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BOUNDARY VALUE PROBLEMS FOR HADAMARD-CAPUTO IMPLICIT FRACTIONAL DIFFERENTIAL INCLUSIONS IN A BANACH SPACE

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Abstract. The authors prove the existence of solutions to a boundary value problem for an implicit fractional differential inclusion of Hadamard-Caputo type in a Banach space. The technique of proof makes use of a set-valued analog of Mönch's fixed point theorem combined with a measure of noncompactness. They present an example to illustrate the main results.

1. Introduction

In this paper we study the existence of solutions to boundary value problems (BVP) for the implicit fractional differential inclusion

$$ {}_{H}^{C}D^{\alpha}y(t) \in F(t, y(t), {}_{H}^{C}D^{\alpha}y(t)) \quad \text{for a.e.} \quad t \in J = [1, T], \ 1 < \alpha \leq 2, \tag{1} $$

$$ y(1) = y_1, \ y(T) = y_T, \tag{2} $$

where ${}_{H}^{C}D^{\alpha}$ is the Hadamard-Caputo fractional derivative, E is a Banach space, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $F : [1, T] \times E \times E \to \mathcal{P}(E)$ is a multivalued map, and y_1, $y_T \in \mathbb{R}$.

Differential equations and inclusions of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, and, as a consequence, have found themselves as the object of many studies. For theoretical details, including some applications and recent results, we refer the reader to the monographs of Kilbas *et al.* [25] and Podlubny [28] as well as the papers of Agarwal *et al.* [6, 7], Momani *et al.* [26], Guerraiche *et al.* [21], and the references therein.

The Caputo left-hand fractional derivative of order $r > 0$ of the function h is defined by

$$ ({}^{c}D^{r}_{a+}h)(t) = \frac{1}{\Gamma(n-r)} \int_{a}^{t} (t-s)^{n-r-1} h^{(n)}(s) ds, $$

where $n = [r] + 1$ and $[r]$ denotes the integer part of r. This derivative is very useful in many applied problems because it satisfies its initial data which contains $y(0)$, $y'(0)$, etc., as well as similar data for the boundary conditions.

The Hadamard fractional derivative was introduced in 1892 [22]; this derivative has the properties that its kernel contains a logarithmic function of arbitrary exponential order and the Hadamard derivative of a constant does not equal 0. The Hadamard-Caputo fractional derivative given by Jarad *et al.* [24] is a modification of the Hadamard fractional derivative, but it keeps the characteristic property of the Caputo fractional derivative in that the derivative of a constant is 0.

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Implicit differential equations involving the classical fractional derivatives have been analyzed by many authors in the last few years, and for recent results we mention the books [2, 3, 4] and the papers [5, 14, 15, 16, 29]. Studies on these types of problems that have involved measures of noncompactness include, for example, the papers [1, 8, 11, 13, 23].

As for contributions to the study of implicit fractional differential equations and inclusions in Banach spaces, we refer the reader to the papers of Agarwal *et al.* [1], Benchohra *et al.* [12, 13], and Graef *et al.* [20].

In this paper, we present existence results for the problem (1)–(2) in the case where the right hand side is convex valued by using a set–valued analog of Mönch's fixed point theorem combined with the measure of noncompactness.

2. Preliminaries

In this section, we introduce those concepts needed in the remainder of this paper. Let $(E, |\cdot|)$ be a Banach space and let $C(J,E)$ denote the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

We take $L^1(J,E)$ to be the space of Lebesgue integrable functions $y : J \longrightarrow E$ with the norm

$$\|y\|_{L^1} = \int_J |y(t)|dt,$$

and $AC(J,E)$ to be the space of functions $y : J \to E$ that are absolutely continuous.

With $\delta = t\frac{d}{dt}$, we set

$$AC^n_\delta(J,E) = \{y : J \longrightarrow E, \delta^{n-1}y(t) \in AC(J,E)\},$$

with $AC^1(J,E)$ being the space of functions $y : J \to E$ that are absolutely continuous and have an absolutely continuous first derivative.

For any Banach space $(X, \|\cdot\|)$, let

$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$,
$P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$,
$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$,
$P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multivalued map $G : X \to \mathcal{P}(X)$ is *convex* (*closed*) valued if $G(X)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B) = \cup_{x\in B}G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

A mapping G is *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. The map G is said to be *completely continuous* if $G(B)$ is relatively compact for every $B \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (that is, $x_n \to x_*$, $y_n \to y_*$, and $y_n \in G(x_n)$ imply $y_* \in G(x_*)$).

We say that a mapping G has a *fixed point* if there is an $x \in X$ such that $x \in G(x)$. The set of all fixed points of a multivalued operator G will be denoted by $Fix\, G$.

A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be *measurable* if for every $y \in \mathbb{R}$, the function

$$t \to d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 1. *A multi-valued map $F : [1, T] \times E \times E \to \mathcal{P}(E)$ is said to Carathéodory if*

(1) *$t \to F(t, u, v)$ is measurable for each u, $v \in E$, and*
(2) *$u \to F(t, u, v)$ is upper semicontinuous for almost all $t \in J$.*

Furthermore, a Carathéodory map is called L^1-Carathéodory if

(3) *for each $\rho > 0$, there exists $\phi_\rho \in L^1([1, T], \mathbb{R}^+)$ such that*

$$\|F(t, u, v)\| = \sup\{|v| : v \in F(t, u, v)\} < \phi_\rho(t) \quad \textit{for all} \quad |v|,\ |u| < \rho.$$

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. The function $H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, B)\}$$

is known as the Hausdorff-Pompeiu metric.

For each $y \in AC(J, E)$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1([1, T], \mathbb{R}) : v(t) \in F(t, y(t), {}^C_H D^\alpha y(t)) \text{ a.e. } t \in [1, T]\}.$$

For additional details on multivalued maps, see the monographs of Aubin and Cellina [9], Aubin and Frankowska [10], Deimling [19], or Castaing and Valadier [18].

For convenience, we first recall the definition of the Kuratowski measure of noncompactness and summarize its main properties.

Definition 2. ([8, 11]) *Let E be a Banach space and let Ω_E be the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\rho : \Omega_E \to [0, \infty)$ defined for $B \in \Omega_E$ by*

$$\rho(B) = \inf\{\epsilon > 0 \ : B \subset \bigcup_{j=1}^{m} B_j \ \textit{and} \ diam(B_j) \leq \epsilon\}.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for more details see [8, 11])

(1) $\rho(B) = 0$ if and only if $\overline{B}$ is compact (B is relatively compact).
(2) $\rho(B) = \rho(\overline{B})$.
(3) $A \subset B$ implies $\rho(A) \leq \rho(B)$.
(4) $\rho(A + B) \leq \rho(A) + \rho(B)$.
(5) $\rho(cB) = |c|\rho(B)$, $c \in \mathbb{R}$.
(6) $\rho(convB) = \rho(B)$.

Here $\overline{B}$ and *conv* B denote the closure and the convex hull of the bounded set B, respectively.

The next theorem will be used in our proofs.

Theorem 1. ([23]) *Let E be a Banach space and C be a countable subset of $L^1(J, E)$ for which there exists $h \in L^1(J, \mathbb{R}_+)$ with $|u(t)| \leq h(t)$ for a.e. $t \in J$ and every $u \in C$. Then the function $\varphi(t) = \rho(C(t))$ belongs to $L^1(J, \mathbb{R}_+)$ and satisfies*

$$\rho\left(\left\{\int_1^T u(s)ds : u \in C\right\}\right) \leq 2\int_1^T \rho(C(s))ds. \tag{3}$$

Next, we define the fractional integrals and derivatives to be used in this paper.

Definition 3. ([25]) *The Hadamard fractional integral of order $r > 0$ for a function $h : [1, +\infty) \to \mathbb{R}$ is defined as*

$${}^H I^r h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds,$$

provided the integral exists.

In what follows we let $[r]$ denote the integer part of r and $\log(\cdot) = \log_e(\cdot)$.

Definition 4. ([25]) *For a function $h : [1, +\infty) \to \mathbb{R}$, the r Hadamard fractional derivative of order r of h is defined by*

$$({}^H D^r h)(t) = \frac{1}{\Gamma(n-r)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-r-1} \frac{h(s)}{s} ds.$$

Definition 5. ([24]) *For a given function $h \in AC^n_\delta([a,b], E)$ with $a > 0$, we define the Caputo–type modification of the left–hand Hadamard fractional derivative by*

$${}^H_C D^r y(t) = {}^H D^r \left[y(s) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log\frac{s}{a}\right)^k \right](t),$$

where $Re(r) \geq 0$ and $n = [Re(r)] + 1$.

Lemma 1. ([24]]) *Let y belong to either $AC^n_\delta([a,b], E)$ or C^n_δ with $r \in \mathbb{C}$. Then*

$${}^H I^r ({}^H_C D^r) y(t) = y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log\frac{t}{a}\right)^k.$$

We now recall Mönch's fixed point theorem.

Theorem 2. ([27]]) *Let K be a closed, convex subset of a Banach space E, U be a relatively open subset of K, and $N : \overline{U} \mapsto \mathcal{P}(K)$. Assume that N has a closed graph and maps compact sets into relatively compact sets. For some $x_0 \in U$ let the following two conditions be satisfied:*

(i) *$M \subset \overline{U}$, $M \subset \operatorname{conv}(x_0 \cup N(M))$, and $\overline{M} = \overline{C}$ with C a countable subset of M, implies $\overline{M}$ is compact;*
(ii) *$x \notin (1-\lambda)x_0 + \lambda N(x)$ for all $x \in \overline{U} \setminus U$ and $\lambda \in (0,1)$.*

Then there exists $x \in \overline{U}$ with $x \in N(x)$.

3. Main Results

We start by defining what we mean by a solution of the problem (1)–(2).

Definition 6. *A function $y \in AC([1,T], E)$ is said to be a solution of (1)–(2) if there exists a function $x \in L^1([1,T], E)$ with $x(t) \in F(t, y(t), {}^C_H D^\alpha y(t))$ for a.e. $t \in [1,T]$ such that ${}^C_H D^\alpha y(t) = x(t)$ and the function y satisfies conditions (2).*

The following lemma allows us to write a solution of the problem (1)–(2) in integral form.

Lemma 2. *For $1 < \alpha \leq 2$, the solution of the BVP (1)–(2) can be written as*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} + y_1 + \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}}{\log(T)} \log(t), \quad (4)$$

where $x \in L^1([1,T],\mathbb{R})$ *is the solution of the functional inclusion*

$$x(t) \in F\Big(t, \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} + y_1 + \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}}{\log(T)} \log(t), x(t)\Big). \quad (5)$$

Proof. Let ${}^C_H D^\alpha y(t) = x(t)$ in equation (1) so that we have

$$x(t) \in F(t, y(t), x(t)) \quad (6)$$

and

$$y(t) = c_1 + c_2 \log(t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s} \quad (7)$$

Letting $t = 1$ in (7), we obtain

$$y(1) = c_1$$

and so

$$y(t) = y_1 + c_2 \log(t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}. \quad (8)$$

For $t = T$, from (2) we see that

$$y(T) = y_1 + c_2 \log(T) + \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} x(s)\frac{ds}{s},$$

and hence

$$c_2 = \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}}{\log(T)}. \quad (9)$$

Now substituting (8) into (7), we obtain (4).

To complete the proof, we need to show that (4) satisfies problem (1)–(2). Differentiating (4), we have

$${}^C_H D^\alpha y(t) = x(t) \in F(t, y(t), {}^C_H D^\alpha y(t)).$$

Letting $t = 1$ and $t = T$ in (4), we see that the boundary conditions (2) are satisfied. This complete the proof of the equivalence between the problem (1)–(2) and (4). □ □

Our main existence result is contained in the following theorem.

Theorem 3. *Assume that:*

(H1) $F : J \times E \times E \to \mathcal{P}_{cp,p}(E)$ *is a Carathéodory multi-valued mapping.*

(H2) *There exist a function* $a \in C(J, E)$ *such that*

$$\|F(t,x,y)\|_{\mathcal{P}} = \sup\{|f| : f \in F(t,x,y)\} \le a(t)$$

for $(t,x,y) \in J \times E \times E$.

(H3) *There exist* l_1, $l_2 > 0$ *such that*

$$H_d(F(t,x,y), F(t,\bar{x},\bar{y})) < l_1|x-\bar{x}| + l_2|y-\bar{y}| \text{ for every } x,\ \bar{x},\ y,\ \bar{y} \in E,$$

(H4) *For each bounded set $B \subset C(J)$ and for each $t \in J$, we have*

$$\rho(F(t, B, {}^{C}_{H}D^{\alpha}B)) \leq a(t)\rho(B),$$

where

$${}^{C}_{H}D^{\alpha}B = \{{}^{C}_{H}D^{\alpha}w : w \in B\},$$

and ρ is a measure of noncompactness on E.

(H5) *The function $\phi = 0$ is the unique solution in $C(J)$ of*

$$\phi(t) \leq 2\left\{\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \psi(s,\phi(s),\phi(s))\frac{ds}{s} + y_1 + \left[y_T - y_1 - \frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} \psi(s,\phi(s),\phi(s))\frac{ds}{s}\right]\frac{\log(t)}{\log(T)}\right\}. \quad (10)$$

Then the BVP (1)–(2) has at least one solution in J.

Proof. In order to transform the problem (1)–(2) into a fixed point problem, define the multivalued operator

$$(Nx)(t) = \left\{h \in C(J,E) : h(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} + y_1 + \left[y_T - y_1 - \frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} v(s)\frac{ds}{s}\right]\frac{\log(t)}{\log(T)}, \quad v \in S_{F,x}\right\}. \quad (11)$$

Clearly, from Lemma 2, the fixed points of N are solutions to (1)–(2). We wish to show that N satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in several steps.

Step 1: *$N(x)$ is convex for each $x \in C(J,E)$.* If h_1, h_2 belong to $N(x)$, then there exist v_1, $v_2 \in S_{F,y}$ such that, for $t \in J$ and $i = 1, 2$, we have

$$h_i(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v_i(s)\frac{ds}{s} + y_1 + \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} v_i(s)\frac{ds}{s}}{\log(T)}\log(t).$$

Let $0 \leq d \leq 1$; then, for $t \in J$,

$$(dh_1 + (1-d)h_2)(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} (dv_1 + (1-d)v_2)(s)\frac{ds}{s} + y_1 + \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} (dv_1 + (1-d)v_2)(s)\frac{ds}{s}}{\log(T)}\log(t).$$

Now $S_{F,y}$ is convex since F has convex values, so

$$dh_1 + (1-d)h_2 \in N(x),$$

i.e., N is convex.

Step 2: *$N(M)$ is relatively compact for each compact $M \subset \overline{U}$.* Let $M \subset \overline{U}$ be a compact set and let $\{h_n\}$ be any sequence of elements in $N(M)$. We will show that $\{h_n\}$

has a convergent subsequence by using the Arzelà-Ascoli compactness criteria in $C(J,B)$. Since $h_n \in N(M)$, there exist $y_n \in M$ and $v_n \in S_{F,y_n}$ such that

$$h_n(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} + y_1 + \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s}}{\log(T)}\log(t),$$

for $n \geq 1$. Using Theorem 1 and the properties of the Kuratowski measure of noncompactness, we have

$$\begin{aligned} \rho(\{h_n(t)\}) &\leq \frac{2}{\Gamma(\alpha)}\int_1^t \rho\left(\left\{\left(\log\frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} : n \geq 1\right\}\right) \\ &+ \frac{2}{\Gamma(\alpha)}\int_1^T \rho\left(\left\{\left(\log\frac{T}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} : n \geq 1\right\}\right). \end{aligned} \tag{12}$$

On the other hand, since M is compact in E, the set $\{v_n(s) : n \geq 1\}$ is compact. Consequently, $\rho(v_n(s) : n \geq 1) = 0$ for a.e. $s \in J$. Furthermore,

$$\rho\left(\left\{\left(\log\frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s}\right\}\right) = \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}\rho(\{v_n(s) : n \geq 1\}) = 0$$

and

$$\rho\left(\left\{\left(\log\frac{T}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s}\right\}\right) = \left(\log\frac{T}{s}\right)^{\alpha-1}\frac{1}{s}\rho(\{v_n(s) : n \geq 1\}) = 0$$

for a.e. $t,\ s \in J$. Now (12) implies that $\{h_n(t) : n \geq 1\}$ is relatively compact in B for each $t \in J$. In addition, for $t_1,\ t_2 \in J$ with $t_1 < t_2$, we have

$$\begin{aligned} |h_n(t_2) - h_n(t_1)| &= \Bigg|(\log t_2 - \log t_1)\left[y_T - y_1 - \frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s}\right]\frac{1}{\log(T)} \\ &+ \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right] v_n(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s}\Bigg| \\ &\leq (\log t_2 - \log t_1)\left|y_T - y_1 - \frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s}\right| \\ &+ \frac{a(t)}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_2}{s}\right)^{\alpha-1}\right]\frac{ds}{s} \\ &+ \frac{a(t)}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s}. \end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. This shows that $\{h_n : n \geq 1\}$ is equicontinuous. Consequently, $\{h_n : n \geq 1\}$ is relatively compact in $C(J,B)$.

Step 3: *N has a closed graph.* Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(x_*)$. Now $h_n \in N(x_n)$ implies there exists $v_n \in S_{F,x_n}$ such that, for each $t \in J$,

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s) \frac{ds}{s} + y_1 + \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} v_n(s) \frac{ds}{s}}{\log(T)} \log(t),$$

We need to show that there exists $v_* \in S_{F,x_*}$ such that, for each $t \in J$,

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_*(s) \frac{ds}{s} + y_1 + \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} v_*(s) \frac{ds}{s}}{\log(T)} \log(t).$$

Since $F(t,\cdot,\cdot)$ is upper semicontinuous, for every $\epsilon > 0$ there exist $n_0(x)$ such that, for every $n \geq n_0$, we have $v_n \in F(t, y(t), x(t)) \subset F(t, y_*(t), x_*(t)) + \epsilon B(0,1)$ a.e. $t \in J$. Since F has compact values, there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot) \to v_* \text{ as } m \to \infty,$$

and

$$v_* \in F(t, y_*(t), x_*(t)) \text{ for } t \in J.$$

For every $w(t) \in F(t, y_*(t), x_*(t))$, we have

$$|v_{n_m} - v_*| \leq |v_{n_m} - w(t)| + |w(t) - v_*|,$$

and so

$$|v_{n_m} - v_*| \leq d(v_{n_m}(t), F(t, y_*(t), x_*(t)))$$

By an analogous relation obtained by interchanging the roles of v_{n_m} and v_*, it follows that

$$\begin{aligned}
|v_{n_m} - v_*| &\leq H_d(F(t, y_{n_m}(t), x_{n_m}(t)), F(t, y_*(t), x_*(t))) \\
&\leq l_1 |y_{n_m} - y_*| + l_2 |x_{n_m} - x_*| \\
&\leq l_1 |{}^H I^\alpha (x_{n_m} - x_*)|_{t=T} + {}^H I^\alpha (x_{n_m} - x_*)| + l_2 |x_{n_m} - x_*| \\
&\leq 2 l_1 |{}^H I^\alpha (x_{n_m} - x_*)| + l_2 |x_{n_m} - x_*|
\end{aligned}$$

Therefore,

$$|h_{n_m}(t) - h_*(t)| \leq \frac{2}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v_{n_m} - v_*| \frac{ds}{s},$$

so

$$\begin{aligned}
\|h_n(t) - h_*(t)\|_\infty \leq{}& \frac{4 l_1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \|x_{n_m} - x_*\|_\infty \\
&+ \frac{4 l_2}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \|x_{n_m} - x_*\|_\infty.
\end{aligned}$$

Thus,

$$\|h_{n_m}(t) - h_*(t)\|_\infty \to 0 \text{ as } m \to \infty$$

as we needed to show.

Step 4: *M is relatively compact in $C(J,B)$.* Suppose $M \subset \overline{U}$, $M \subset conv(\{0\} \cup N(M))$, and $M = C$ for some countable set $C \subset M$, and $\overline{M} = \overline{C}$. Using an estimation of type (12), we see that $N(M)$ is equicontinuous. Then, since $M \subset conv(\{0\} \cup N(M))$, we see that M is equicontinuous as well. To apply the Arzelà-Ascoli theorem, it remains to show that $M(t)$ is relatively compact in E for each $t \in J$. Since $C \subset M \subset conv(\{0\} \cup N(M))$ and C is countable, we can find a countable set $H = \{h_n : n \geq 1\} \subset N(M)$ with $C \subset conv(\{0\} \cup H)$. Then, there exist $x_n \in M$ and $v_n \in S_{F,x_n}$ such that

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} + y_1 + \frac{(\log t)}{(\log T)} \left[y_T - y_1 - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} \right].$$

From the fact that $M \subset C \subset conv(\{0\} \cup H))$, by Theorem 1, we have

$$\rho(M(t)) \leq (\rho(C(t)) \leq \rho(H(t)) = \rho(\{h_n(t) : n \geq 1\}).$$

From (3), we obtain

$$\rho(M(t)) \leq 2\left\{ \frac{1}{\Gamma(r)} \int_1^t \rho\left(\left\{ \left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} \right\} + y_1 \right) + \frac{(\log t)}{(\log T)} \left[y_T - y_1 - \frac{1}{\Gamma(\alpha)} \int_1^T \rho\left(\left\{ \left(\log \frac{T}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} \right\} \right) \right] \right\}.$$

Now, since $v_n(s) \in M(s)$, we have

$$\rho(M(t)) \leq 2\left\{ \frac{1}{\Gamma(\alpha)} \int_1^t \rho\left(\left\{ \left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} : n \geq 1 \right\} + y_1 \right) + \frac{(\log t)}{(\log T)} \left[y_T - y_1 - \frac{1}{\Gamma(\alpha)} \int_1^T \rho\left(\left\{ \left(\log \frac{T}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} : n \geq 1 \right\} \right) \right] \right\}.$$

Also,

$$\rho\left(\left\{ \left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} : n \geq 1 \right\} \right) = \left(\log \frac{t}{s}\right)^{\alpha-1} \rho(M(s)) = 0$$

and

$$\rho\left(\left\{ \left(\log \frac{T}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} : n \geq 1 \right\} \right) = \left(\log \frac{T}{s}\right)^{\alpha-1} \rho(M(s))) = 0.$$

It follows that

$$\begin{aligned}
\rho(M(t)) &\leq 2\left\{ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \rho(M(s))\frac{ds}{s} + y_1 \right. \\
&\quad \left. + \frac{(\log t)}{(\log T)} \left[y_T - y_1 - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \rho(M(s))\frac{ds}{s} \} \right] \right\} \\
&\leq 2\left\{ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \psi(s, \rho(M(s)), \rho(M(s)))\frac{ds}{s} + y_1 \right. \\
&\quad \left. + \frac{(\log t)}{(\log T)} \left[y_T - y_1 - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \psi(s, \rho(M(s)), \rho(M(s)))\frac{ds}{s}) \right] \right\}.
\end{aligned}$$

Note that the function φ given by $\varphi(t) = \rho(M(t))$ belongs to $C(J, E)$. Consequently, by $(H5)$, $\varphi \equiv 0$; that is, $\rho(M(t)) = 0$ for all $t \in J$. Now, by the Arzelà-Ascoli theorem, M is relatively compact in $C(J, E)$.

Step 5: *An a priori estimate.* Let $h \in C(J, E)$ such that $x \in \lambda N(x)$ for some $0 < \lambda < 1$. Then,

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} + y_1 + \frac{y_T - y_1 - \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} v(s)\frac{ds}{s}}{\log(T)} \log(t), \quad v \in S_{F,x}$$

For each $t \in J$, we have

$$\begin{aligned}
\|h\| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v_n(s)|\frac{ds}{s} + |y_1| \\
&\quad + \frac{(\log t)}{(\log T)} \left[|y_T| + |y_1| + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |v_n(s)|\frac{ds}{s} \right] \\
&\leq \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v_n(s)|\frac{ds}{s} + |y_1| \\
&\quad + |y_T| + |y_1| + \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} |v_n(s)|\frac{ds}{s} \\
&\leq 2\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \int_1^T a(s)ds + 2|y_1| + |y_T| \\
&\leq 2\frac{a^*(\log T)^\alpha}{\Gamma(\alpha+1)} + 2|y_1| + |y_T|,
\end{aligned}$$

where

$$a^* = \int_1^T |a(s)|ds.$$

Then,

$$\|x\| \leq 2\frac{a^*(\log T)^\alpha}{\Gamma(\alpha+1)} + 2|y_1| + |y_T| := R$$

Hence, condition (ii) in Theorem 2 is satisfied. As a consequence of Steps 1–5 and Theorem 2, we conclude that N has a fixed point $x \in C(J, E)$ that in turn is a solution of problem (1)–(2). This completes the proof of the theorem. □

4. An Example

We conclude this paper with an example to illustrate our main result. Let

$$E = l^1 = \left\{ (y_1, y_2, \dots, y_n, \dots) : \sum_1^\infty |y_n| < \infty \right\}.$$

be our Banach space with the norm

$$\|y\|_E = \sum_1^\infty |y_n|.$$

We will apply Theorem 3 to the implicit fractional differential boundary value inclusion

$${}^C_H D^\alpha y_n(t) \in F_n(t, y(t), {}^C_H D^\alpha y(t)), \text{ for a.e. } t \in J = [1, e],\ 1 < \alpha \leq 2, \tag{13}$$

with

$$y(1) = 0,\ y(T) = 1, \tag{14}$$

where

$$F_n(t, y(t), {}^C_H D^\alpha y(t)) = \{v \in E : f_n(t, y(t), {}^C_H D^\alpha y(t)) \le v \le g_n(t, y(t), {}^C_H D^\alpha y(t))\}$$

and $f_n,\ g_n : J \times E \times E \mapsto E$. We assume that for each $t \in [1, e]$, $f_n(t, \cdot, \cdot)$ is lower semi-continuous (i.e., the set $\{y \in E : f_n(t, y(t), {}^c D^\alpha y(t)) > \mu_1\}$ is open for each $\mu_1 \in \mathbb{R}$), and for each $t \in [1, e]$, $g_n(t, \cdot, \cdot)$ is upper semi-continuous (i.e., the set $\{y \in E : g_n(t, y(t), {}^c D^\alpha y(t)) < \mu_2\}$ is open for each $\mu_2 \in \mathbb{R}$), and $y = (y_1, y_2, \dots, y_n, \dots)$. Set $F = (F_1, F_2, \dots, F_n, \dots)$, $f = (f_1, f_2, \dots, f_n, \dots)$, $g = (g_1, g_2, \dots, g_n, \dots)$, and assume that there exists $a \in C([1, e], \mathbb{R}^+)$ such that

$$\begin{aligned} \|F(t, u_1, u_2)\|_{\mathcal{P}} &= \sup\{|v| : v(t) \in F(t, y(t), {}^C_H D^\alpha y(t))\} \\ &= \max(|f_n(t, y(t), {}^c D^\alpha y(t))|, |g_n(t, y(t), {}^c D^\alpha y(t))|) \\ &\le a(t) \ \text{ for each } t \in [1, e] \text{ and } y \in E. \end{aligned}$$

It is clear that F is compact, convex, and upper semi-continuous. In addition, we assume that for $(t, y, x) \in \times E \times E$. We also assume that for each bounded set $B \subset C(J)$ and for each $t \in J$, we have

$$\rho(F(t, B, {}^C_H D^\alpha B)) \le a(t)\rho(B),$$

where

$${}^C_H D^\alpha B = \{{}^C_H D^\alpha w : w \in B\}$$

and ρ is a measure of noncompactness on E. In addition, we assume that the function $\phi = 0$ is the unique solution in $C(J)$ of

$$\phi(t) \le 2\left\{\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \psi(s, \phi(s), \phi(s)) \frac{ds}{s}\right. \tag{15}$$

$$\left. + \left[1 - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1} \psi(s, \phi(s), \phi(s)) \frac{ds}{s}\right] \log(t)\right\}. \tag{16}$$

Since all the conditions of the Theorem 3 are satisfied, problem (13)–(14) has at least one solution y on J.

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