 3 follows from (RZ2). Since X is closed, the assumption (7) in Theorem 3 is deduced from (RZ0). Using Theorem 1, we know that (RZ3) implies the assumption (4) in Theorem 3. Therefore it is enough to verify the assumption (5) in Theorem 3. Let $\{x_n\}$ be a sequence in X such that $x_n \to y$ and $(x_n, x_{n+1}) \in R$ for all $n \in \mathbb{N}$. Since X is closed, it follows that $y \in X$. Let $m \in \mathbb{N}$ be fixed. Then it is easy to check that $(x_m, x_n) \in R$ for all $n \in \mathbb{N}$ with $m \leq n$. Since $\{(x_m, x_n)\}_{n \geq m}$ converges to (x_m, y) in $X \times X$ and R is closed in $X \times X$, we see that $(x_m, y) \in R$. Hence $(x_m, y) \in R$ for all $m \in \mathbb{N}$. Set $A = \{n \in \mathbb{N} : x_n = y\}$. Suppose that A is an infinite set. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = y$ for all $k \in \mathbb{N}$, and hence $Tx_{n_k} \to Ty$ as $k \to \infty$. On the other hand, suppose that A is not a infinite set. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \neq y$ for all $k \in \mathbb{N}$. Since $(x_{n_k}, y) \in R$ and $d(x_{n_k}, y) > 0$ for all $k \in \mathbb{N}$, it follows from (RZ3) that

$$d(Tx_{n_k}, Ty) \leq \psi\big(d(x_{n_k}, y)\big) < d(x_{n_k}, y) \to 0$$

as $k \to \infty$. Therefore the assumption (5) in Theorem 3 holds. Consequently, Theorem 3 implies the conclusion. □

5. Lemmas

In this section, we prove lemmas which are used in the proof of Theorem 2.

In what follows, let K be a nonempty set and let $f: K \to \mathbb{R}_+$ and $g: K \to \mathbb{R}_+$ be functions.

Lemma 4. *The conditions* (1) *and* (5) *in Theorem 2 are equivalent. Moreover, in* (5), *one can choose l to be a right continuous and nondecreasing function such that $l(s) > 0$ for all $s > 0$.*

Proof. We first prove (5) $\Rightarrow$ (1). Let $\epsilon > 0$. Since l is of type (L), there exists $\delta > 0$ such that $l(t) \le \epsilon$ for all $t \in [\epsilon, \epsilon + \delta]$. Let $x \in K$ with $\epsilon \le g(x) < \epsilon + \delta$. Then $g(x) \ne 0$. Thus it follows from (5) that $f(x) < l\big(g(x)\big) \le \epsilon$.

We next prove (1) $\Rightarrow$ (5) and the "Moreover" part. We follow the proof of [8, Proposition 1]. By assumption, for any $\epsilon > 0$ there exists $\alpha(\epsilon) > 0$ such that

$$x \in K,\ \epsilon \le g(x) < \epsilon + 2\alpha(\epsilon) \Rightarrow f(x) < \epsilon. \tag{5}$$

Since $\{\epsilon > 0 \colon t \le \epsilon + \alpha(\epsilon)\} \ne \emptyset$ for all $t > 0$, we can define a function $\beta \colon (0, \infty) \to [0, \infty)$ by

$$\beta(t) = \inf\{\epsilon > 0 \colon t \le \epsilon + \alpha(\epsilon)\}$$

for $t > 0$. Then it is clear that β is nondecreasing, $\beta(t) \le t$ for all $t > 0$, and moreover, $\min\{\epsilon > 0 \colon t \le \epsilon + \alpha(\epsilon)\}$ exists for all $t > 0$ with $\beta(t) = t$. Let $\phi_1 \colon (0, \infty) \to [0, \infty)$ be a function defined by

$$\phi_1(t) = \begin{cases} \beta(t) & \text{if } \min\{\epsilon > 0 \colon t \le \epsilon + \alpha(\epsilon)\} \text{ exists;} \\ \dfrac{\beta(t) + t}{2} & \text{otherwise} \end{cases}$$

for $t > 0$. Then we verify the following:

(i) $\phi_1(t) > 0$ for all $t > 0$;
(ii) ϕ_1 is of type (L);
(iii) $f(x) < \phi_1\big(g(x)\big)$ for all $x \in K$ with $g(x) \ne 0$.

By the definition of ϕ_1, (i) is clear. We show (ii). Let $s > 0$ be fixed. Suppose that $\phi_1(t) \le s$ for all $t \in (s, s + \alpha(s)]$. Then setting $\delta = \alpha(s)$, we conclude that

$$t \in [s, s + \delta] \Rightarrow \phi_1(t) \le s. \tag{6}$$

On the other hand, suppose that there exists $\sigma \in (s, s + \alpha(s)]$ such that $\phi_1(\sigma) > s$. Then $s \in \{\epsilon > 0 \colon \sigma \le \epsilon + \alpha(\epsilon)\}$, and hence $\beta(\sigma) \le s$. If $\beta(\sigma) = s$, then we have $\beta(\sigma) = \min\{\epsilon > 0 \colon \sigma \le \epsilon + \alpha(\epsilon)\}$, and thus

$$\phi_1(\sigma) = \beta(\sigma) = s < \phi_1(\sigma),$$

which is a contradiction. Consequently, we know that

$$\beta(\sigma) < s < \phi_1(\sigma) = \frac{\beta(\sigma) + \sigma}{2}.$$

Taking into account the definition of $\beta(\sigma)$, we can choose $u \in [\beta(\sigma), s)$ with $\sigma \le u + \alpha(u)$. Then set $\delta = s - u$ and let $t \in [s, s + \delta]$. Since

$$t \le s + \delta = 2s - u < 2 \cdot \frac{\beta(\sigma) + \sigma}{2} - \beta(\sigma) = \sigma \le u + \alpha(u),$$

it follows that $\beta(t) \le u$. Therefore we have

$$\phi_1(t) \le \frac{\beta(t) + t}{2} \le \frac{u + s + \delta}{2} = s.$$

Thus (6) holds, and hence ϕ_1 is of type (L). We next show (iii). Let $x \in K$ with $g(x) \ne 0$. Taking into account the definition of ϕ_1, we know that for any $t > 0$ there exists $\epsilon \in (0, \phi_1(t)]$ such that $\epsilon \le t \le \epsilon + \alpha(\epsilon)$, and thus there exists $\epsilon \in \big(0, \phi_1\big(g(x)\big)\big]$ such that $\epsilon \le g(x) \le \epsilon + \alpha(\epsilon)$. Hence we deduce from (5) that $f(x) < \epsilon \le \phi_1\big(g(x)\big)$. Consequently, (iii) holds. Now let us define functions $\phi_2 \colon (0, \infty) \to \mathbb{R}_+$ and $l \colon (0, \infty) \to \mathbb{R}_+$ by

$$\phi_2(t) = \sup\{\phi_1(s) \colon s \le t\} \text{ and } l(t) = \inf\{\phi_2(s) \colon s > t\}$$

for $t \in (0, \infty)$. Then it is not hard to check that ϕ_2 and l are well-defined and nondecreasing, and moreover,

$$0 < \phi_1(t) \le \phi_2(t) \le l(t) \le t$$

for all $t > 0$. Thus it follows from (ii) and (iii) that l is of type (L) and $f(x) < l\big(g(x)\big)$ for all $x \in K$ with $g(x) \ne 0$. We can also verify that l is right continuous. This completes the proof. $\square$

Lemma 5. *The condition* (2) *in Theorem 2 implies the condition* (3) *in Theorem 2.*

Proof. Define a function $\gamma\colon \mathbb{R}_+ \to [0, \infty]$ by

$$\gamma(t) = \inf\{g(x) \colon x \in K,\ f(x) \ge t\}$$

for $t \in \mathbb{R}_+$, where $\inf \emptyset = \infty$. Then the function γ is well-defined and nondecreasing, and moreover, $\gamma\big(f(x)\big) \le g(x)$ for all $x \in K$. Hence it is enough to show that $\gamma(t) > t$ for all $t > 0$. Suppose that $\gamma(t) \le t$ for some $t > 0$. Then, by assumption, there exists $\delta > 0$ such that $x \in K$ and $g(x) < t + \delta$ imply $f(x) < t$. Since $\gamma(t) < t + \delta$, there exists $y \in K$ such that $f(y) \ge t$ and $g(y) < t + \delta$. Therefore we have $t \le f(y) < t$, which is a contradiction. $\square$

Lemma 6. *The condition* (3) *in Theorem 2 implies the condition* (4) *in Theorem 2.*

Proof. We follow the idea of the proof of [3, Theorem 1]. If $\{t \in \mathbb{R}_+ \colon \gamma(t) = \infty\}$ is empty, then we easily obtain the conclusion. Thus we may assume that $\{t \in \mathbb{R}_+ \colon \gamma(t) = \infty\}$ is nonempty. Set $t_0 = \inf\{t \in \mathbb{R}_+ \colon \gamma(t) = \infty\}$. In the case of $\gamma(t_0) < \infty$, let $w_1 \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a function defined by

$$w_1(t) = \begin{cases} \gamma(t) & \text{if } t \in [0, t_0]; \\ \gamma(t_0) + t - t_0 & \text{otherwise.} \end{cases}$$

Then it is clear that $w_1(s) > s$ for all $s > 0$. Since w_1 is nondecreasing, we know that w_1 is right lower semicontinuous on $(0, \infty)$. We can also check that $w_1\big(f(x)\big) \le g(x)$ for all $x \in K$. On the other hand, in the case of $\gamma(t_0) = \infty$, let $w_2 \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a function defined by

$$w_2(t) = \begin{cases} \gamma(t) & \text{if } t \in [0, t_0), \\ 2t & \text{otherwise.} \end{cases}$$

Then it is clear that $w_2(s) > s$ for all $s > 0$. Since w_2 is nondecreasing on $(0, t_0)$ and continuous on $[t_0, \infty)$, we know that w_2 is right lower semicontinuous on $(0, \infty)$. We can also check that $w_2\big(f(x)\big) \le g(x)$ for all $x \in K$. $\square$

Lemma 7. *The condition* (4) *in Theorem 2 implies the condition* (2) *in Theorem 2.*

Proof. Suppose that (2) does not hold. Then there exist $\epsilon > 0$ and a sequence $\{x_n\}$ in K such that $g(x_n) < \epsilon + 1/n$ and $f(x_n) \ge \epsilon$ for all $n \in \mathbb{N}$. Since $f(x_n) > 0$, it follows from the properties of w that

$$\epsilon \le f(x_n) < w\big(f(x_n)\big) \le g(x_n) < \epsilon + 1/n$$

for all $n \in \mathbb{N}$. Hence $f(x_n) \to \epsilon$ and $w\big(f(x_n)\big) \to \epsilon$. Since w is right lower semicontinuous at ϵ and $\epsilon < w(\epsilon)$, we have $\epsilon < w(\epsilon) \le \liminf_n w\big(f(x_n)\big) = \epsilon$, which is a contradiction. $\square$

Lemma 8. *Suppose $g^{-1}(0) \subset f^{-1}(0)$. Then the condition* (1) *in Theorem 2 implies the condition* (2) *in Theorem 2.*

Proof. Let $\epsilon > 0$ be given. Then, by (1), there exists $\delta > 0$ such that $x \in K$ and $\epsilon \le g(x) < \epsilon + \delta$ imply $f(x) < \epsilon$. Let $x \in K$ such that $g(x) < \epsilon$. It is enough to show that $f(x) < \epsilon$. Suppose that $g(x) = 0$. Then, by assumption, $f(x) = 0 < \epsilon$. On the other hand, suppose that $0 < g(x) < \epsilon$. Set $\epsilon' = g(x)$. Then, by (1), there exists $\delta' > 0$ such that $y \in K$ and $\epsilon' \le g(y) < \epsilon' + \delta'$ imply $f(y) < \epsilon'$. Since $\epsilon' = g(x) < \epsilon' + \delta'$, we have $f(x) < \epsilon' = g(x) < \epsilon$. □

Lemma 9. *The condition* (6) *in Theorem 2 implies the condition* (1) *in Theorem 2.*

Proof. Suppose that (1) does not hold. Then there exist $\epsilon > 0$ and a sequence $\{x_n\}$ in K such that $\epsilon \le g(x_n) < \epsilon + 1/n$ and $f(x_n) \ge \epsilon$ for all $n \in \mathbb{N}$. Thus $g(x_n) \to \epsilon$ and, by assumption,

$$\psi(\epsilon) < \phi(\epsilon) \le \phi\big(f(x_n)\big) \le \psi\big(g(x_n)\big)$$

for all $n \in \mathbb{N}$. Since ψ is right upper semicontinuous at ϵ, we conclude that $\psi(\epsilon) < \phi(\epsilon) \le \limsup_n \psi\big(g(x_n)\big) \le \psi(\epsilon)$, which is a contradiction. □

Acknowledgment

The authors would like to express their deep appreciation to the anonymous referee for helpful comments on the manuscript.

The first author would like to acknowledge the financial support from Professor Kaoru Shimizu of Chiba University.

References

[1] H. Ben-El-Mechaiekh, *The Ran-Reurings fixed point theorem without partial order: a simple proof*, J. Fixed Point Theory Appl. **16** (2014), 373–383.

[2] L. Gavruta, P. Gavruta, and F. Khojasteh, *Two classes of meir-keeler contractions*, arXiv preprint arXiv:1405.5034 (2014).

[3] T.-C. Lim, *On characterizations of Meir-Keeler contractive maps*, Nonlinear Anal. **46** (2001), 113–120.

[4] A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. **28** (1969), 326–329.

[5] J. J. Nieto and R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005), 223–239 (2006).

[6] A. C. M. Ran and M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2004), 1435–1443.

[7] S. Reich and A. J. Zaslavski, *Monotone contractive mappings*, J. Nonlinear Var. Anal **1** (2017), 391–401.

[8] T. Suzuki, *Fixed-point theorem for asymptotic contractions of Meir-Keeler type in complete metric spaces*, Nonlinear Anal. **64** (2006), 971–978.

[9] C. S. Wong, *Characterizations of certain maps of contractive type*, Pacific J. Math. **68** (1977), 293–296.

(K. Aoyama) Chiba University
Aoyama Mathematical Laboratory
Yayoi-cho, Inage-ku, Chiba, Chiba 263-8522, Japan
E-mail address, Corresponding author: aoyama@bm.skr.jp

(M. Toyoda) Toho University
Department of Information Science
Miyama, Funabashi, Chiba 274-8510, Japan
E-mail address: mss-toyoda@is.sci.toho-u.ac.jp