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# FIXED POINT RESULTS FOR SOME CONTRACTION TYPE MAPPINGS IN $A$-METRIC SPACES

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ABSTRACT. In the present paper, we define an analogue of Hardy-Rogers type contraction in $A$-metric spaces and prove a fixed point theorem for such mappings under appropriate conditions in such spaces. Also, we give some results for Kannan, Chatterjea, Reich, Ciric type contraction mappings in $A$-metric spaces. Our results generalize many known results in the fixed point theory.

## 1. Introduction and Preliminaries

Banach fixed point theory is one of the cornerstones of mathematics and many other sciences. Various studies have been made using different generalizations of the contraction mappings in this theory. Definitions of some contractive type mappings are as follows;

Let $(U, d)$ be a metric space and let $T : U \to U$ be a mapping. The mapping $T$ is said to be

i) Kannan ([6]) contraction mapping if

$$d(Tu, Tv) \leq \alpha \left[d(u, Tu) + d(v, Tv)\right], \tag{1}$$

for all $u, v \in U$, where $\alpha < \frac{1}{2}$;

ii) Chatterjea ([3]) contraction mapping if

$$d(Tu, Tv) \leq \beta \left[d(u, Tv) + d(v, Tu)\right], \tag{2}$$

or all $u, v \in U$, where $\beta < \frac{1}{2}$;

iii) Reich ([8]) contraction mapping if

$$d(Tu, Tv) \leq \alpha d(u, v) + \beta d(u, Tu) + \gamma d(v, Tv), \tag{3}$$

for all $u, v \in U$, where $\alpha + \beta + \gamma < 1$;

iv) Ciric ([4]) contraction mapping if

$$d(Tu, Tv) \leq \alpha d(u, v) + \beta d(u, Tu) + \gamma d(v, Tv) + \delta \left[d(u, Tv) + d(v, Tu)\right], \tag{4}$$

for all $u, v \in U$, where $\alpha + \beta + \gamma + 2\delta < 1$;

From the above definitions, it is easy to see that the contractive condition (4) generalizes the contractive conditions (1)-(3). The mapping $T$ has a fixed point in $U$ if $U$ is a complete metric space and $T$ is a mapping that satisfies any of the above contractive conditions.

In 1973, Hardy and Rogers ([5]) obtained a generalization of Reich fixed point theorem as follows:

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**Theorem 1.** *Let $(U,d)$ be a complete metric space and let $T:U\to U$. Suppose that there exists constants $\alpha,\beta,\gamma\geq 0$ such that*

$$d(Tu,Tv)\leq \alpha d(u,v)+\beta\left[d(u,Tu)+d(v,Tv)\right]+\gamma\left[d(u,Tv)+d(v,Tu)\right] \qquad (5)$$

*for all $u,v\in U$, where $\alpha+2\beta+2\gamma<1$; then $T$ has a unique fixed point in $U$.*

After then, many authors have been used different Hardy-Rogers contractive type conditions in order to obtain fixed point results.

In this paper, we will introduce the definition of Hardy-Rogers type contraction in $A$-metric spaces and give some fixed point results for such mappings.

Since we will work in $A$-metric spaces, now we will give some notations in such metric spaces.

**Definition 1.** *([1]) Let $U$ be nonempty set. Suppose a mapping $A:U^t\to\mathbb{R}$ satisfy the following conditions:*

*$(A_1)$ $A(u_1,u_2,...,u_{t-1},u_t)\geq 0$ ,*

*$(A_2)$ $A(u_1,u_2,...,u_{t-1},u_t)=0$ if and only if $u_1=u_2=...=u_{t-1}=u_t$,*

*$(A_3)$ $A(u_1,u_2,...,u_{t-1},u_t)\leq A(u_1,u_1,...,(u_1)_{t-1},v)+A(u_2,u_2,...,(u_2)_{t-1},v)+...+A(u_{t-1},u_{t-1},...,(u_{t-1})_{t-1},v)+A(u_t,u_t,...,(u_t)_{t-1},v)$*

*for any $u_i,v\in U$, $(i=1,2,...,t)$. Then, $(U,A)$ is said to be an $A$-metric space.*

Taking $t=2$ and $t=3$ in $A$-metric space, it reduce to ordinary metric $d$ and $S$-metric spaces, respectively. So, an $A$-metric space is a generalization of the $G$-metric space ([7]), the $D^*$-metric space ([10]) and the $S$-metric space ([9]).

**Example 1.** *([1]) Let $U=\mathbb{R}$. Define a function $A:U^t\to\mathbb{R}$ by*

$$A(x_1,x_2,...,x_{t-1},x_t)=\sum_{i=1}^{t}\sum_{i<j}\left|x_i-x_j\right|.$$

*Then $(U,A)$ is an $A$-metric space.*

**Lemma 1.** *([1]) Let $(U,A)$ be $A$-metric space. Then $A(u,u,\ldots,u,v)=A(v,v,\ldots,v,u)$ for all $u,v\in U$.*

**Lemma 2.** *([1]) Let $(U,A)$ be $A$-metric space. Then for all for all $u,v\in U$ we have $A(u,u,\ldots,u,y)\leq(t-1)A(u,u,\ldots,u,v)+A(y,y,\ldots,y,v)$ and $A(u,u,\ldots,u,y)\leq(t-1)A(u,u,\ldots,u,v)+A(v,v,\ldots,v,y)$.*

**Definition 2.** *([1]) Let $(U,A)$ be $A$-metric space.*

*(i) A sequence $\{u_k\}$ in $U$ is said to converge to a point $u\in U$ if $A(u_k,u_k,\ldots,u_k,u)\to 0$ as $k\to\infty$.*

*(ii) A sequence $\{u_k\}$ in $U$ is called a Cauchy sequence if $A(u_k,u_k,\ldots,u_k,u_m)\to 0$ as $k,m\to\infty$.*

*(iii) The $A$-metric space $(U,A)$ is said to be complete if every Cauchy sequence in $U$ is convergent.*

## 2. Main Results

In this section, following the ideas of Hardy and Rogers ([5]) we first introduce the notion of Hardy-Rogers type contraction mappings in $A$-metric space as follows:

**Definition 3.** *Let $(U,A)$ be $A$-metric space and $T:U\to U$ be a mapping. $T$ is called a Hardy-Rogers type contraction mapping, if and only if, there exist $\alpha,\beta,\gamma\in\mathbb{R}^+$ with*

$\alpha+2\beta+t\gamma<1$ *such that for all* $u,v\in U$,

$$A(Tu,Tu,\ldots,Tu,Tv) \leq \alpha A(u,u,\ldots,u,v)+\beta\left[A(Tu,Tu,\ldots,Tu,u) +A(Tv,Tv,\ldots,Tv,v)\right]+\gamma\left[A(Tu,Tu,\ldots,Tu,v) +A(Tv,Tv,\ldots,Tv,u)\right]. \tag{6}$$

It is clear that if we take $t=2$ in the Definition 3, we obtain the contractive definition of Hardy-Rogers (5) in ordinary metric space.
Our main result concerning Hardy-Rogers's fixed point theorem is as follows.

**Theorem 2.** *Let* $(U,A)$ *be a complete* $A$*-metric space and* $T:U\to U$ *be a Hardy-Rogers type contraction mapping as Definition 3. Then* $T$ *has a unique fixed point in* $U$ *and Picard iteration process* $\{u_n\}$ *defined by* $u_{n+1}=Tu_n$ *converges to a fixed point of* $T$.

*Proof.* Let $u_0\in U$ and a sequence $\{u_n\}$ be defined by $u_{n+1}=Tu_n$. Assume that $u_n\neq u_{n+1}$ for all $n$.

If we take $u=u_n$ and $v=v_{n+1}$ at the inequality (6), we obtain that

$$\begin{aligned} A(u_n,u_n,\ldots,u_n,u_{n+1}) &= A(Tu_{n-1},Tu_{n-1},\ldots,Tu_{n-1},Tu_n)\\ &\leq \alpha A(u_{n-1},u_{n-1},\ldots,u_{n-1},u_n)\\ &\quad +\beta\begin{bmatrix} A(Tu_{n-1},Tu_{n-1},\ldots,Tu_{n-1},u_{n-1})\\ +A(Tu_n,Tu_n,\ldots,Tu_n,u_n)\end{bmatrix}\\ &\quad +\gamma\begin{bmatrix} A(Tu_{n-1},Tu_{n-1},\ldots,Tu_{n-1},u_n)\\ +A(Tu_n,Tu_n,\ldots,Tu_n,u_{n-1})\end{bmatrix}. \end{aligned}$$

Using conditions in Definition 1 and above the inequality, we obtain

$$\begin{aligned} A(u_n,u_n,\ldots,u_n,u_{n+1}) &\leq \alpha A(u_{n-1},u_{n-1},\ldots,u_{n-1},u_n)\\ &\quad +\beta\begin{bmatrix} A(u_n,u_n,\ldots,u_n,u_{n-1})\\ +A(u_{n+1},u_{n+1},\ldots,u_{n+1},u_n)\end{bmatrix}\\ &\quad +\gamma\begin{bmatrix} A(u_n,u_n,\ldots,u_n,u_n)\\ +A(u_{n+1},u_{n+1},\ldots,u_{n+1},u_{n-1})\end{bmatrix}\\ &= \alpha A(u_{n-1},u_{n-1},\ldots,u_{n-1},u_n)\\ &\quad +\beta\begin{bmatrix} A(u_n,u_n,\ldots,u_n,u_{n-1})\\ +A(u_{n+1},u_{n+1},\ldots,u_{n+1},u_n)\end{bmatrix}\\ &\quad +\gamma A(u_{n+1},u_{n+1},\ldots,u_{n+1},u_{n-1}). \end{aligned} \tag{7}$$

Using Lemmas 1-2 and the inequality (7), we get

$$\begin{aligned} A(u_n,u_n,\ldots,u_n,u_{n+1}) &\leq \alpha A(u_{n-1},u_{n-1},\ldots,u_{n-1},u_n)\\ &\quad +\beta\begin{bmatrix} A(u_{n-1},u_{n-1},\ldots,u_{n-1},u_n)\\ +A(u_n,u_n,\ldots,u_n,u_{n+1})\end{bmatrix}\\ &\quad +\gamma A(u_{n+1},u_{n+1},\ldots,u_{n+1},u_{n-1})\\ &\leq \alpha A(u_{n-1},u_{n-1},\ldots,u_{n-1},u_n)\\ &\quad +\beta\begin{bmatrix} A(u_{n-1},u_{n-1},\ldots,u_{n-1},u_n)\\ +A(u_n,u_n,\ldots,u_n,u_{n+1})\end{bmatrix}\\ &\quad +\gamma(t-1)A(u_{n+1},u_{n+1},\ldots,u_{n+1},u_n)\\ &\quad +\gamma A(u_{n-1},u_{n-1},\ldots,u_{n-1},u_n)\\ &\leq (\alpha+\beta+\gamma)A(u_{n-1},u_{n-1},\ldots,u_{n-1},u_n)\\ &\quad +(\beta+\gamma(t-1))A(u_n,u_n,\ldots,u_n,u_{n+1}) \end{aligned} \tag{8}$$

It follows from (8) that

$$[1-\beta-\gamma(t-1)]\, A\left(u_n,u_n,\ldots,u_n,u_{n+1}\right) \leq (\alpha+\beta+\gamma)\, A\left(u_{n-1},u_{n-1},\ldots,u_{n-1},u_n\right).$$

This implies

$$A\left(u_n,u_n,\ldots,u_n,u_{n+1}\right) \leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma(t-1)}\right) A\left(u_{n-1},u_{n-1},\ldots,u_{n-1},u_n\right).$$

From above inequality, we write

$$A\left(u_n,u_n,\ldots,u_n,u_{n+1}\right) \leq \delta A\left(u_{n-1},u_{n-1},\ldots,u_{n-1},u_n\right)$$

where

$$\delta = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma(t-1)} < 1, \text{as } \alpha+2\beta+t\gamma<1. \tag{9}$$

Repeating iteratively, we have

$$\begin{aligned}
A\left(u_n,u_n,\ldots,u_n,u_{n+1}\right) &\leq \delta A\left(u_{n-1},u_{n-1},\ldots,u_{n-1},u_n\right)\\
&\leq \delta^2 A\left(u_{n-2},u_{n-2},\ldots,u_{n-2},u_{n-1}\right)\\
&\ \ \vdots\\
&\leq \delta^n A\left(u_0,u_0,\ldots,u_0,u_1\right).
\end{aligned}$$

Suppose that $m>n$. From Lemmas 1-2 and the above inequality, we have

$$\begin{aligned}
A\left(u_n,u_n,\ldots,u_n,u_m\right) &\leq (t-1)\, A\left(u_n,u_n,\ldots,u_n,u_{n+1}\right) + A\left(u_{n+1},u_{n+1},\ldots,u_{n+1},u_m\right)\\
&\leq (t-1)\, A\left(u_n,u_n,\ldots,u_n,u_{n+1}\right) + (t-1)\, A\left(u_{n+1},u_{n+1},\ldots,u_{n+1},u_{n+2}\right)\\
&\quad + A\left(u_{n+2},u_{n+2},\ldots,u_{n+2},u_m\right)\\
&\leq (t-1)\, A\left(u_n,u_n,\ldots,u_n,u_{n+1}\right) + (t-1)\, A\left(u_{n+1},u_{n+1},\ldots,u_{n+1},u_{n+2}\right)\\
&\quad + \cdots + (t-1)\, A\left(u_{m-2},u_{m-2},\ldots,u_{m-2},u_{m-1}\right)\\
&\quad + A\left(u_{m-1},u_{m-1},\ldots,u_{m-1},u_m\right)\\
&= (t-1)\left[\delta^n A\left(u_0,u_0,\ldots,u_0,u_1\right) + \delta^{n+1} A\left(u_0,u_0,\ldots,u_0,u_1\right)\right.\\
&\quad \left. + \cdots + \delta^{m-2} A\left(u_0,u_0,\ldots,u_0,u_1\right)\right] + \delta^{m-1} A\left(u_0,u_0,\ldots,u_0,u_1\right)\\
&= (t-1)\,\delta^n\left[1+\delta+\delta^2+\cdots+\delta^{m-n-2}\right] A\left(u_0,u_0,\ldots,u_0,u_1\right)\\
&\quad + \delta^{m-1} A\left(u_0,u_0,\ldots,u_0,u_1\right)\\
&\leq (t-1)\,\delta^n\left[1+\delta+\delta^2+\delta^3+\cdots\right] A\left(u_0,u_0,\ldots,u_0,u_1\right)\\
&\quad + \delta^{m-1} A\left(u_0,u_0,\ldots,u_0,u_1\right)\\
&= \left[(t-1)\,\frac{\delta^n}{1-\delta} + \delta^{m-1}\right] A\left(u_0,u_0,\ldots,u_0,u_1\right).
\end{aligned}$$

From the inequality (9), we know that $0 \leq \delta < 1$. Let $A\left(u_0,u_0,\ldots,u_0,u_1\right) > 0$. Taking limit as $m,n\to\infty$ in above inequality, we obtain that

$$\lim_{n,m\to\infty} A\left(u_n,u_n,\ldots,u_n,u_m\right) = 0.$$

Thus $\{u_n\}$ is a Cauchy sequence in $U$. Also, suppose that $A\left(u_0,u_0,\ldots,u_0,u_1\right)=0$, then $A\left(u_n,u_n,\ldots,u_n,u_m\right)=0$ for all $m>n$ and $\{u_n\}$ is a Cauchy sequence in $U$. Since $(U,A)$ is a complete metric space, $u_n \to x^* \in X$ as $n\to\infty$.

We shall show that $x^*$ is a fixed point of $T$. Using (6), we have

$$\begin{aligned} A(x^*, x^*, \ldots, x^*, Tx^*) &\leq (t-1) A(x^*, x^*, \ldots, x^*, Tu_n) + A(Tu_n, Tu_n, \ldots, Tu_n, Tx^*) \\ &\leq (t-1) A(x^*, x^*, \ldots, x^*, u_{n+1}) + \alpha A(u_n, u_n, \ldots, u_n, x^*) \\ &\quad +\beta [A(Tu_n, Tu_n, \ldots, Tu_n, u_n) + A(Tx^*, Tx^*, \ldots, Tx^*, x^*)] \\ &\quad +\gamma [A(Tu_n, Tu_n, \ldots, Tu_n, x^*) + A(Tx^*, Tx^*, \ldots, Tx^*, u_n)] \\ &= (t-1) A(x^*, x^*, \ldots, x^*, u_{n+1}) + \alpha A(u_n, u_n, \ldots, u_n, x^*) \\ &\quad +\beta A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_n) + \beta A(x^*, x^*, \ldots, x^*, Tx^*) \\ &\quad +\gamma A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, x^*) + \gamma A(u_n, u_n, \ldots, u_n, Tx^*) \\ &\leq [t-1+\gamma] A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, x^*) + \alpha A(u_n, u_n, \ldots, u_n, x^*) \\ &\quad +\beta A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_n) + \beta A(x^*, x^*, \ldots, x^*, Tx^*) \\ &\quad +\gamma [(t-1) A(u_n, u_n, \ldots, u_n, x^*) + A(x^*, x^*, \ldots, x^*, Tx^*)] \\ &= [t-1+\gamma] A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, x^*) + [\alpha + \gamma (t-1)] \\ &\quad A(u_n, u_n, \ldots, u_n, x^*) + \beta A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_n) \\ &\quad + (\beta+\gamma) A(x^*, x^*, \ldots, x^*, Tx^*). \end{aligned} \tag{10}$$

Using (10), we have

$$\begin{aligned} [1-(\beta+\gamma)] A(x^*, x^*, \ldots, x^*, Tx^*) &\leq [t-1+\gamma] A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, x^*) \\ &\quad + [\alpha + \gamma (t-1)] A(u_n, u_n, \ldots, u_n, x^*) \\ &\quad +\beta A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_n). \end{aligned}$$

This implies that

$$\begin{aligned} A(x^*, x^*, \ldots, x^*, Tx^*) &\leq \frac{1}{1-(\beta+\gamma)} \{[t-1+\gamma] A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, x^*) \\ &\quad + [\alpha + \gamma (t-1)] A(u_n, u_n, \ldots, u_n, x^*) \\ &\quad +\beta A(u_{n+1}, u_{n+1}, \ldots, u_{n+1}, u_n)\}. \end{aligned} \tag{11}$$

It follows from $\alpha + 2\beta + t\gamma < 1$ that

$$\begin{aligned} \beta + \gamma &< 2\beta + t\gamma < \alpha + 2\beta + t\gamma < 1 \\ &\Longrightarrow 1 - (\beta+\gamma) > 0. \end{aligned}$$

Taking limit for $n \to \infty$ in above inequality (11), we get $A(x^*, x^*, \ldots, x^*, Tx^*) = 0$, that is, $Tx^* = x^*$. This implies that $x^*$ is a fixed point of the mapping $T$. Now, we shall show that the uniqueness of fixed point of $T$. Assume that $y^*$ is another fixed point of $T$. That is, $Tx^* = x^*$ and $Ty^* = y^*$. Applying (6), we obtain

$$\begin{aligned} A(x^*, x^*, \ldots, x^*, y^*) &= A(Tx^*, Tx^*, \ldots, Tx^*, Ty^*) \leq \alpha A(x^*, x^*, \ldots, x^*, y^*) \\ &\quad +\beta [A(Tx^*, Tx^*, \ldots, Tx^*, x^*) + A(Ty^*, Ty^*, \ldots, Ty^*, y^*)] \\ &\quad +\gamma [A(Tx^*, Tx^*, \ldots, Tx^*, y^*) + A(Ty^*, Ty^*, \ldots, Ty^*, x^*)] \\ &= \alpha A(x^*, x^*, \ldots, x^*, y^*) + \gamma \left[ \begin{array}{c} A(x^*, x^*, \ldots, x^*, y^*) + \\ A(y^*, y^*, \ldots, y^*, x^*) \end{array} \right] \\ &= (\alpha + 2\gamma) A(x^*, x^*, \ldots, x^*, y^*) \end{aligned}$$

thus,

$$(1-\alpha-2\gamma) A(x^*, x^*, \ldots, x^*, y^*) \leq 0,$$

where $1-\alpha-2\gamma > 0$, as $\alpha + 2\beta + t\gamma < 1$. This implies that $A(x^*, x^*, \ldots, x^*, y^*) = 0 \Longrightarrow x^* = y^*$ and hence, $T$ has a unique fixed point in $U$. □

Choosing $\beta = \gamma = 0$ in Theorem 2, the following result for Banach type contraction in $A$-metric space is obvious. The following Corollary also generalizes the results of Banach ([2]) to $A$-metric space.

**Corollary 1.** *([11]) Let $(U, A)$ be a complete $A$-metric space. Suppose the mapping $T : U \to U$ satisfies the following condition:*

$$A(Tu, Tu, \ldots, Tu, Tv) \leq \alpha A(u, u, \ldots, u, v)$$

*for all $u, v \in U$, where $0 \leq \alpha < 1$. Then $T$ has a unique fixed point in $U$.*

Choosing $\alpha = \gamma = 0$ in Theorem 2, the following result for Kannan type contraction in $A$-metric space is obvious. The following Corollary generalizes the results of Kannan [6] to $A$-metric space.

**Corollary 2.** *Let $(U, A)$ be a complete $A$-metric space. Suppose the mapping $T : U \to U$ satisfies the following condition:*

$$A(Tu, Tu, \ldots, Tu, Tv) \leq \beta \left[A(Tu, Tu, \ldots, Tu, u) + A(Tv, Tv, \ldots, Tv, v)\right]$$

*for all $u, v \in U$, where $0 \leq \beta < \frac{1}{2}$. Then $T$ has a unique fixed point in $U$.*

**Remark 1.** *Corollary 2 expands the Theorem 2.3 in ([11]), relaxed the contraction condition from $\beta \in \left[0, \frac{1}{t}\right)$ to $\beta \in \left[0, \frac{1}{2}\right)$.*

Putting $\alpha = \beta = 0$ in Theorem 2, we obtain the following result for Chatterjea type contraction in $A$-metric space. The following Corollary also generalizes the results of Chatterjea ([3]) to $A$-metric space.

**Corollary 3.** *([11]) Let $(U, A)$ be a complete $A$-metric space. Suppose the mapping $T : U \to U$ satisfies the following condition:*

$$A(Tu, Tu, \ldots, Tu, Tv) \leq \gamma \left[A(Tu, Tu, \ldots, Tu, v) + A(Tv, Tv, \ldots, Tv, u)\right]$$

*for all $u, v \in U$, where $0 \leq \gamma < \frac{1}{t}$. Then $T$ has a unique fixed point in $U$.*

From our main Theorem 2, we have the following corollaries for Reich and Ciric type contraction in $A$-metric space. The following Corollaries generalize the results of Reich ([8]) and Ciric ([4]) to $A$-metric space.

**Corollary 4.** *Let $(U, A)$ be a complete $A$-metric space. Suppose the mapping $T : U \to U$ satisfies the following condition:*

$$\begin{aligned} A(Tu, Tu, \ldots, Tu, Tv) \leq\ & \alpha A(u, u, \ldots, u, v) + \beta A(Tu, Tu, \ldots, Tu, u) \\ & + \theta A(Tv, Tv, \ldots, Tv, v) \end{aligned}$$

*for all $u, v \in U$, where $0 \leq \alpha + \beta + \theta < 1$. Then $T$ has a unique fixed point in $U$.*

**Corollary 5.** *Let $(U, A)$ be a complete $A$-metric space. Suppose the mapping $T : U \to U$ satisfies the following condition:*

$$\begin{aligned} A(Tu, Tu, \ldots, Tu, Tv) \leq\ & \alpha A(u, u, \ldots, u, v) + \beta A(Tu, Tu, \ldots, Tu, u) \\ & + \theta A(Tv, Tv, \ldots, Tv, v) + \gamma A(Tu, Tu, \ldots, Tu, v) \\ & + \eta A(Tv, Tv, \ldots, Tv, u) \end{aligned}$$

*for all $u, v \in U$, where $0 \leq \alpha + \beta + \theta + \left(\frac{\gamma+\eta}{2}\right) t < 1$. Then $T$ has a unique fixed point in $U$.*

**Remark 2.** *Taking $t = 2$ in Theorem 2, $A$-metric space reduces to a ordinary metric space $(U, d)$. Therefore, we obtain the result of Hardy-Rogers ([5]). So, Theorem 2 extends the result of Hardy-Rogers ([5]) to $A$-metric space.*

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