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ULAM-HYERS STABILITY OF SOME INTEGRAL EQUATIONS

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Abstract. This paper contains some results regarding the property of the Ulam-Hyers stability of a Fedholm type, a Volterra type and a Fredholm-Volterra type respectively, integral equation. The results presented in this paper were obtained using the Picard operator technique and complete the study of the solution of these integral equations.

1. Introduction

In Mathematical Modelling the integral equations have had and continue to play an important role. Many mathematical models of various phenomena in engineering, economics, biology, physics, even in mathematics and other fields of science are governed by integral equations. For some examples of such mathematical models governed by integral equations can be consulted papers, among which we mention the papers [1]-[15] and also, the references therein.

In this paper we aim to study the property of Ulam-Hyers stability of the following integral equations with modified argument:

$$x(t) = \int_a^b K_1(t, s, x(s), x(g_1(s)))ds + f_1(t), \tag{1}$$

where $t \in [a, b]$, $K_1 : [a, b] \times [a, b] \times \mathbb{R}^2 \to \mathbb{R}$, $g_1 : [a, b] \to [a, b]$, $f_1 : [a, b] \to \mathbb{R}$;

$$x(t) = \int_a^t K_2(t, s, x(s), x(g_2(s)))ds + f_2(t), \tag{2}$$

where $t \in [a, b]$, $K_2 : [a, b] \times [a, b] \times \mathbb{R}^2 \to \mathbb{R}$, $g_2 : [a, b] \to [a, b]$, $f_2 : [a, b] \to \mathbb{R}$;

$$x(t) = F\Big(t, x(a), \int_a^b K_1(t, s, x(s), x(g_1(s)))ds, \int_a^t K_2(t, s, x(s), x(g_2(s)))ds\Big), \tag{3}$$

where $t \in [a, b]$, $F : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$, $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R}^2 \to \mathbb{R}$, $g_1, g_2 : [a, b] \to [a, b]$.

We studied some integral equations of these types and obtained some results regarding the existence and uniqueness, comparison, continuous dependence on data, differentiability and approximation of their solutions and these results can be consulted in the papers [2]-[8].

In this paper we studied the property of Ulam-Hyers stability of the integral equations (1), (2) and (3) and the obtained results complete the study of their solutions. The results obtained for the integral equation (1) were communicated to the conference [3] and we publish it in this paper.

The paper contains three sections. In section 2, we recall some definitions and results concerning the obtained properties. The section 3 contains the results regarding the Ulam-Hyers stability of the equations (1), (2) and (3).

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2. Preliminaries

In what follows, we present some definitions and results that were used in obtaining the results of Ulam stability of the integral equations (1), (2) and (3). These preliminaries were used to obtaining the results which were communicated to the conference [3]. We present them again in this paper, to help the reader to more easily understand the new obtained results.

First of all, we present the Ulam stability for a coincidence equation in a metric space (see [14]).

Let (X,d) and (Y,ρ), with $d(x,y),\rho(x,y)\in E_+$, be two metric spaces and let $f,g: X\longrightarrow Y$ be two operators. Let us consider the coincidence equation:

$$f(x)=g(x) \tag{4}$$

Definition 1. *([12], [14]) The equation (4) is Ulam-Hyers stable if there exists a linear increasing operator $c_{fg}:E\longrightarrow E$ such that for each $\varepsilon\in E_+^*$ and each solution $y^*\in X$ of the inequality*

$$\rho(f(x),g(x))\leq\varepsilon \tag{5}$$

there exists a solution $x^*\in X$ of (5) such that

$$d(y^*,x^*)\leq c_{f,g}(\varepsilon).$$

Remark 1. *If $Y:=X$ and $g:=1_X$, then we have the notion of Ulam-Hyers stability for a fixed point equation.*

Remark 2. *If $E:=\mathbb{R}$ is endowed with the usual structure, then then instead of the Definition 1 we will have the definition below.*

Definition 2. *([12], [14]) The equation (4) is Ulam-Hyers stable if there exists a positive real number $c_{f,g}>0$ such that for each $\varepsilon\in\mathbb{R}_+^*$ and each solution $y^*\in X$ of (5), there exists a solution $x^*\in X$ of the equation (4) such that*

$$d(y^*,x^*)\leq c_{f,g}(\varepsilon).$$

Next, we present the definition of the Ulam-Hyers stability of the fixed point equation

$$x=A(x). \tag{6}$$

Definition 3. *([12], [14]) Let (X,d) be a metric space and $A:X\longrightarrow X$ an operator. The equation of fixed point (6) is Ulam-Hyers stable if there exists a real number $c_A>0$ such that for each $\varepsilon>0$ and each solution y^* of the inequation $d(y,A(y))\leq\varepsilon$ there exists a solution x^* of equation (6), such that*

$$d(y^*,x^*)\leq c_A\varepsilon.$$

Also, in this paper we will use the Remark 2.1 from [14], that you can find below.

Remark 3. *([14], Remark 2.1) If f is a c-weakly Picard operator, then the fixed point equation (6) is Ulam-Hyers stable.*

Indeed, let $\varepsilon>0$ and y^* a solution of $d(y,A(y))\leq\varepsilon$. Since A is c-weakly Picard operator, we have that

$$d(x,A^\infty(x))\leq c\cdot d(x,A(x)),\forall x\in X.$$

If we take $x:=y^*$ and $x^*:=A^\infty(y)$, we have that $d(y^*,x^*)\leq c_A\varepsilon$. ([14])

3. Ulam-Hyers stability

A. For the Fredholm nonlinear integral equation (1) we suppose that the following conditions hold:

(a_1) $K_1 \in C([a,b] \times [a,b] \times \mathbb{R}^2)$, $g_1 \in C([a,b],[a,b])$;

(a_2) $f_1 \in C[a,b]$.

Theorem 1 presents the conditions of existence and uniqueness of the solution of the integral equation (1) ([7], [8]). Theorem 2 gives us the conditions of Ulam-Hyers stability of this integral equation.

Theorem 1. *([7], [8]) Suppose that the conditions (a_1) and (a_2) are satisfied and the following conditions hold:*

(a_3) *there exists $L_1 > 0$ such that:*

$$|K_1(t,s,u_1,v_1) - K_1(t,s,u_2,v_2)| \leq L_1 \left(|u_1 - u_2| + |v_1 - v_2|\right),$$

for all $t, s \in [a,b], u_i, v_i \in \mathbb{R}, i = 1,2;$

(a_4) $2L_1(b-a) < 1.$

Then the Fredholm integral equation (1) has a unique solution $x^* \in C[a,b]$.

Theorem 2. *Under the conditions of the Theorem 1, the Fredholm integral equation (1) is Ulam-Hyers stable, i.e. for $\varepsilon > 0$ and $y^* \in C[a,b]$ a solution of the inequation*

$$\left| y(t) - \int_a^b K_1(t,s,y(s),y(g_1(s)))ds - f_1(t) \right| \leq \varepsilon, \quad \textit{for all } t \in [a,b],$$

there exists a solution of the integral equation (1), $x^* \in C[a,b]$, such that

$$|y^*(t) - x^*(t)| \leq \frac{1}{1 - L_{A_1}} \varepsilon, \quad \textit{for all } t \in [a,b],$$

where $L_{A_1} = 2L_1(b-a)$.

In the proof of the Theorem 2, the Picard operators technique is applied and the Remark 1.2 from [14] is used.

The Theorem 1 and Theorem 2 with its proof were communicated to the conference [3].

B. For the Volterra nonlinear integral equation (2) we suppose that the following conditions hold:

(b_1) $K_2 \in C([a,b] \times [a,b] \times \mathbb{R}^2)$, $g_2 \in C([a,b],[a,b])$;

(b_2) $f_2 \in C[a,b]$.

Theorem 3 presents the conditions of existence and uniqueness of the solution of the integral equation (2). Theorem 4 gives us the conditions of Ulam-Hyers stability of this integral equation.

Theorem 3. *Suppose that the conditions (b_1) and (b_2) are satisfied and the following conditions hold:*

(b_3) *there exists $L_2 > 0$ such that:*

$$|K_2(t,s,u_1,v_1) - K_2(t,s,u_2,v_2)| \leq L_2 \left(|u_1 - u_2| + |v_1 - v_2|\right),$$

for all $t, s \in [a,b], u_i, v_i \in \mathbb{R}, i = 1,2;$

(b_4) $2L_2(b-a) < 1.$

Then the Volterra integral equation (2) has a unique solution $x^* \in C[a,b]$.

Remark 4. *Chebyshev's norm was used to prove this theorem. Instead of the Chebyshev's norm, the Bielecki's norm can be used.*

Theorem 4. *Under the conditions of the Theorem 3, the Volterra integral equation* (2) *is Ulam-Hyers stable, i.e. for* $\varepsilon > 0$ *and* $y^* \in C[a,b]$ *a solution of the inequation*

$$\left| y(t) - \int_a^t K_2(t,s,y(s),y(g_2(s)))ds - f_2(t) \right| \le \varepsilon, \quad for\, all\ t \in [a,b],$$

there exists a solution of the integral equation (2)*,* $x^* \in C[a,b]$*, such that*

$$|y^*(t) - x^*(t)| \le \frac{1}{1 - L_{A_2}} \varepsilon, \quad for\, all\ t \in [a,b],$$

where $L_{A_2} = 2L_2(b-a)$*.*

Proof. We consider the operator $A_2 : C[a,b] \to C[a,b]$, defined by the relation:

$$A_2(x)(t) = \int_a^t K_2(t,s,x(s),x(g_2(s)))ds + f_2(t), \quad for\, all\ t \in [a,b]. \tag{7}$$

Under the conditions of Theorem 3, it results that the operator A_2 is a contraction and therefore, A_2 is c-Picard operator (PO) with the constant $c_2 = \frac{1}{1-L_{A_2}}$. Consequently, the conclusion of this theorem it results as an application of the Remark 2.1 from [14] and the proof is complete. □

C. We consider the Fredholm-Volterra nonlinear integral equation (3) and suppose that the following conditions hold:

(c_1) $F \in C([a,b] \times \mathbb{R}^3)$;
(c_2) $K_1 \in C([a,b] \times [a,b] \times \mathbb{R}^2)$, $g_1 \in C([a,b],[a,b])$;
(c_3) $K_2 \in C([a,b] \times [a,b] \times \mathbb{R}^2)$, $g_2 \in C([a,b],[a,b])$.

Theorem 5 presents the conditions of existence and uniqueness of the solution of the integral equation (3) (see [4]). Theorem 6 gives us the conditions of Ulam-Hyers stability of this integral equation.

Theorem 5. *(*[4]*) Suppose that the conditions* (c_1) $-$ (c_3) *are satisfied and the following conditions hold:*

(c_4) *there exist* $\alpha, \beta, \gamma > 0$ *such that:*

$$|F(t,u_1,v_1,w_1) - F(t,u_2,v_2,w_2)| \le \alpha|u_1 - u_2| + \beta|v_1 - v_2| + \gamma|w_1 - w_2|,$$

for all $t \in [a,b], u_i, v_i, w_i \in \mathbb{R}, i = 1,2$;

(c_5) *there exists* $L_1 > 0$ *such that:*

$$|K_1(t,s,u_1,v_1) - K_1(t,s,u_2,v_2)| \le L_1(|u_1 - u_2| + |v_1 - v_2|),$$

for all $t, s \in [a,b], u_i, v_i \in \mathbb{R}, i = 1,2$;

(c_6) *there exists* $L_2 > 0$ *such that:*

$$|K_2(t,s,u_1,v_1) - K_2(t,s,u_2,v_2)| \le L_2(|u_1 - u_2| + |v_1 - v_2|),$$

for all $t, s \in [a,b], u_i, v_i \in \mathbb{R}, i = 1,2$;

(c_7) $\alpha + (\beta L_1 + \gamma L_2)(b-a) < 1$.

Then the Fredholm-Volterra integral equation (3) *has a unique solution* $x^* \in C[a,b]$*.*

Proof. We attach to the integral equation (3) the operator $A_3 : C[a,b] \to C[a,b]$, defined by:

$$A_3(x)(t) = F\Big(t, x(a), \int_a^b K_1(t,s,x(s),x(g_1(s)))ds, \int_a^t K_2(t,s,x(s),x(g_2(s)))ds\Big), \tag{8}$$

for all $t \in [a,b]$.

The set of the solutions of the integral equation (3) coincide with the set of fixed points of the operator A_3.

In order to use the Contraction Principle, the operator A_3 must be a contraction. We have:

$$\Big|A_3(x_1)(t) - A_3(x_2)(t)\Big| \leq$$

$$\Big|F\Big(t, x_1(a), \int_a^b K_1(t,s,x_1(s),x_1(g_1(s)))ds, \int_a^t K_2(t,s,x_1(s),x_1(g_2(s)))ds\Big)$$

$$-F\Big(t, x_2(a), \int_a^b K_1(t,s,x_2(s),x_2(g_1(s)))ds, \int_a^t K_2(t,s,x_2(s),x_2(g_2(s)))ds\Big)\Big|.$$

Using (c_4), (c_5) and (c_6) and the Chebyshev norm it result:

$$\big|\big|A_3(x_1) - A_3(x_2)\big|\big|_{C[a,b]} \leq \big[\alpha + (\beta L_1 + \gamma L_2)(b-a)\big] \cdot \big|\big|x_1 - x_2\big|\big|_{C[a,b]}$$

Consequently, from (c_7) it result that the operator A_3 is a contraction with the coefficient $\alpha + (\beta L_1 + \gamma L_2)(b-a) < 1$.

Now, we will apply the Contraction Principle and it results that the Fredholm-Volterra integral equation (3) has a unique solution $x^* \in C[a,b]$. The proof is complete. □

Theorem 6. *Under the conditions of the Theorem 5, the Fredholm-Volterra integral equation* (3) *is Ulam-Hyers stable, i.e. for* $\varepsilon > 0$ *and* $y^* \in C[a,b]$ *a solution of the inequation:*

$$\Big|y(t) - F\Big(t, y(a), \int_a^b K_1(t,s,y(s),y(g_1(s)))ds, \int_a^t K_2(t,s,y(s),y(g_2(s)))ds\Big)\Big| \leq \varepsilon,$$

for all $t \in [a,b]$,

there exists a solution of the integral equation (3), $x^* \in C[a,b]$, *such that*

$$|y^*(t) - x^*(t)| \leq \frac{1}{1 - L_{A_3}}\varepsilon, \quad \textit{for all } t \in [a,b],$$

where $L_{A_3} = \alpha + (\beta L_1 + \gamma L_2)(b-a)$.

Proof. We consider the operator $A_3 : C[a,b] \longrightarrow C[a,b]$, defined by the relation (8).

Under the conditions of the Theorem 5, it results that the operator A_3 is a contraction and therefore, A_3 is c-Picard operator (PO) with the constant $c = \frac{1}{1-L_{A_3}}$.

Finally, the conclusion of the Theorem 6 it results as an application of the Remark 2.1 from [14] and the proof is complete. □

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