TJMM
13 (2021), No. 1-2, 89-96

ON THE STABILITY AND CONVERGENCE OF MANN ITERATION PROCESS IN CONVEX A-METRIC SPACES

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(Dedicated to Assoc. Prof. Dr. Birol Gunduz who passed away on the 3rd of April, 2019.)

Abstract. In this paper, firstly, we introduce the concept of convexity in A-metric spaces and show that Mann iteration process converges to the unique fixed point of Zamfirescu type contractions in this newly defined convex A-metric space. We give also an example concerning with this convergence. Secondly, we define the concept of stability in convex A-metric spaces and establish stability result for the Mann iteration process considered in such spaces. Our results carry some well-known results from the literature to convex A-metric spaces.

1. Introduction and Preliminaries

The Banach Fixed Point Theorem which is the one of the most important theorem in all analysis. It plays a key role for many applications in nonlinear analysis. For example, in the areas such as optimization, mathematical models, and economic theories. Due to this, the result has been generalized in various directions. As a generalization of metric space, Mustafa and Sims introduced a new class of generalized metric spaces called G-metric spaces (see [9], [10]) as a generalization of metric spaces (X, d). This was done to introduce and develop a new fixed point theory for a variety of mappings in this new setting. This helped to extend some known metric space results to this more general setting. The G-metric space is defined as follows:

Definition 1. [10] *Let X be a nonempty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:*
(i) $G(x, y, z) = 0$ if $x = y = z$
(ii) $0 < G(x, x, y)$ for all $x, y \in X$, with $x \neq y$
(iii) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$
(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$, (symmetry in all three variables); and
(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).
Then the function G is called a generalized metric or more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Mustafa et al. studied many fixed point results for a self-mapping in G-metric space. [11]-[13] can be cited for reference.

On the other hand, Abbas et al. [1] introduced the concept of an A-metric space as follows:

Definition 2. *Let X be nonempty set. Suppose a mapping $A : X^t \to \mathbb{R}$ satisfy the following conditions:*
(A_1) $A(x_1, x_2, ..., x_{t-1}, x_t) \geq 0$,

2010 *Mathematics Subject Classification.* 47H09, 47H10.
Key words and phrases. Convex structure, Convex A-metric space, Mann iteration process, Stability.

(A_2) $A(x_1, x_2, ..., x_{t-1}, x_t) = 0$ *if and only if* $x_1 = x_2 = ... = x_{t-1} = x_t$,
(A_3) $A(x_1, x_2, ..., x_{t-1}, x_t) \leq A(x_1, x_1, ..., (x_1)_{t-1}, y) + A(x_2, x_2, ..., (x_2)_{t-1}, y) +$
$... + A(x_{t-1}, x_{t-1}, ..., (x_{t-1})_{t-1}, y) + A(x_t, x_t, ..., (x_t)_{t-1}, y)$
for any $x_i, y \in X$, $(i = 1, 2, ..., t)$. *Then,* (X, A) *is said to be an* A*-metric space.*

It is clear that the an A-metric space for $t = 2$ reduces to ordinary metric d. Also, an A-metric space is a generalization of the G-metric space.

Example 1. [1] *Let* $X = \mathbb{R}$. *Define a function* $A : X^t \to \mathbb{R}$ *by*

$$\begin{aligned} A(x_1, x_2, ..., x_{t-1}, x_t) &= |x_1 - x_2| + |x_1 - x_3| + ... + |x_1 - x_t| \\ &\quad + |x_2 - x_3| + |x_2 - x_4| + ... + |x_2 - x_t| \\ &\quad \vdots \\ &\quad + |x_{t-2} - x_{t-1}| + |x_{t-2} - x_t| + |x_{t-1} - x_t| \\ &= \sum_{i=1}^{t} \sum_{i<j} |x_i - x_j| . \end{aligned}$$

Then (X, A) *is an* A*-metric space..*

Lemma 1. [1] *Let* (X, A) *be* A*-metric space. Then* $A(x, x, \ldots, x, y) = A(y, y, \ldots, y, x)$ *for all* $x, y \in X$.

Lemma 2. [1] *Let* (X, A) *be* A*-metric space. Then for all for all* $x, y \in X$ *we have* $A(x, x, \ldots, x, z) \leq (t-1) A(x, x, \ldots, x, y) + A(z, z, \ldots, z, y)$ *and* $A(x, x, \ldots, x, z) \leq (t-1) A(x, x, \ldots, x, y) + A(y, y, \ldots, y, z)$.

Definition 3. [1] *Let* (X, A) *be* A*-metric space.*

(i) A sequence $\{x_n\}$ *in* X *is said to converge to a point* $u \in X$ *if* $A(x_n, x_n, \ldots, x_n, u) \to 0$ *as* $n \to \infty$.

(ii) A sequence $\{x_n\}$ *in* X *is called a Cauchy sequence if* $A(x_n, x_n, \ldots, x_n, u_m) \to 0$ *as* $n, m \to \infty$.

(iii) The A*-metric space* (X, A) *is said to be complete if every Cauchy sequence in* X *is convergent.*

Recently, Yildirim [19] introduced the notion of Zamfirescu mappings in A-metric space as follows:

Definition 4. *Let* (X, A) *be* A*-metric space and* $f : X \to X$ *be a mapping.* f *is called a* A*-Zamfirescu mapping (AZ mapping), if and only if, there are real numbers,* $0 \leq a < 1$, $0 \leq b, c < \frac{1}{t}$ *such that for all* $x, y \in X$, *at least one of the next conditions is true:*

$$(AZ_1)\, A(fx, fx, \ldots, fx, fy) \leq aA(x, x, \ldots, x, y)$$

$$(AZ_2)\, A(fx, fx, \ldots, fx, fy) \leq b\,[A(fx, fx, \ldots, fx, x) + A(fy, fy, \ldots, fy, y)]$$

$$(AZ_3)\, A(fx, fx, \ldots, fx, fy) \leq c\,[A(fx, fx, \ldots, fx, y) + A(fy, fy, \ldots, fy, x)]$$

Yildirim [19] also extended the Zamfirescu results [21] to A-metric spaces and he obtained the following results on fixed point theorems for such mappings.

Lemma 3. [19] *Let* (X, A) *be* A*-metric space and* $f : X \to X$ *be a mapping. If* f *is a* AZ *mapping, then there is* $0 \leq \delta < 1$ *such that*

$$A(fx, fx, \ldots, fx, fy) \leq \delta A(x, x, \ldots, x, y) + t\delta A(fx, fx, \ldots, fx, x) \quad (1)$$

and

$$A(fx, fx, \ldots, fx, fy) \leq \delta A(x, x, \ldots, x, y) + t\delta A(fy, fy, \ldots, fy, x) \quad (2)$$

for all $x, y \in X$.

Theorem 1. [19] *Let (X, A) be complete A-metric space and $f : X \to X$ be an AZ mapping. Then f has a unique fixed point and Picard iteration process $\{x_n\}$ defined by $x_{n+1} = fx_n$ converges to a fixed point of f.*

Studies in metric spaces are related to the existence of fixed point without approximating them. The reason behind is the unavailablity of convex structure in metric spaces. To solve this problem, Takahashi [15] introduced the notion of convex metric spaces and studied the approximation of fixed points for nonexpansive mappings in this setting. Inspired by this, Yildirim and Khan [20] defined convex structure in G-metric spaces and they transformed the Mann iterative process to a convex G-metric space as follows. And they also proved some fixed point theorems deal with convergence of Mann iteration process for some class of mappings.

Definition 5. [20] *Let (X, G) be a G-metric space. A mapping $W : X^2 \times I^2 \to X$ is termed as a convex structure on X if $G(W(x, y; \lambda, \beta), u, v) \leq \lambda G(x, u, v) + \beta G(y, u, v)$ for real numbers λ and β in $I = [0, 1]$ satisfying $\lambda + \beta = 1$ and x, y, u and $v \in X$.*

A G-metric space (X, G) with a convex structure W is called a convex G-metric space and denoted as (X, G, W).

A nonempty subset C of a convex G-metric space (X, G, W) is said to be convex if $W(x, y; a, b) \in C$ for all $x, y \in C$ and $a, b \in I$.

Definition 6. [20] *Let (X, G, W) be convex G-metric space with convex structure W and $f : X \to X$ be a mapping. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ for $n \in \mathbb{N}$. Then for any given $x_0 \in X$, the iterative process defined by the sequence $\{x_n\}$ as*

$$x_{n+1} = W\left(x_n, fx_n; 1 - \alpha_n, \alpha_n\right), \qquad n \in \mathbb{N}, \tag{3}$$

is called Mann iterative process in the convex metric space (X, G, W).

The iterative approximation of a fixed point for certain classes of mappings is one of the main tools in the fixed point theory. Many authors ([3, 4, 5, 6, 7, 8, 14, 16, 17, 18]) discussed the existence of fixed points and convergence of different iterative processes for various mappings in convex metric spaces.

Keeping the above in mind, in this paper, we first define the concept of convexity in A-metric spaces. Then, we use Mann iteration in this newly defined convex A-metric space to prove some convergence results for approximating fixed points of some classes of mappings. We also discuss stability result for the Mann iteration process. Results in this paper show that different iteration methods can be used to approximate fixed points of different class of mappings in A-metric spaces. Our results are just new in the setting.

Now, we define convex structure in A-metric spaces as follows.

Definition 7. *Let (X, A) be a A-metric space and $I = [0, 1]$. A mapping $W : X^t \times I^t \to X$ is termed as a convex structure on X if*

$$\begin{aligned} & A(u_1, u_2, ..., u_{t-1}, W(x_1, x_2, ..., x_{t-1}, x_t; a_1, a_2, ..., a_t)) \\ \leq\ & a_1 A(u_1, u_2, ..., u_{t-1}, x_1) + a_2 A(u_1, u_2, ..., u_{t-1}, x_2) \\ & +... + a_t A(u_1, u_2, ..., u_{t-1}, x_t) \\ =\ & \sum_{i=1}^{t} a_i A(u_1, u_2, ..., u_{t-1}, x_i) \end{aligned} \tag{4}$$

for real numbers $a_1, a_2, ..., a_t$ in $I = [0, 1]$ satisfying $\sum_{i=1}^{t} a_i = 1$ and u_i, $x_i \in X$ for all $i = 1, 2, ..., t$.

An A-metric space (X, A) with a convex structure W is called a convex A-metric space and denoted as (X, A, W).

A nonempty subset C of a convex A-metric space (X, A, W) is said to be convex if $W(x_1, x_2, ..., x_{t-1}, x_t; a_1, a_2, ..., a_t) \in C$ for all $x_i \in C$ and $a_i \in I$, $i = 1, 2, ..., t$.

Next, we transform the Mann iteration process to a convex A-metric space as follows.

Definition 8. *Let (X, A, W) be convex A-metric space with convex structure W and $f : X \to X$ be a mapping. Let $\{\alpha_i^n\}$ be sequences in $[0, 1]$ for all $i = 1, 2, ..., t$ and $n \in \mathbb{N}$. Then for any given $x_0 \in X$, the iteration process defined by the sequence $\{x_n\}$ as*

$$x_{n+1} = W\left(x_n, x_n, ..., x_n, fx_n; \alpha_1^n, \alpha_2^n, ..., \alpha_t^n\right), \tag{5}$$

is called Mann iteration process in the convex metric space (X, A, W).

If we take $t = 2$ in (5), this structures reduces to (3).

The following Lemma shall be used in the proof of the stability result.

Lemma 4. [2] *If δ is a real number such that $0 \leq \delta < 1$ and $\{\varepsilon_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}$ satisfying*

$$u_{n+1} \leq \delta u_n + \varepsilon_n \ , \ n = 0, 1, \ldots$$

we have

$$\lim_{n\to\infty} u_n = 0 \ .$$

2. Main Results

2.1 Convergence Result: In this section, we prove the Mann iteration process converges to fixed point of Zamfirescu mappings in complete convex metric space (X, A, W).

Theorem 2. *Let (X, A, W) be a complete convex A-metric space with a convex structure W and, $f : X \to X$ be an AZ mapping. Let $\{x_n\}$ be defined iteratively by (5) and $x_0 \in X$, with $\{\alpha_t^n\} \subset [0, 1], \sum_{i=1}^t a_i = 1$ satisfying $\sum_{n=0}^\infty \alpha_t^n = \infty$ for all $n \in \mathbb{N}$ and $i = 1, 2, ..., t$. Then $\{x_n\}$ converges to a unique fixed point of f.*

Proof. From Theorem 1, we know that an AZ mapping has a unique fixed point in X. Call it u and consider $x_i \in X$, $i = 1, 2, ..., t$.

At least one of (AZ_1), (AZ_2) and (AZ_3) is satisfied. If (AZ_1), (AZ_2) or (AZ_3) holds, we know that the following inequality from Lemma 3

$$A\left(fx, fx, \ldots, fx, fy\right) \leq \delta A\left(x, x, \ldots, x, y\right) + t\delta A(fx, fx, \ldots, fx, x) \tag{6}$$

for all $x, y \in X$.

Let $\{x_n\}$ be the Mann iteration process (5), with $x_0 \in X$ arbitrary. Then

$$\begin{aligned} A(u, u, ..., u, x_{n+1}) &= A(u, u, ..., u, W\left(x_n, x_n, ..., x_n, fx_n; \alpha_1^n, \alpha_2^n, ..., \alpha_t^n\right)) \\ &\leq \alpha_1^n A(u, u, ..., u, x_n) + \alpha_2^n A(u, u, ..., u, x_n) \\ &\quad +... + \alpha_t^n A(u, u, ..., u, fx_n) \\ &= \left(1 - \alpha_t^n\right) A(u, u, ..., u, x_n) + \alpha_t^n A(u, u, ..., u, fx_n). \end{aligned}$$

Take $x = u$ and $y = x_n$ in (6) to obtain

$$\begin{aligned} A(u, u, ..., u, fx_n) &= A(fu, fu, ..., fu, fx_n) \\ &\leq \delta A(u, u, ..., u, x_n) + t\delta A(fu, fu, \ldots, fu, u) \\ &= \delta A(u, u, ..., u, x_n) \end{aligned} \tag{7}$$

which together with (7) yields

$$\begin{aligned} A(u, u, ..., u, x_{n+1}) &\leq \left(1 - \alpha_t^n\right) A(u, u, ..., u, x_n) + \alpha_t^n \delta A(u, u, ..., u, x_n) \\ &= \left[1 - (1 - \delta)\,\alpha_t^n\right] A(u, u, ..., u, x_n). \end{aligned} \tag{8}$$

Inductively we get

$$\begin{aligned} A(u,u,...,u,x_{n+1}) &\leq [1-(1-\delta)\,\alpha_t^n]\,A(u,u,...,u,x_n) \quad (9)\\ &\leq [1-(1-\delta)\,\alpha_t^n]\left[1-(1-\delta)\,\alpha_t^{n-1}\right]A(u,u,...,u,x_{n-1})\\ &\vdots\\ &\leq \prod_{k=0}^{n}\left[1-(1-\delta)\,\alpha_t^k\right]A(u,u,...,u,x_0) \end{aligned}$$

As $0\leq\delta<1$, $\left\{\alpha_t^k\right\}\subset[0,1]$ and $\sum_{k=0}^{\infty}\alpha_t^k=\infty$, we have

$$\lim_{n\to\infty}\prod_{k=0}^{n}\left[1-(1-\delta)\,\alpha_t^k\right]=0,$$

which by (9) implies

$$\lim_{n\to\infty}A(u,u,...,u,x_{n+1})=\lim_{n\to\infty}A(x_{n+1},x_{n+1},...,x_{n+1},u)=0.$$

Hence the sequence $\{x_n\}$ defined iteratively by (5) converges to the fixed point of f. □

Example 2. *Let $X=\mathbb{R}$. For all $x_i\in X$, $i=1,2,...,t$, we define a function $A:X^t\to[0,\infty)$ by*

$$A(x_1,x_2,...,x_{t-1},x_t)=\sum_{i=1}^{t}\sum_{i<j}|x_i-x_j|$$

while the mapping $W:X^t\times I^t\to X$ is defined as

$$W(x_1,x_2,...,x_{t-1},x_t;a_1,a_2,...,a_t)=a_1x_1+a_2x_2+\cdots+a_tx_t$$

for real numbers $a_1,a_2,...,a_t$ in $I=[0,1]$ satisfying $\sum_{i=1}^{t}a_i=1$. Then (X,A,W) is a complete convex A-metric space. If we define $f:X\to X$ by $f(x)=\frac{2x}{5}$, then the mapping f holds conditions of Theorem 1. That is, f has a fixed point and the Mann iteration process (5) converges to fixed point of this mapping. Indeed, set

$$x_{n+1}=W\left(x_n,x_n,...,x_n,fx_n;\alpha_1^n,\alpha_2^n,...,\alpha_t^n\right)$$

and $\alpha_i^n=\frac{1}{t}$ for all $i=1,2,...,t$. Combining with $x_{n+1}=W\left(x_n,x_n,...,x_n,fx_n;\alpha_1^n,\alpha_2^n,...,\alpha_t^n\right)$, $\alpha_i^n=\frac{1}{2n+1}$ for all $i=1,2,...,t$ and $f(x)=\frac{2x}{5}$, we obtain

$$\begin{aligned} x_{n+1} &= W\left(x_n,x_n,...,x_n,,fx_n;\alpha_1^n,\alpha_2^n,...,\alpha_t^n\right)\\ &= W\left(x_n,x_n,...,x_n,\frac{2x_n}{5};\frac{1}{t},\frac{1}{t},...,\frac{1}{t}\right)\\ &= \frac{1}{t}x_n+\frac{1}{t}x_n+\cdots+\frac{1}{t}x_n+\frac{1}{t}\frac{2x_n}{5}\\ &= \frac{t-1}{t}x_n+\frac{1}{t}\frac{2x_n}{5}\\ &= \frac{5t-3}{5t}x_n. \end{aligned}$$

Similarly, we have

$$x_n=\frac{5t-3}{5t}x_{n-1},x_{n-1}=\frac{5t-3}{5t}x_{n-2},\cdots,x_1=\frac{5t-3}{5t}x_0.$$

Therefore

$$x_{n+1}=\left(\frac{5t-3}{5t}\right)^{n+1}x_0 \text{ and } fx_{n+1}=\frac{2}{5}\left(\frac{5t-3}{5t}\right)^{n+1}x_0.$$

If we take limits of above sequences as $n \to \infty$*, we have that* $x_{n+1} \to 0$ *and* $fx_{n+1} \to 0$*. That is* 0 *is a fixed point of* f*.*

Definition 9. *Let* (X, A, W) *be a convex A-metric space with a convex structure* W *and,* $f : X \to X$ *be a mapping,* $x_0 \in X$ *and let us assume that the iteration process* (5)*, that is, the sequence* $\{x_n\}$ *defined by* (5)*, converges to a fixed point* u *of* f*.*

Let $\{y_n\}$ *be an arbitrary sequence in* X *and set*

$$\epsilon_n = A\left(y_{n+1}, y_{n+1}, ..., y_{n+1}, g\left(f, y_n\right)\right) \text{ for } n = 0, 1, 2, ...$$

where $g\left(f, y_n\right) = W\left(y_n, y_n, ..., y_n, fy_n; \alpha_1^n, \alpha_2^n, ..., \alpha_t^n\right)$ *and* $\{\alpha_i^n\}$ *are real sequences in* $[0, 1]$ *for* $i = 1, 2, ..., t$*.*

We say that the Mann iteration process (5) *is* f*-stable or stable with respect to* f *if and only if*

$$\lim_{n\to\infty} \epsilon_n = 0 \Longleftrightarrow \lim_{n\to\infty} y_n = u.$$

Theorem 3. *Let* (X, A, W) *be a complete convex A-metric space with a convex structure* W *and,* $f : X \to X$ *be an AZ mapping.Let* $\{x_n\}$ *be defined iteratively by* (5) *and* $x_0 \in X$*, with* $\{\alpha_t^n\} \subset [0, 1], \sum_{i=1}^{t} a_i = 1$ *satisfying* $0 < \alpha \leq \alpha_n$ *and* $\sum_{n=0}^{\infty} \alpha_t^n = \infty$ *for all* $n \in \mathbb{N}$ *and* $i = 1, 2, ..., t$*. Then the Mann iteration process* (5) *is* f*-stable.*

Proof. From Theorem 1, we know that f has a unique fixed point. Suppose that $u \in X$. From from Lemma 3, we also know that

$$A\left(fx, fx, \ldots, fx, fy\right) \leq \delta A\left(x, x, \ldots, x, y\right) + t\delta A(fx, fx, \ldots, fx, x). \tag{10}$$

Let $\{y_n\} \subset X$ and $\epsilon_n = A\left(y_{n+1}, y_{n+1}, ..., y_{n+1}, g\left(f, y_n\right)\right)$. Assume that $\lim_{n\to\infty} \epsilon_n = 0$. Then, we will show that $\lim_{n\to\infty} y_n = u$. From (10) and triangle inequality, we get

$$\begin{aligned}
& A\left(y_{n+1}, y_{n+1}, \ldots, y_{n+1}, u\right) \qquad (11)\\
\leq\ & (t-1)\, A\left(y_{n+1}, y_{n+1}, \ldots, y_{n+1}, g\left(f, y_n\right)\right) \\
& + A\left(g\left(f, y_n\right), g\left(f, y_n\right), \ldots, g\left(f, y_n\right), u\right) \\
=\ & (t-1)\, A\left(y_{n+1}, y_{n+1}, \ldots, y_{n+1}, g\left(f, y_n\right)\right) \\
& + A\left(u, u, \ldots, u, g\left(f, y_n\right)\right) \\
\leq\ & (t-1)\, \varepsilon_n + A\left(u, u, \ldots, u, g\left(f, y_n\right)\right) \\
=\ & (t-1)\, \varepsilon_n + A\left(u, u, \ldots, u, W\left(y_n, y_n, \ldots, y_n, fy_n; \alpha_1^n, \alpha_2^n, \ldots, \alpha_t^n\right)\right) \\
\leq\ & (t-1)\, \varepsilon_n + \alpha_1^n A\left(u, u, \ldots, u, y_n\right) + \alpha_2^n A\left(u, u, \ldots, u, y_n\right) \\
& + \cdots + \alpha_t^n A\left(u, u, \ldots, u, fy_n\right) \\
=\ & (t-1)\, \varepsilon_n + \left(1 - \alpha_t^n\right) A\left(u, u, \ldots, u, y_n\right) + \alpha_t^n A\left(u, u, \ldots, u, fy_n\right) \\
\leq\ & (t-1)\, \varepsilon_n + \left(1 - \alpha_t^n\right) A\left(u, u, \ldots, u, y_n\right) \\
& + \alpha_t^n \left[\delta A\left(u, u, \ldots, u, y_n\right) + t\delta A\left(fu, fu, \ldots, fu, u\right)\right] \\
=\ & \left[1 - \left(1 - \delta\right) \alpha_t^n\right] A\left(y_n, y_n, \ldots, y_n, u\right) + (t-1)\, \varepsilon_n.
\end{aligned}$$

Since $0 \leq 1 - \left(1 - \delta\right) \alpha_t^n < 1 - \left(1 - \delta\right) \alpha < 1$, using Lemma 4 in (11) yields

$$\lim_{n\to\infty} A\left(y_n, y_n, \ldots, y_n, u\right) = 0,$$

that is,

$$\lim_{n\to\infty} y_n = u\ .$$

Conversely, let $\lim_{n\to\infty} y_n = u$. Then,

$$\begin{aligned}
\varepsilon_n &= A(y_{n+1}, y_{n+1}, \ldots, y_{n+1}, g(f, y_n)) \\
&\leq (t-1) A(y_{n+1}, y_{n+1}, \ldots, y_{n+1}, u) + A(u, u, \ldots, u, g(f, y_n)) \\
&= (t-1) A(y_{n+1}, y_{n+1}, \ldots, y_{n+1}, u) \\
&\quad + A(u, u, \ldots, u, W(y_n, y_n, \ldots, y_n, fy_n; \alpha_1^n, \alpha_2^n, \ldots, \alpha_t^n)) \\
&\leq (t-1) A(y_{n+1}, y_{n+1}, \ldots, y_{n+1}, u) + \alpha_1^n A(u, u, \ldots, u, y_n) \\
&\quad + \alpha_2^n A(u, u, \ldots, u, y_n) + \cdots + \alpha_t^n A(u, u, \ldots, u, fy_n) \\
&= (t-1) A(y_{n+1}, y_{n+1}, \ldots, y_{n+1}, u) + (1-\alpha_t^n) A(u, u, \ldots, u, y_n) \\
&\quad + \alpha_t^n A(u, u, \ldots, u, fy_n) \\
&\leq (t-1) A(y_{n+1}, y_{n+1}, \ldots, y_{n+1}, u) + (1-\alpha_t^n) A(u, u, \ldots, u, y_n) \\
&\quad + \alpha_t^n [\delta A(u, u, \ldots, u, y_n) + t\delta A(fu, fu, \ldots, fu, u)] \\
&\leq (t-1) A(y_{n+1}, y_{n+1}, \ldots, y_{n+1}, u) + (1-\alpha_t^n) A(u, u, \ldots, u, y_n) \\
&\quad + \alpha_t^n \delta A(u, u, \ldots, u, y_n)
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we have $\lim_{n\to\infty} \epsilon_n = 0$. □

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