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ANALYTICAL ANALYSIS AND GRAPHICAL PRESENTATIONS OF THREE DIMENSIONAL INFLUENCE FUNCTIONS WITHIN A THERMOELASTIC HALF-LAYER

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Abstract. This paper is devoted to analytical analysis of the solution for a boundary value problem given partially in the works: [18] V. Șeremet, A three-dimensional generalized BVP of thermoelasticity for a layer: Green's functions and integration formula, TJMM, vol. 10, no. 2, p. 121-129, 2018 and [19] V. Șeremet and I. Crețu, Three-dimensional influence functions and integration formula for many boundary value problems within a thermoelastic half-layer, TJMM, vol. 12, no, 1, p. 45-58, 2020. The mentioned above solutions satisfy equations of thermoeasticity and boundary conditions only, but derived Green's functions for thermoelastic displacements, caused by a unit point heat source do not vanish at infinity. Thus, the application simultaneous of analytical and graphical methods permed us to obtain exact analytical solutions which vanish at the infinity. In fact, to derive exacts analytical expressions for main termoelastic displacements Green's functions (MTDGFs) was necessary to be omitted in the solutions given in [18] and [19] the terms which do not vanish at infinity without affecting the other main equations. Graphical presentation of derived exact analytical expressions for a three-dimensional MTDGFs, plotted by using soft Maple 18 is included.

Abbreviations

MTDGFs – main thermoelastic displacements Green's functions;
3D – three dimensional;
BVP – boundary value problem;
GFPE – Green's function for Poisson equation;
HIRM – harmonic integral representations method;
HIR - harmonic integral representations;
TVD – thermoelastic volume dilatation.

1. INTRODUCTION

As is well known the theory of thermal stresses [1]-[8] is the most used theory for practical engineering calculations. In the limit of this theory was proposed a new efficient method to solve BVPs of thermoelasticity, called harmonic integral representations method (HIRM) [9]. This method is based on new approach [10] and new general harmonic integral representations (HIR) [11] for main thermoelastic displacements Green's functions (MTDGFs) via Green's functions for Poisson equation (GFPE) [12]. HIRM was used successfully [13]-[19] to constructing structural formulas and analytical expressions for MTDGFs expressed via GFPE for many generalized and particular BVPs of thermoelasticity. The mentioned in [13]-[19] analytical expressions for solutions (thermoelastic

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displacements) of particular BVPs were derived by using proposed special integration formulas. These formulas express the searched thermoelastic displacements via integral from the product of MTDGFs and given on the surfaces thermal boundary conditions, without preliminary determination of the inner temperature field, as in classical methods [5],[6]. This paper is devoted to continuation of the research given in the works [18],[19]. Note that made by us analytical and graphical analysis the analytical expressions for MTDGFs given in the works [18],[19] do not vanish at infinity. So, the main objective of this paper is to obtain exact analytical expressions for MTDGFs within thermoelastic layer and half-layer by simultaneous using analytical and graphical analysis.

2. MATHEMATICAL FORMULATION OF THE RESEARCH PROBLEM FOR HALF-LAYER AND ANALYTICAL EXPRESSIONS FOR MTDGFs GIVEN IN [19]

The BVP to uncoupled thermoelasticity for determining analytical expressions for MTDGFs $U_i(x,\xi); i = 1,2,3$ within the half-layer consists from Lame's and Poisson's equations:

$$\begin{aligned}&\mu \nabla_\xi^2 U_i(x,\xi) + (\lambda+\mu)\Theta_{,\xi_i}(x,\xi) - \gamma G_{T,\xi_i}(x,\xi) = 0; i = 1,2,3;\\ &\nabla_\xi^2 G_T(x,\xi) = -\delta(x-\xi); x \equiv (x_1,x_2,x_3), \xi \equiv (\xi_1,\xi_2,\xi_3),\end{aligned} \tag{1}$$

where λ and μ are Lame constants of elasticity; $\gamma = \alpha_T(3\lambda+2\mu)$ is the thermoelastic constant; α_T is the coefficient of linear thermal dilatation; $\Theta(x,\xi)$ is thermoelastic volume dilatation (TVD); $\delta(x-\xi)$ is delta Dirac function; $x \equiv (x_1,x_2,x_3)$ is the point of application of the unit interior heat source; $\xi \equiv (\xi_1,\xi_2,\xi_3)$ is the point in which MTDGFs $U_i(x,\xi)$ appeared. So the MTDGFs are generated by the unitary inner point heat source described by Dirac function. In addition on the surface of the half-layer are given suitable boundary conditions for normal derivative of temperature GFPE $\partial G_T(x,y)/\partial n_\Gamma$, for MTDGFs $U_i(x,y); i = 1,2,3$ and thermal stresses $\sigma^*_{ij}(x,y); i,j = 1,2,3$, which are determined by the Duhamel-Newman law [5],[6]:

$$\sigma^*_{ij} = \mu(U_{i,j} + U_{j,i}) + \delta_{ij}(\lambda\Theta - \gamma G_T), \tag{2}$$

where δ_{ij} is the Kronecker's symbol and G_T is GFPE for temperature.

These suitable mechanical and thermal boundary conditions look as follows (see Figure 1 also):

$$\begin{gathered}U_3(x,y) = \sigma^*_{31}(x,y) = \sigma^*_{32}(x,y) = 0, \partial G_T(x,y)/\partial n_{\Gamma_{30}} = 0;\\ x \in V; y \equiv (y_1,y_2,0) \in \Gamma_{30};\end{gathered} \tag{3}$$

- on the boundary half-plane $\Gamma_{30}(0 \le y_1 < \infty, -\infty < y_2 < \infty, y_3 = 0)$,

$$\begin{gathered}U_3(x,y) = \sigma^*_{31}(x,y) = \sigma^*_{32}(x,y) = 0, \partial G_T(x,y)/\partial n_{\Gamma_{31}} = 0;\\ x \in V; y \equiv (y_1,y_2,a_3) \in \Gamma_{31};\end{gathered} \tag{4}$$

- on the boundary half-plane $\Gamma_{31}(0 \le y_1 < \infty, -\infty < y_2 < \infty, y_3 = a_3)$, and

$$\begin{gathered}U_1(x,y) = \sigma^*_{12}(x,y) = \sigma^*_{13}(x,y) = 0, \partial G_T(x,y)/\partial n_{\Gamma_{10}} = 0;\\ x \in V; y \equiv (0,y_2,y_3) \in \Gamma_{10};\end{gathered} \tag{5}$$

- on the boundary strip $\Gamma_{10}(y_1 = 0, -\infty < y_2 < \infty, 0 \le y_3 \le a_3)$.

The analytical expressions for MTDGFs for BVP (1) - (5) are the following [19]:

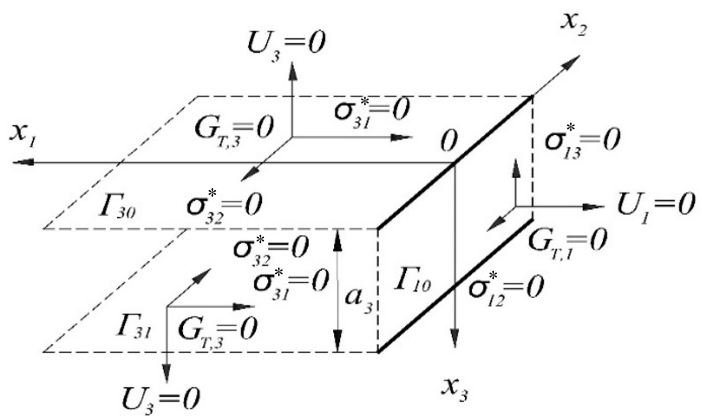

FIGURE 1. The scheme of the half-layer $V \equiv (0 \leq x_1 < \infty, -\infty < x_2 < \infty, 0 \leq x_3 \leq a_3)$ with boundary half-planes Γ_{30}, Γ_{31} and boundary strip Γ_{10}, on which are given the homogeneous mechanical locally-mixed boundary conditions as normal displacements, tangential stresses and homogeneous external normal derivatives from GFPE for temperature G_T.

$$U_1(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)}\left\{\xi_1 b + \frac{1}{\pi a_3}\left[(x_1-\xi_1)\left(\ln r - \sum_{n=1}^{\infty} K_0(\mu_1 r)\cos\mu_1 x_3\cos\mu_1\xi_3\right)\right.\right.$$
$$\left.\left. -\ (x_1+\xi_1)\left(\ln r_1 - \sum_{n=1}^{\infty} K_0(\mu_1 r_1)\cos\mu_1 x_3\cos\mu_1\xi_3\right)\right]\right\}; \qquad (6)$$
$$U_2(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)}(\xi_2 - x_2)$$
$$\times\left\{b - \frac{1}{\pi a_3}\ln r r_1 + \frac{1}{\pi a_3}\sum_{n=1}^{\infty}\left[K_0(\mu_1 r) + K_0(\mu_1 r_1)\right]\cos\mu_1 x_3\cos\mu_1\xi_3\right\}; \qquad (7)$$
$$U_3(x,\xi) = \frac{\gamma}{2\pi a_3(\lambda+2\mu)}$$
$$\times\left\{\xi_3(r+r_1) + \sum_{n=1}^{\infty}\left[rK_1(\mu_1 r) + r_1 K_1(\mu_1 r_1)\right]\cos\mu_1 x_3\sin\mu_1\xi_3\right\}; \qquad (8)$$

- for MTDGFs $U_i(x,\xi)$.

Here, and after here $\mu_1 = \dfrac{n\pi}{a_3}$; b is an arbitrary constant; $K_0(\mu_1 r)$ and $K_0(\mu_1 r_1)$ are modified Bessel functions (or cylindrical functions) of the zero-order of the second type, respectively:

$K_1(\mu_1 r) = -\partial K_0(\mu_1 r)/\partial(\mu_1 r)$ and $K_1(\mu_1 r_1) = -\partial K_0(\mu_1 r_1)/\partial(\mu_1 r_1)$;

$$r = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}; r_1 = \sqrt{(x_1+\xi_1)^2 + (x_2-\xi_2)^2}.$$

Also, for TVD in [19] is given the following analytical expression:

$$\Theta(x,\xi) = \frac{\gamma}{\lambda+2\mu}$$

$$\times\left\{b-\frac{1}{\pi a_3}\ln rr_1+\frac{1}{\pi a_3}\sum_{n=1}^{\infty}[K_0(\mu_1 r)+K_0(\mu_1 r_1)]\cos\mu_1 x_3\cos\mu_1\xi_3\right\}. \qquad (9)$$

The analytical and graphical analyses of the MTDGFs and TVD in Eqns. (6) - (9) showed that their analytical expressions do not vanish at infinity. Thus, the MTDGFs and TVD in Eqns. (6) - (9) are not yet exact solutions for half-layer. So, we have to find exact expressions for MTDGFs and TVD by using simultaneous the analytical and graphical analyzes of expressions (6) - (9).

3. EXACT ANALYTICAL EXPRESSIONS FOR MTDGFs AND TVD WITHIN THERMOELASTIC HALF-LAYER

The analytical and graphical analyses of the MTDGFs and TVD in Eqns. (6) - (9) showed that they may became exact solutions, if will be omitted the following terms: $\xi_1 b$; $\ln r$; $\ln r_1$ - in the Eqn. (6), $b-(\pi a_3)^{-1}\ln rr_1$ - in the Eqns. (7), (9) and $\xi_3(r+r_1)$ - in the Eqn. (8).

In fact, to obtain exact analytical expressions for MTDGFs and TVD which vanish at infinity is necessary to be omitted the terms showed before. In this case the expressions (6) - (9) will vanish at infinity without affecting the equations (1) - (5).

Thus, after making mentioned omitting in the equations (6) - (9) they became the following final exact solutions for thermoelastic half-layer:

$$U_1(x,\xi)=\frac{\gamma}{2\pi a_3(\lambda+2\mu)}\left\{(\xi_1-x_1)\left[\sum_{n=1}^{\infty}K_0(\mu_1 r)\cos\mu_1 x_3\cos\mu_1\xi_3\right]\right.$$

$$\left.+\,(x_1+\xi_1)\left[\sum_{n=1}^{\infty}K_0(\mu_1 r_1)\cos\mu_1 x_3\cos\mu_1\xi_3\right]\right\}; \qquad (10)$$

$$U_2(x,\xi)=\frac{\gamma}{2\pi a_3(\lambda+2\mu)}(\xi_2-x_2)$$

$$\times\left\{\sum_{n=1}^{\infty}[K_0(\mu_1 r)+K_0(\mu_1 r_1)]\cos\mu_1 x_3\cos\mu_1\xi_3\right\}; \qquad (11)$$

$$U_3(x,\xi)=\frac{\gamma}{2\pi a_3(\lambda+2\mu)}$$

$$\times\left\{\sum_{n=1}^{\infty}[rK_1(\mu_1 r)+r_1K_1(\mu_1 r_1)]\cos\mu_1 x_3\sin\mu_1\xi_3\right\}; \qquad (12)$$

- for MTDGFs, and

$$\Theta(x,\xi)=\frac{\gamma}{\pi a_3(\lambda+2\mu)}\left\{\sum_{n=1}^{\infty}[K_0(\mu_1 r)+K_0(\mu_1 r_1)]\cos\mu_1 x_3\cos\mu_1\xi_3\right\}; \qquad (13)$$

- for TVD.

So, analytical expressions (10) - (13) satisfy Eqns. (1) - (5) to BVP of thermoelasticity and vanish at infinity.

Thus, obtained in this paper analytical expressions (10) - (13) present the final exact expressions for MTDGFs and TVD within thermoelastic half-layer.

Note that the respective exact analytical expressions for MTDGFs and TVD within a thermoelastic layer can be obtained from expressions (10) - (13) for half-layer, if will be omitted terms containing distance r_1.

4. GRAPHICAL PRESENTATION OF MTDGFs FOR THERMOELASTIC HALF-LAYER

Graphs of the thermoelastic displacements $Ui(x, \xi)$ within the thermoelastic half-layer V, caused by a unit heat source applied in the point $(x \equiv (x_1, x_2, x_3))$ were plotted by using the soft Maple 18 and exact expressions (10) - (12) for MTDGFs at the following values of the constants: Poisson ratio $\nu = 0,3$; modulus of elasticity $E = 2,1 \cdot 10^5 MPa$ and coefficient of linear thermal dilatation $\alpha_T = 1,2 \cdot 10^{-5}(K^{-1})$.

Graphs of the MTDGFs $U_i(x, \xi)$ in dependence of ξ_1, ξ_3, within the thermoelastic half-layer V for $0 \leq \xi_1 \leq 10m$; $\xi_2 = 0,1m$; $0 \leq \xi_3 \leq 2m$, caused by a unit heat source applied in the point $x_1 = 5m, x_2 = 0, x_3 = 1m$ are presented in the Figure 2 ($U_1(x, \xi)$ - Figure 2a); $U_2(x, \xi)$ - Figure 2b) and $U_3(x, \xi)$ - Figure 2c)).

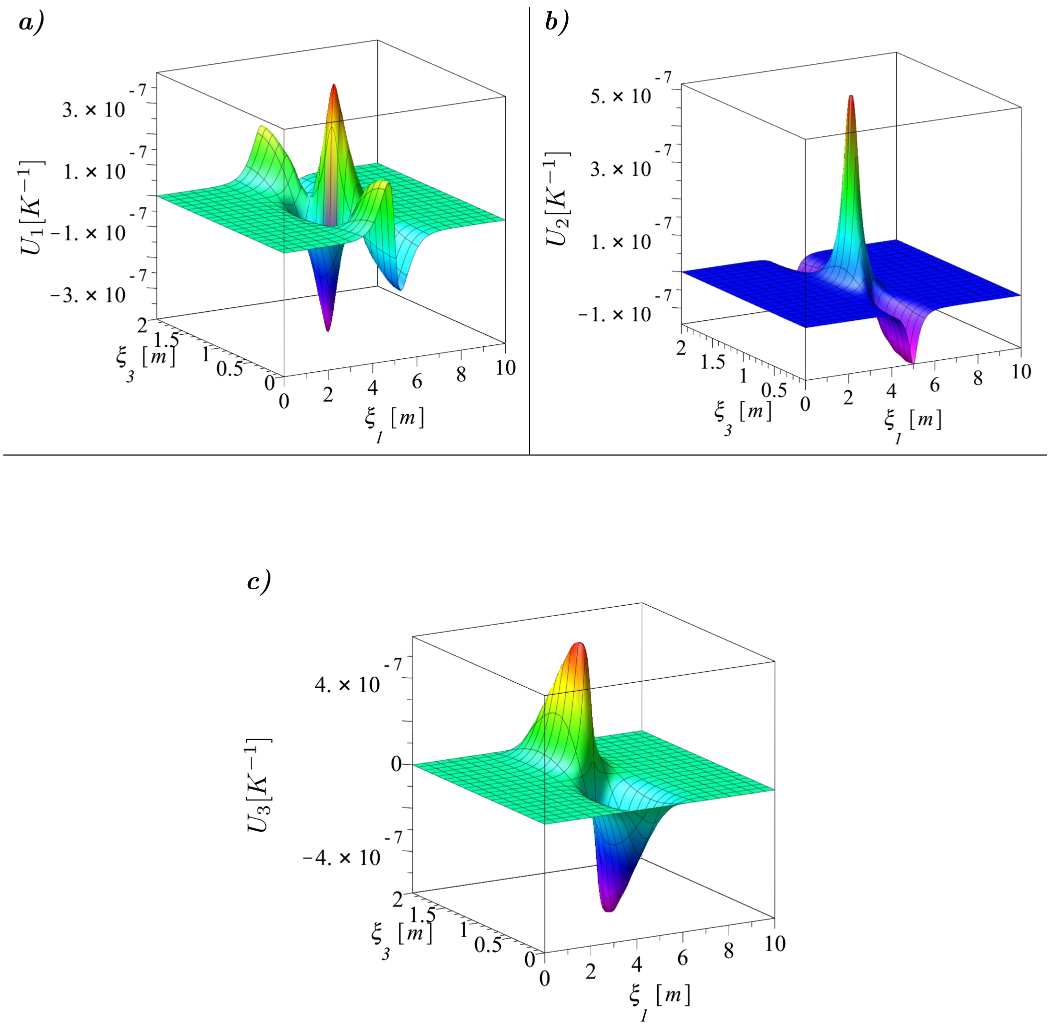

FIGURE 2. Graphs of MTDGFs $U_i(x, \xi)$ within the thermoelastic half-layer V for $0 \leq \xi_1 \leq 10m$; $\xi_2 = 0,1m$; $0 \leq \xi_3 \leq 2m$, caused by a unit heat source applied in the point $x_1 = 5m, x_2 = 0, x_3 = 1m$.

In the Figure 2 can be observed:

1) All graphs were plotted by the soft Maple 18;
2) In the Figure 2, all graphs have jumps in the point $x_1 = 5m, x_2 = 0, x_3 = 1m$ of application of the unit point heat source. In this point the MTDGFs achieve maximal values;
3) All graphs of MTDGFs at infinity vanish;
4) The graph in the Figure 2*a)* is symmetrical in rapport with the plane $\xi_3 = 1m$ and asymmetrical in rapport to the planes $U_1 = 0; \xi_1 = 5m$. The boundary condition $U_1 = 0$ for $\xi_1 = 0$ is met (see eqn. (5) or Figure 1);
5) The graph in the Figure 2*b)* is symmetrical in rapport with the planes $\xi_3 = 1m$ and $\xi_1 = 5m$;
6) The graph in the Figure 2*c)* is symmetrical in rapport with the plane $\xi_1 = 5m$ and asymmetrical in rapport to the planes $U_3 = 0; \xi_3 = 1m$. The boundary conditions $U_3 = 0$ for $\xi_3 = 0$ and $\xi_3 = 2m$ are met (see eqns. (3)-(4) or Figure 1).

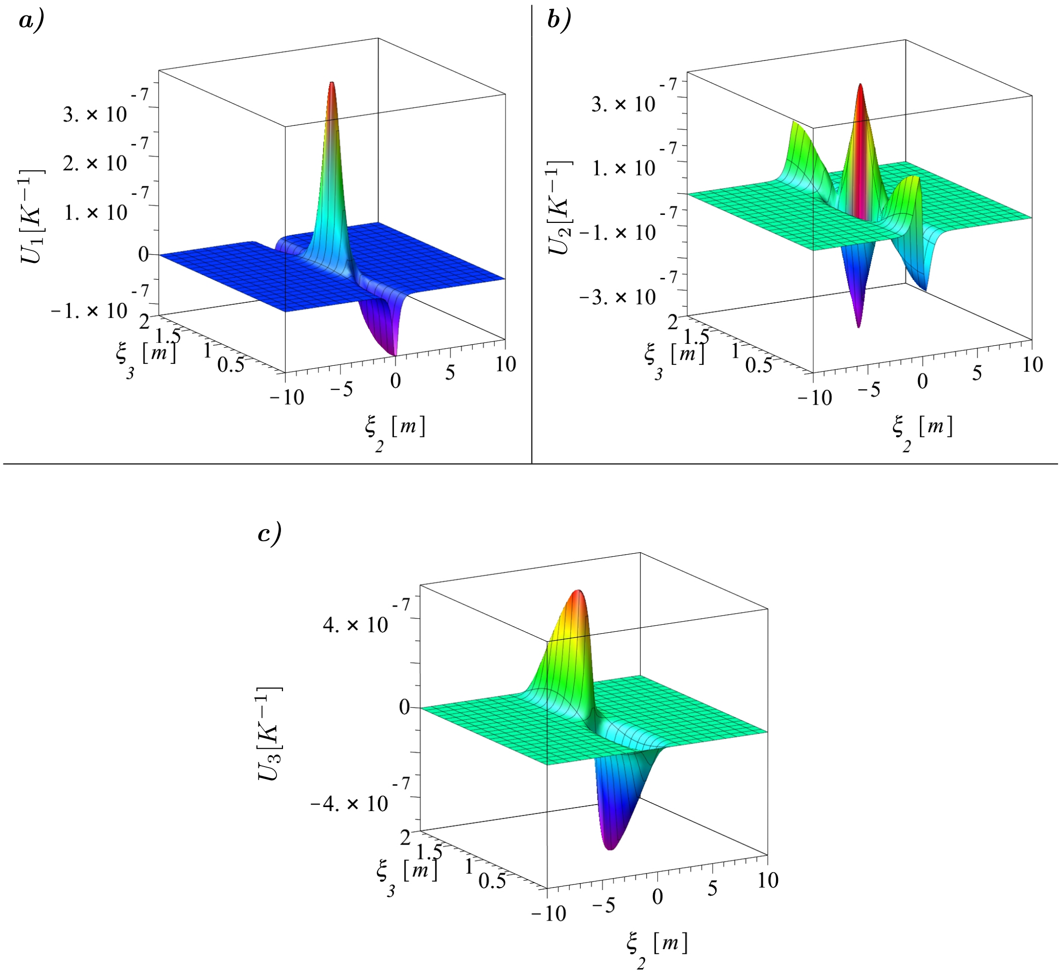

FIGURE 3. Graphs of MTDGFs $U_i(x,\xi)$ within the thermoelastic half-layer V for $\xi_1 = 5, 1m$; $-10m \le \xi_2 \le 10m$; $0 \le \xi_3 \le 2m$, caused by a unit heat source applied in the point $x_1 = 5m, x_2 = 0, x_3 = 1m$.

Graphs of the MTDGFs $U_i(x,\xi)$ in dependence of ξ_2, ξ_3, within the thermoelastic half-layer V for $\xi_1 = 5,1m$; $-10m \leq \xi_2 \leq 10m$; $0 \leq \xi_3 \leq 2m$, caused by a unit heat source applied in the point $x_1 = 5m, x_2 = 0, x_3 = 1m$ are presented in the Figure 3 ($U_1(x,\xi)$ - Figure 3a); $U_2(x,\xi)$ - Figure 3b) and $U_3(x,\xi)$ - Figure 3c)).

In the Figure 3 can be observed:

1) All graphs were plotted by the soft Maple 18;
2) In the Figure 3, all graphs have jumps in the point $x_1 = 5m, x_2 = 0, x_3 = 1m$ of application of the unit point heat source. In this point the MTDGFs achieve maximal values.
3) All graphs of MTDGFs at infinity vanish;
4) The graph in the Figure 3a) is symmetrical in rapport with the planes $\xi_3 = 1m$ and $\xi_2 = 0$;
5) The graph in the Figure 3b) is symmetrical in rapport with the plane $\xi_3 = 1m$ and asymmetrical in rapport to the plane $\xi_2 = 0$;
6) The graph in the Figure 3c) is symmetrical in rapport with the plane $\xi_2 = 0$ and asymmetrical in rapport to the planes $U_3 = 0; \xi_3 = 1m$. The boundary conditions $U_3 = 0$ for $\xi_3 = 0$ and $\xi_3 = 2m$ are met (see eqns. (3)-(4) or Figure 1).

5. CONCLUSION

Simultaneous application of the analytical and graphical analysis permitted us to obtain exact MTDGFs and TVD for a BVP of thermoelasticity within half-layer and layer. The proposed analyzes can be applied to other seven BVPs for thermoelastic half-layer given in [19] in order to obtain for them respective exact analytical expressions for MTDFGs and TVD.

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