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STRONGLY GENERAL BIVARIATIONAL INEQUALITIES

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Abstract. In this work, we introduce and study some new classes of biconvex functions involving an arbitrary bifunction, which are called strongly biconvex functions. Some new relationships among various concepts of strongly biconvex functions have been established. We have shown that the optimality conditions for the biconvex functions can be characterized by a class of bivariational inequalities. An auxiliary principle technique is used to propose proximal point methods for solving bivariational inequalities. We also discussed the conversance criteria for the suggested methods under pseudo-monotonicity. Several special cases are discussed as applications of our main concepts and results.

1. Introduction

Convexity theory is a branch of mathematical sciences with a wide range of applications in industry, physics, social, regional and engineering sciences. The general theory of the convexity started soon after the introduction of differential and integral calculus by Newton and Leibnitz, although some individual optimization problems had been investigated before that. It is worth mentioning that variational inequalities represent the optimality conditions for the differentiable convex functions on the convex sets. Variational inequality theory provides us with a simple, natural, unified, novel and general framework to study an extensive range of unilateral, obstacle, free, moving and equilibrium problems arising in fluid flow through porous media, elasticity, circuit analysis, transportation, oceanography, operations research, finance, economics, and optimization. Variational inequalities were introduced and considered in early 1960s by Stampacchia [20], and they combine both theoretical and algorithmic advances with new and novel domains of applications. The analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis.

Inspired by the research work going on in this field, we introduce and consider a new class of nonconvex functions with respect to an arbitrary function and bifunction. This class of nonconvex functions is called the strongly general biconvex functions. We have shown that the strongly general biconvex functions enjoy some nice properties, which convex have. We establish the relationship between these classes and derive some new results under some mild conditions. It is shown that the optimality conditions of the differentiable biconvex functions can be characterized by a class of variational inequalities, which is called general bivariational inequalities. Some iterative methods are suggested for solving the bivariational

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inequalities using the auxiliary principle technique [3, 11, 12, 13, 14, 16, 17, 23] involving Bregman distance functions. A convergence criteria is also discussed using the pseudo monotonicity which is a weaker condition than monotonicity. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2. Preliminary and basic Results

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and by $\| \cdot \|$ the inner product and norm, respectively. Let $F : K \to R$ be a continuous function and let $\beta(. - .) : K \times K \to R$ be an arbitrary continuous bifunction.

Definition 1. *The set K in H is said to be a general biconvex set with respect to an arbitrary function g and the bifunction $\beta(\cdot - \cdot)$ if*

$$g(u) + \lambda\beta(g(v) - g(u)) \in K, \qquad \forall u, v \in K, \lambda \in [0, 1].$$

Remark 1. *The biconvex set K is also called $g\beta$-convex set. In fact, a research on this "connectivity" is necessary to understand whether the new convexity implies connectivity or is of a non-connected type. Of course, there are connected particular cases, as when $\beta(g(v) - g(u)) = g(v) - g(u)$, which implies that set $g(K)$ is connected, if K has the property of strong biconvexity with respect to g and $\beta(. - .)$. This new "connectivity" uses an object that may be an arch of some curve instead of straight-line segment, as in case of the classical convexity. For more details, see Cristescu and Lupsa* [2].

Note that the biconvex set with $\beta(g(v), g(u)) = v - u$ is a convex set, but the converse is not true. For example, the set $K = R - (-\frac{1}{2}, \frac{1}{2})$ is an biconvex set with respect to g taken as the identity map, and β as follows

$$\beta(v - u) = \begin{cases} v - u, & \text{for} \quad v > 0, u > 0 \quad \text{or} \quad v < 0, u < 0 \\ u - v, & \text{for} \quad v < 0, u > 0 \quad \text{or} \quad v < 0, u < 0. \end{cases}$$

It is clear that K is not a convex set.
If $\beta(g(v) - g(u)) = g(v) - g(u), \forall u, v \in K$, then the Definition 1 reduces to

Definition 2. *The set K in H is said to be a general convex set with respect to an arbitrary function g, if*

$$g(u) + \lambda(g(v) - g(u)) \in K, \qquad \forall u, v \in K, \lambda \in [0, 1].$$

This definition was introduced by Youness [22]. For the applications and properties of the general convex sets and general convex functions, see [10, 11, 16, 17].

From now onward K is a nonempty, closed, biconvex set in H with respect to the arbitrary function g and the bifunction $\beta(\cdot - \cdot)$, unless otherwise specified.

We now introduce some new concepts of general biconvex functions and their variant forms, which is the main motivation of this paper.

Definition 3. *The function F on the biconvex set K is said to be strongly biconvex with respect to an arbitrary function g and the bifunction $\beta(\cdot - \cdot)$, if there exists a constant $\mu \geq 0$, such that*

$$\begin{aligned} F(g(u) + \lambda\beta(g(v) - g(u))) &\leq (1-\lambda)F(g(u)) + \lambda F(g(v)) \\ &- \mu\lambda(1-\lambda)\|\beta(g(v) - g(u))\|^2, \\ &\qquad \forall u, v \in K, \lambda \in [0,1]. \end{aligned} \quad (1)$$

The function F is said to be strongly general biconcave, if and only if, $-F$ is a strongly general biconvex function. Consequently, we have a new concept.

Definition 4. *A function F is said to be strongly general affine biconvex involving an arbitrary function g and the bifunction $\beta(\cdot - \cdot)$, if there exists a constant $\mu \geq 0$, such that*

$$\begin{aligned} F(g(u) + \lambda\beta(g(v) - g(u))) &= (1-\lambda)F(g(u)) + \lambda F(g(v)) \\ &- \mu\lambda(1-\lambda)\|\beta(g(v) - g(u))\|^2, \\ &\qquad \forall u, v \in K, \lambda \in [0,1]. \end{aligned} \quad (2)$$

If $\beta(g(v) - g(u)) = v - u$, then the strongly general biconvex function becomes strongly convex functions, that is,

$$F(u + \lambda(v-u)) \leq (1-\lambda)F(u) + \lambda F(v) - \mu\lambda(1-\lambda)\|\beta(v-u)\|^2, \quad \forall u, v \in K, \lambda \in [0,1],$$

which were introduced by Polyak [19] and Karmardian [5], who used these strongly convex functions in the study of complementarity problems. For the properties of the convex functions in variational inequalities and equilibrium problems, see Noor [8, 9, 11, 12, 13] and Noor et al [15, 16, 17]. Note that every strongly convex function is strongly biconvex, but the converse is not true.

Definition 5. *The function F on the biconvex set K is said to be strongly general quasi biconvex with respect to an arbitrary function g and the bifunction $\beta(\cdot - \cdot)$, if there exists a constant $\mu \geq 0$, such that*

$$\begin{aligned} F(g(u) + \lambda\beta(g(v) - g(u))) &\leq \max\{F(g(u)), F(g(v))\} \\ &\quad -\mu\lambda(1-\lambda)\|\beta(g(v) - g(u))\|^2, \quad \forall u, v \in K, \lambda \in [0,1]. \end{aligned}$$

Definition 6. *The function F on the general biconvex set K is said to be strongly general* log*-biconvex with respect to an arbitrary function g and the bifunction $\beta(\cdot - \cdot)$, if there exists a constant $\mu \geq 0$, such that*

$$\begin{aligned} F(g(u) + \lambda\beta(g(v) - g(u))) &\leq (F(g(u)))^{1-\lambda}(F(g(v)))^{\lambda} \\ &\quad -\mu\lambda(1-\lambda)\|\beta(g(v) - g(u))\|^2, \quad \forall u, v \in K, \lambda \in [0,1]. \end{aligned}$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} &F(g(u) + \lambda\beta(g(v) - g(u))) \\ \leq\ &(F(g(u)))^{1-\lambda}(F(g(v))))^{\lambda} - \mu\lambda(1-\lambda)\|\beta(g(v) - g(u))\|^2 \\ \leq\ &(1-\lambda)F(g(u)) + \lambda F(g(v)) - \mu\lambda(1-\lambda)\|\beta(g(v) - g(u))\|^2 \\ \leq\ &\max\{F(g(u)), F(g(v))\} - \mu\lambda(1-\lambda)\|\beta(g(v) - g(u))\|^2, \quad \forall u, v \in K, \lambda \in [0,1]. \end{aligned}$$

This shows that every strongly general log-biconvex function is a strongly general biconvex function and every strongly general biconvex function is a strongly general quasi-biconvex function. However, the converse is not true.

Definition 7. *The differentiable function F is said to be strongly general biconvex function with respect to g, and β if there exists a constant $\mu \geq 0$, such that*

$$\begin{aligned} F(g(v)) - F(g(u)) \quad &\geq \quad \langle F'(g(u)), \beta(g(v) - g(u))\rangle \\ &\quad +\mu\|\beta(g(v) - g(u))\|^2, \forall u, v \in K. \end{aligned} \tag{3}$$

For $\lambda = 1$, Definition 3 and 6 reduce to the following condition.
Condition A.

$$F(g(u) + \beta(g(v) - g(u))) \quad \leq \quad F(g(v)), \quad \forall v \in K,$$

which plays an important role in the derivation of the results.

If $\mu = 0$, then Definition 3 reduces to

Definition 8. *The function F on the biconvex set $K_{g\beta}$ is said to be general biconvex with respect to the bifunction $\beta(\cdot - \cdot)$, if*

$$F(g(u) + \lambda\beta(g(v) - g(u))) \leq (1 - \lambda)F(g(u)) + \lambda F(g(v)), \forall u, v \in K, \lambda \in [0, 1]. \tag{4}$$

This class of functions was introduced and studied by Noor [14].

We now consider some basic properties of strongly general biconvex functions with respect to g and β and their variant forms. For this purpose, we need the following assumption regarding the bifunction $\beta(\cdot - \cdot)$.

Condition M. The bifunction $\beta(. - .)$ is required to satisfy the assumptions:

$$\begin{aligned} (i). \quad \beta(\gamma\beta(g(v) - g(u))) \quad &= \quad \gamma\beta(g(v) - g(u)), \forall u, v \in K, \gamma \in R. \\ (ii). \quad \beta(g(v) - g(u) - \lambda\beta(g(v) - g(u)) \quad &= \quad (1 - \lambda)\beta(g(v) - g(u)), \\ &\quad\quad \forall u, v \in K, \lambda \in [0, 1]. \end{aligned}$$

Remark 2. *Let $\beta(\cdot - \cdot) : K \times K \to H$ satisfy the assumption*

$$\beta(g(v) - g(u)) = \beta(g(v) - g(z)) + \beta(g(z) - g(u)), \forall u, v, z \in K. \tag{5}$$

One can easily show that $\beta(g(v) - g(u)) = 0 \quad \forall u, v \in K$. Consequently $\beta(0) = 0$, for $v = u \in K$. Also $\beta(g(v) - g(u)) + \beta(g(u) - g(v)) = 0$. This implies that the bifunction $\beta(. - .)$ is skew symmetric.

Theorem 1. *Let F be a differentiable general biconvex function on the general biconvex set K with respect to g and β in H, and let the condition M holds. Then*

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), \beta(g(v) - g(u)))\rangle + \mu\|\beta(g(v) - g(u))\|^2, \tag{6}$$

if and only if, F is a strongly general biconvex function.

Proof. Let F be a strongly general biconvex function on the general biconvex set K.
Then

$$\begin{aligned} F(g(u)+\lambda\beta(g(v)-g(u))) &\leq (1-\lambda)F(g(u))+\lambda F(g(v)) \\ &\quad -\mu\lambda(1-\lambda)\|\beta(g(v)-u)\|^2 \forall u,v\in K, \lambda\in[0,1]. \end{aligned}$$

which can be written as

$$\begin{aligned} F(g(v))-F(g(u)) \geq \frac{F(g(u)+\lambda\beta(g(v)-g(u)))-F(g(u))}{\lambda} \\ +\mu(1-\lambda)\|\beta(g(v)-g(u))\|^2. \end{aligned}$$

Taking the limit in the above inequality as $\lambda\to 0$, we have

$$F(g(v))-F(g(u)) \geq \langle F'(g(u)), \beta(g(v)-g(u)))\rangle + \mu\|\beta(g(v)-g(u))\|^2,$$

which is the required relation (6).
Conversely, let F be a strongly general biconvex function with respect to g and β on the biconvex set K. Then, $\forall u,v\in K, \lambda\in[0,1]$, $g(v_\lambda)=g(u)+\lambda\beta(g(v)-g(u))\in K$ and using the condition M, we have

$$\begin{aligned} & F(g(v))-F(g(u)+\lambda\beta(g(v)-g(u))) \\ \geq\ & \langle F'(g(u)+\lambda\beta(g(v)-g(u))), \beta(g(v)-g(u)+\lambda\beta(g(v)-g(u)))\rangle \\ & +\mu\|\beta(g(v)-g(v_\lambda))\|^2 \\ =\ & (1-\lambda)\angle F'(g(u)+\lambda\beta(g(v)-g(u))), \beta(g(v)-g(u))\rangle \\ & +(1-\lambda)^2\mu\|\beta(g(v)-g(u))\|^2. \end{aligned} \tag{7}$$

In a similar way, we have

$$\begin{aligned} & F(g(u))-F(g(u)+\lambda\beta(g(v)-g(u))) \\ \geq\ & \langle F'(g(u)+\lambda\beta(g(v)-g(u))), \beta(g(u)-g(u)+\lambda\beta(g(v)-g(u)))\rangle \\ & +\mu\|\beta(g(u)-g(v_\lambda))\|^2 \\ =\ & -\lambda\langle F'(g(u)+\lambda\beta(g(v)-g(u))), \beta(g(v)-g(u))\rangle \\ & +\lambda^2\mu\|\beta(g(v)-g(u))\|^2. \end{aligned} \tag{8}$$

Multiplying (7) by λ and (8) by $(1-\lambda)$ and adding the resultants, we have

$$\begin{aligned} F(g(u)+\lambda\beta(g(v)-g(u))) &\leq (1-\lambda)F(g(u))+\lambda F(g(v)) \\ &\quad -\mu\lambda(1-\lambda)\|\beta(g(v)-g(u))\|^2, \end{aligned}$$

which shows that F is a strongly general biconvex function. □

Theorem 2. *Let F be a differentiable strongly general biconvex function with respect to g and β on the biconvex set K. Assume that β satisfies the Condition M. If F is a strongly general biconvex function then*

$$\begin{aligned} & \langle F'(g(u)), \beta(g(v)-g(u)))\rangle + \langle F'(g(v)), \beta(g(u)-g(v))\rangle \\ & \leq -2\mu\|\beta(g(v)-g(u))\|, \forall u,v\in K. \end{aligned} \tag{9}$$

If, additionally, the condition A is satisfied, then the converse statement holds true.

Proof. Let F be a strongly general biconvex function on the general biconvex set K with respect to g and β. Then

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), \beta(g(v) - g(u)))\rangle + \mu\|\beta(g(v) - g(u))\|^2, \quad \forall u, v \in K. \tag{10}$$

Changing the role of u and v in (10), we have

$$F(g(u)) - F(g(v)) \geq \langle F'(g(v)), \beta(g(u) - g(v))\rangle + \mu\|\beta(g(u) - g(v))\|^2, \quad \forall u, v \in K. \tag{11}$$

Adding (10) and (11), we have

$$\langle F'(g(u)), \beta(g(v) - g(u)))\rangle + \langle F'(g(v)), \beta(g(u) - g(v))\rangle \leq -2\mu\{\|\beta(g(v) - g(u))\|^2, \forall u, v \in K,$$

which is the required result (10).

Let $F'(.)$ satisfy inequality (9). Since K is a general biconvex set, $\forall u, v \in K, \quad \lambda \in [0,1]$ $g(v_\lambda) = g(u) + \lambda\beta(g(v) - g(u)) \in K$. Taking $g(v) = g(v_\lambda)$ in (9) and using the Condition M, we have

$$\begin{aligned}\langle F'(g(v_\lambda)), \beta(-\lambda\beta((g(v) - g(u)))\rangle &\leq \langle F'(g(u)), \beta(\lambda\beta(g(v) - g(u)))\rangle \\ &\quad -2\mu\lambda^2\|\beta(g(v) - g(u))\|^2 \\ &= -\lambda\langle F'(g(u)), \beta(g(v) - g(u))\rangle \\ &\quad -2\mu\lambda^2\|\beta(g(v) - g(u))\|^2,\end{aligned}$$

which implies that

$$\langle F'(g(v_\lambda)), \beta(g(v) - g(u))\rangle \geq \langle F'(g(u)), \beta(g(v) - g(u))\rangle + 2\mu\lambda\|\beta(g(v) - g(u)\|^2. \tag{12}$$

We now consider the auxiliary function

$$\xi(\lambda) = F(g(u) + \lambda\beta(g(v) - g(u))) = F(g(v_\lambda)).$$

Then, from (12), we have

$$\begin{aligned}\xi'(\lambda) &= \langle F'(g(u) + \lambda\beta(g(v) - g(u))), \beta(g(v) - g(u))\rangle \\ &\geq \langle F'(g(u)), \beta(g(v) - g(u))\rangle + 2\mu\|\beta(g(v) - g(u))\|^2.\end{aligned} \tag{13}$$

Integrating (13) between 0 and 1, we have

$$\xi(1) - \xi(0) \geq \langle F'(g(u)), \beta(g(v) - g(u))\rangle + 2\mu\int_0^1 \lambda\|\beta(g(v) - g(u))\|^2 d\lambda,$$

that is,

$$F(g(u) + \beta(g(v) - g(u))) - F(g(u)) \geq \langle F'(g(u)), \beta(g(v) - u)\rangle + \mu\|\beta(g(v) - g(u))\|^2.$$

By using Condition A, we have

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), \beta(g(v) - g(u))\rangle + \mu\|\beta(g(v) - g(u))\|^2.$$

the required result. □

3. General bivariational inequalities

In this section, we consider strongly general bivariational inequalities and suggest some iterative methods by using the auxiliary principle techniques involving the Bregman distance functions.

For the readers convenience, we recall some basic properties of the Bregman convex functions [1]. For strongly convex function F, we define the Bregman distance function as

$$\begin{aligned} \mathcal{B}(v,u) &= F(v) - F(u) - \langle F'(u), v-u \rangle \\ &\geq \alpha \|v-u\|^2, \forall u,v \in K. \end{aligned} \tag{14}$$

It is important to emphasize that various types of function F give different Bregman distance function. We give the following important examples of some practical important types of function F and their corresponding Bregman distance functions, see [4, 21].

Examples

(1) If $f(v) = \|v\|^2$, then $\mathcal{B}(v,u) = \|v-u\|$, which is the squared Euclidean distance (SE).

(2) If $f(v) = \sum_{i=1}^{n} a_i \log v_i$, which is known as Shannon entropy, then its corresponding Bregman distance is given as

$$\mathcal{B}(v,u) = \sum_{i=1}^{n} \left(v_i \log(\frac{v_i}{u_i}) + u_i - v_i \right),$$

This distance is called Kullback-Leibler distance (KL) and has become a very important tool in several areas of applied mathematics such as machine learning.

(3) If $f(v) = -\sum_{i=1}^{n} \log v_i$, which is called Burg entropy, then its corresponding Bregman distance is given as

$$\mathcal{B}(v,u) = \sum_{i=1}^{n} \left(\log \frac{v_i}{u_i} + \frac{v_i}{u_i} - 1 \right).$$

This is called Itakura–Saito distance (IS),which is very important in the information theory, data analysis and machine learning.

Remark 3. *It is a challenging problem to explore the applications of Bregman distance for other types of nonconvex functions such as biconvex, k-convex functions and harmonic functions.*

We now discuss the optimality conditions for the differentiable strongly general biconvex functions.

Theorem 3. *Let F be a differentiable strongly general biconvex function with respect to g and β, with modulus $\mu > 0$. If $u \in K$ is the minimum of the function F, then $u \in K$ satisfies the inequality*

$$F(g(v)) - F(g(u)) \geq \mu \|\beta(g(v) - g(u))\|^2, \quad \forall u, v \in K. \tag{15}$$

Proof. Let $u \in K$ be a minimum of the function F. Then

$$F(g(u)) \leq F(g(v)), \forall v \in K. \tag{16}$$

Since K is a biconvex set, $\forall u, v \in K, \quad \lambda \in [0,1]$,

$$g(v_\lambda) = g(u) + \lambda\beta(g(v) - g(u)) \in K.$$

Taking $g(v) = g(v_\lambda)$ in (16), we have

$$\begin{aligned} 0 &\leq \lim_{\lambda \to 0}\{\frac{F(g(u) + \lambda\beta(g(v) - g(u))) - F(g(u))}{\lambda}\} \\ &= \langle F'(g(u)), \beta(g(v) - g(u))\rangle. \end{aligned} \tag{17}$$

Since F is a differentiable general biconvex function, it follows that

$$\begin{aligned} F(g(u) + \lambda\beta(g(v) - g(u))) &\leq F(g(u)) + \lambda(F(g(v)) - F(g(u))) \\ &\quad -\mu\lambda(1-\lambda)\|\beta(g(v) - g(u))\|^2, \forall u, v \in K, \end{aligned}$$

from which, using (17), we have

$$\begin{aligned} F(g(v)) - F(g(u)) &\geq \lim_{\lambda \to 0} \frac{F(g(u) + \lambda\beta(g(v) - g(u))) - F(g(u))}{\lambda} \\ &\quad + \|\beta(g(v) - g(u))\|^2 \\ &= \langle F'(g(u)), \beta(g(v), g(u))\rangle + \mu\|\beta(g(v) - g(u))\|^2 \\ &\geq \mu\|\beta(g(v) - g(u))\|^2, \end{aligned}$$

which is the required result (15). □

Remark: We would like to mention that, if $u \in K$ satisfies the inequality

$$\langle F'(g(u)), \beta(g(v) - g(u))\rangle + \mu\|\beta(g(v) - g(u))\|^2 \geq 0, \quad \forall u, v \in K, \tag{18}$$

then $u \in K$ is the minimum of the differentiable strongly general biconvex function F.

The inequality of the type (18) is called the bivariational inequality and appears to be a new one. It is worth mentioning that inequalities of the type (18) may not arise as the minimization of the biconvex functions. This motivated us to consider a more general bivariational inequality of which (18) is a special case.

For given operators T, g and bifunction $\beta(. - .)$, consider the problem of finding $u \in K_{g\beta}$, such that

$$\langle Tu, \beta(g(v) - g(u))\rangle + \mu\|\beta(g(v) - g(u))\|^2 \geq 0, \forall v \in K, \tag{19}$$

which is called strongly general bivariational inequality.

It is worth mentioning that for suitable and appropriate choice of the operators, biconvex sets, biconvex functions and spaces, one can obtain a wide class of variational inequalities and optimization problems. This shows that the strongly bivariational inequalities are quite flexible and unified ones.

We would like to mention that the projection method and its variant form can not be used to suggest the iterative methods for solving these bivariational inequalities. To overcome these drawback, one may use the auxiliary principle technique of Glowinski et al.[3] as developed by Noor [8, 9, 11, 12, 13] and Noor et al. [14, 15, 16, 17] to suggest and analyze some iterative methods for solving the bivariational inequalities(19). We again use the auxiliary principle technique coupled with Bergman distance functions. These applications are based on the type of convex functions associated with the Bregman distance. We now suggest and analyze some iterative methods for bivariational inequalities (19) using the auxiliary principle technique coupled with Bregman functions as developed by Noor [12, 13, 14].

For given $u \in K$ satisfying the general bivariational inequality (19), we consider the auxiliary problem of finding $w \in K$ such that

$$\begin{aligned}\langle \rho Tw, \beta(g(v) - g(w))\rangle \quad &+ \quad \langle E'(g(w)) - E'(g(u)), \beta(g(v) - g(w))\rangle \\ &+\rho\mu\|\beta(g(v) - g(w))\|^2 \geq 0, \quad \forall v \in K, \qquad (20)\end{aligned}$$

where $\rho > 0$ is a constant and $E'(g(u))$ is the differential of a strongly general biconvex function $E(g(u))$ at $u \in K$. Since $E(g(u))$ is a strongly general biconvex function, this implies that its differential $E'(u)$ is strongly β-monotone. Consequently, it follows that problem (19) has a unique solution.

Remark 3.1: The function

$$\begin{aligned}\mathcal{B}(w, u) \quad &= \quad E(g(w)) - E(g(u)) - \langle E'(g(u)), \beta(g(w) - g(u))\rangle \\ &\geq \quad \mu\|\beta(g(w) - g(u))\|^2, \forall u, w \in K,\end{aligned}$$

associated with the strongly general biconvex function $E(g(u))$ is called the generalized Bregman distance function. By the strongly general biconvexity of the function $E(g(u))$, the Bregman function $\mathcal{B}(.,.)$ is nonnegative and $\mathcal{B}(g(w), g(u)) = 0$, if and only if, $g(u) = g(w), \forall u, w \in K$. For the applications of the Bregman distance function in solving variational inequalities and complementarity problems, see [12, 13, 15, 16, 17, 23].

We note that, if $w = u$, then clearly w is solution of the strongly general bivariational inequality (19). This observation enables us to suggest and analyze the following iterative method for solving (19).

Algorithm 3.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\langle \rho Tu_{n+1}, \beta(g(v) - g(u_{n+1}))\rangle \quad &+ \quad \langle E'(g(u_{n+1})) - E'(g(u_n)), \beta(g(v) - g(u_{n+1}))\rangle \\ &+\rho\mu\|\beta(g(v) - g(u_{n+1}))\|^2 \geq 0, \quad \forall v \in K, \ (21)\end{aligned}$$

where $\rho > 0$ is a constant. Algorithm 3.1 is called the proximal method for solving bivariational inequalities (19). We remark that the proximal point method was suggested in the context of convex programming problems as a regularization technique.

If $\beta(g(v) - g(u)) = v - u$, then Algorithm 3.1 collapses to:

Algorithm 3.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\langle \rho T(u_{n+1}), v - u_{n+1}\rangle \quad + \quad & \langle E'(u_{n+1}) - E'(u), v - u_{n+1}\rangle \\ & + \rho\mu\|v - u_{n+1}\|^2 \geq 0, \quad \forall v \in K,\end{aligned}$$

for solving the strongly variational inequality and appears to be a new one.

For suitable and appropriate choice of the operators and the spaces, one can obtain a number of known and new algorithms for solving variational inequalities and related problems.

For the convergence analysis of Algorithm 3.1, we need the following concept.

Definition 9. *The operator T is said to be strongly pseudo $g\beta$-monotone with respect to the $\mu\|\beta(g(v) - g(u))\|^2$, if*

$$\begin{aligned}&\langle Tu, \beta(g(v) - g(u))\rangle + \mu\|\beta(g(v) - g(u))\|^2 \geq 0 \\ &\Rightarrow \\ &-\langle Tv, \beta(g(u) - g(v))\rangle - \mu\|\beta(g(u) - g(v))\|^2 \geq 0, \quad \forall v, u \in K.\end{aligned}$$

Theorem 4. *Let the operator T be strongly $g\beta$- pseudomonotone with respect to $\mu\|\beta(g(v) \; -g(u))\|^2$. If E be differentiable strongly general biconvex function with module $\beta > 0$, g^{-1} exits and Condition M holds, then the approximate solution u_{n+1} obtained from Algorithm 3.1 converges to a solution $u \in K$ satisfying the general bivariational inequality (19).*

Proof. Let $u \in K$ be a solution of general bivariational inequality (19). Then

$$\langle Tu, \beta(g(v) - g(u))\rangle + \mu\|\beta(g(v) - g(u))\|^2 \geq 0, \quad \forall v \in K,$$

implies that

$$-\langle Tv, \beta(g(u) - g(v)))\rangle - \mu\|\beta(g(u) - g(v))\|^2 \geq 0, \quad \forall v \in K, \tag{22}$$

since T is a strongly pseudo $g\beta$-monotone operator.

Taking $v = u$ in (21) and $v = u_{n+1}$ in (22), we have

$$\begin{aligned}&\langle \rho T(u_{n+1}), \beta(g(u), g(u) - g(u_{n+1}))\rangle \\ &+\langle E'(g(u_{n+1})) - E'(g(u_n)), \beta(g(u) - g(u_{n+1}))\rangle \\ &+\rho\mu\|\beta(g(u) - g(u_{n+1})\|^2 \geq 0.\end{aligned} \tag{23}$$

and

$$-\langle Tu_{n+1}, \beta(g(u) - g(u_{n+1}))\rangle - \mu\|\beta(g(u) - g(u_{n+1})\|^2 \geq 0. \tag{24}$$

We now consider the Bregman function

$$\begin{aligned}\mathcal{B}(u,w) &= E(g(u)) - E(g(w)) - \langle E'(g(w)), \beta(g(u)-g(w))\rangle \\ &\geq \mu\|\beta(g(u)-g(w))\|^2, \end{aligned} \tag{25}$$

using the strongly general biconvexity of E.
Now combining (23), (24) and (25), we have

$$\begin{aligned}& \mathcal{B}(u,u_n) - \mathcal{B}(u,u_{n+1}) \\ = \;& E(g(u_{n+1})) - E(g(u_n)) - \langle E'(g(u_n)), \beta(g(u)-g(u_n))\rangle \\ + \;& \langle E'(g(u_{n+1})), \beta(g(u)-g(u_{n+1}))\rangle \\ = \;& E(g(u_{n+1})) - E(g(u_n)) - \langle E'(g(u_n)) - E'(g(u_{n+1}), \beta(g(u)-g(u_{n+1}))\rangle \\ & -\langle E'(g(u_n), g(u_{n+1}) - g(u_n)\rangle \\ \geq \;& \mu\|\beta(g(u_{n+1})-g(u_n))\|^2 + \langle E'(g(u_{n+1})) - E'(g(u_n)), \beta(g(u)-g(u_{n+1}))\rangle \\ \geq \;& \mu\|\beta(g(u_{n+1})-g(u_n))\|^2 - \rho\langle T(u_{n+1}), \beta(g(u)-g(u_{n+1}))\rangle \\ & -\rho\mu\|\beta(g(u)-g(u_{n+1}))\|^2 \\ \geq \;& \mu\|\beta(g(u_{n+1})-g(u_n))\|^2. \end{aligned}$$

If $g(u_{n+1}) = g(u_n)$, then clearly u_n is a solution to problem (19). Otherwise, it follows that $\mathcal{B}(u,u_n) - \mathcal{B}(u,u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty} \|\beta(g(u_{n+1}) - g(u_n))\| = 0.$$

from which, we have

$$\lim_{n\to\infty} \|g(u_{n+1}) - g(u_n)\| = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} u_{n+1} = u_n,$$

where we used the fact that g^{-1} exits. It follows that the sequence $\{u_n\}$ is bounded. Let $\bar{u}$ be a cluster point of the subsequence $\{u_{n_i}\}$, and let $\{u_{n_i}\}$ be a subsequence converging toward $\bar{u}$. Now using the technique of Zhu and Marcotte [23], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\bar{u}$ satisfying the general bivariational inequality (19).

□

It is well-known that to implement the proximal point methods, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we now consider another method for solving the general bivariational inequality (19) using the auxiliary principle technique.

For a given $u \in K$ satisfying the general bivariational inequality (19), find $w \in K$ such that

$$\begin{aligned}\langle \rho T(u, \beta(g(v)-g(w))\rangle \;&+\; \langle E'(g(w)) - E'g(u), \beta(g(v)-g(w))\rangle \\ & +\mu\|\beta(g(v)-g(w))\|^2 \geq 0, \quad \forall v \in K, \end{aligned} \tag{26}$$

where $E'(g(u))$ is the differential of a biconvex function $E(g(u))$ at $u \in K$. Problem (19) has a unique solution, since E is strongly biconvex function. Note that problems (26) and (21) are quite different problems.
It is clear that for $w = u$, w is a solution of (19). This fact allows us to suggest and

analyze another iterative method for solving the general bivariational inequality (19).

Algorithm 3.3. *For a given* $u_0 \in H$, *compute the approximate solution* u_{n+1} *by the iterative scheme*

$$\langle \rho T u_n, \beta(g(v) - g(u_{n+1}))\rangle + \langle E'(g(u_{n+1})) - E'(g(u_n)), \beta(g(v) - g(u_{n+1}))\rangle + \mu\|\beta(g(v) - g(u_{n+1}))\|^2 \geq 0, \quad \forall v \in K, \quad (27)$$

for solving the strongly general bivariational inequality (19).

If $\beta(g(v), g(u)) = v - u$, Algorithm 3.3 collapses to:

Algorithm 3.4. *For a given* $u_0 \in H$, *compute the approximate solution* u_{n+1} *by the iterative schemes*

$$\rho\langle Tu_n, v - u_{n+1}\rangle + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle + \mu\|v - u_{n+1}\|^2 \geq 0, \quad \forall v \in K,$$

for solving the strongly variational inequalities and appears to be a new one.

Remark 4. *For suitable and appropriate choice of the operators and the spaces, one can obtain various known and new algorithms for solving strongly bivariational inequality (19) and related optimization problems. we have only give some glimpse of the applications of the auxiliary principle techniques. It is an interesting problem from both analytically and numerically point of views.*

Conclusion

In this paper, we have introduced and studied some new classes of strongly biconvex functions. These concepts are more general and unifying ones. Several new properties of these strongly biconvex functions are discussed and their relations with previously known results are highlighted. It is shown that the optimality conditions of the differentiable strongly biconvex functions can be characterised by a class of bivariational inequalities. This result is used to introduce a more general class of strongly general bivariational inequalities (19). Auxiliary principle techniques is used to suggest and analyze some iterative methods for solving the general bivariational inequalities. Convergence analysis of the proposed methods is condition using the pseudo monotonicity which is a weaker condition than monotonicity. It is itself an interesting problem to develop some efficient numerical methods for solving strongly bivariational inequalities along with applications in pure and applied sciences. Despite the current activities in these fields, much clearly remains to be done in these fields. It is expected that the ideas and techniques of this paper may be starting point for future research activities.

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