TJMM
13 (2021), No. 1-2, 17-25

GRÜSS TYPE INEQUALITIES FOR VARIOUS KINDS OF FRACTIONAL INTEGRATION

GABRIELA CRISTESCU AND SORIN-HORAŢIU HOARĂ

Abstract. Some new and useful Grüss type inequalities using Riemann-Liouville fractional integral, Erdélyi-Kober integral operator and Katugampola fractional integral are established. The inequalities are sharp.

1. Introduction

A very useful tool in assessing the size of the error in various approximation processes involving two function with known mutual behaviour is the inequality proved by G. Grüss proved in 1934 ([10]). This kind of approximation processes are used, for example, in assessing the reliability of two dual networks having two terminals (see [4], [5]). Fractional integration is a largely used tool in describing the overall mutual behaviour of two reliability polynomials during the last twenty years. In this paper we extend the Grüss inequality using few kinds of fractional integrals.
Let $\mathbb{R}$ be the set of real numbers and $[a, b] \subseteq \mathbb{R}$ be an interval. G. Grüss proved in 1934 ([10]) the following result:

Theorem 1. *Let $f, g : [a, b] \to \mathbb{R}$ be two Riemann-integrable functions on interval $[a, b]$ having the property that there are $m_f, M_f, m_g, M_g \in \mathbb{R}$ such as $m_f \leq f(x) \leq M_f$, $m_g \leq g(x) \leq M_g$ for all $x \in [a, b]$. If*

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \tag{1}$$

then the following inequality holds:

$$|T(f,g)| \leq \frac{1}{4}(M_f - m_f)(M_g - m_g). \tag{2}$$

Inequality (2) is known as Grüss inequality. More authors recently established versions of inequality (2) in the framework provided by the presence of additional properties as differentiability (see [16], [8]), convexity ([2], [15]), fractional integration ([6], [7], [3]) with various applications (see [1], [8]). We study, in this paper, what does inequality (2) become in the framework created by the use of few fractional integral operators. We take into account the fractional integration of Riemann-Liouville type (see [13]), the generalized version of fractional integration of Katugampola ([11], [12]), and the Erdélyi-Kober operator (see [9], [14]).

2. Grüss type inequalities in Riemann-Liouville fractional integration

In this section we derive new inequalities of Grüss type using fractional integration considered after the results published during the 19th century by Bernhard Riemann (1847) and Joseph Liouville (1832). Let us consider the interval $[a, b] \subseteq \mathbb{R}$.

2010 *Mathematics Subject Classification.* 26A33, 26D15.
Key words and phrases. fractional integral, Grüss inequality.

Definition 1. *Let $f \in L_1[a,b]$. Then the Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a, \tag{3}$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b, \tag{4}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} x^{\alpha-1} dx,$$

is the Gamma function.

Consider two functions $f, g : [a,b] \to \mathbb{R}$, $f, g \in L_1[a,b]$ and the real numbers $\alpha, \mu, \nu > 0$. In this section we denote:

$$TJ_{a+}^{\alpha,\mu,\nu}(f,g)(x) = \frac{1}{x-a} J_{a+}^{\alpha}(fg)(x) - \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\alpha)} \left(\frac{1}{x-a} J_{a+}^{\mu} f(x) \right) \left(\frac{1}{x-a} J_{a+}^{\nu} g(x) \right), \tag{5}$$

$$TJ_{b-}^{\alpha,\mu,\nu}(f,g)(x) = \frac{1}{b-x} J_{b-}^{\alpha}(fg)(x) - \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\alpha)} \left(\frac{1}{b-x} J_{b-}^{\mu} f(x) \right) \left(\frac{1}{b-x} J_{b-}^{\nu} g(x) \right). \tag{6}$$

Establishing upper bounds of the absolute value of these integral differences is the main result of this section. This result was communicated to the conference [3].

Theorem 2. *Consider $f, g \in L_1[a,b]$ and two points $x, y \in [a,b]$. Suppose that the three numbers $\alpha, \mu, \nu > 0$ are chosen such as $\alpha = \mu + \nu - 1$.*

(1) If functions $u_x, v_x : [a,x] \to \mathbb{R}$, defined by

$$u_x(t) = (x-t)^{\mu-1} f(t),$$
$$v_x(t) = (x-t)^{\nu-1} g(t),$$

for $t \in [a,x]$, are integrable and if there are four numbers m_f, M_f, m_g, M_g such that

$$m_f \leq u_x(t) \leq M_f,$$
$$m_g \leq v_x(t) \leq M_g,$$

for all $t \in [a,x]$, then

$$|TJ_{a+}^{\alpha,\mu,\nu}(f,g)(x)| \leq \frac{1}{4\Gamma(\alpha)} (M_f - m_f)(M_g - m_g). \tag{7}$$

(2) If functions $p_y, q_y : [y,b] \to \mathbb{R}$, defined by

$$p_y(t) = (t-y)^{\mu-1} f(t),$$
$$q_y(t) = (t-y)^{\nu-1} g(t),$$

for $t \in [y,b]$, are integrable and if there are four numbers m_f, M_f, m_g, M_g such that

$$m_f \leq p_y(t) \leq M_f,$$
$$m_g \leq q_y(t) \leq M_g,$$

for all $t \in [y, b]$, *then*

$$|TJ_{b-}^{\alpha,\mu,\nu}(f,g)(y)| \leq \frac{1}{4\Gamma(\alpha)}(M_f - m_f)(M_g - m_g). \tag{8}$$

Proof. First of all we prove inequality (7). Let us compute

$$J_{a+}^{\alpha}(fg)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)g(t)dt$$

$$= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\mu+\nu-2} f(t)g(t)dt$$

$$= \frac{1}{\Gamma(\alpha)}\int_a^x \left((x-t)^{\mu-1} f(t)\right)\left((x-t)^{\nu-1} g(t)\right) dt.$$

Now we consider the functions u_x and v_x defined above, which are integrable, getting

$$J_{a+}^{\alpha}(fg)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x u_x(t)v_x(t)dt.$$

Inequality (2) on interval $[a, x]$ gives, according to the hypothesis of statement 1,

$$|T(u_x, v_x)| = \left|\frac{1}{x-a}\int_a^x u_x(t)v_x(t)dt - \left(\frac{1}{x-a}\int_a^x u_x(t)dt\right)\left(\frac{1}{x-a}\int_a^x v_x(t)dx\right)\right| \tag{9}$$

$$\leq \frac{1}{4}(M_f - m_f)(M_g - m_g).$$

We remark that

$$\int_a^x u_x(t)dt = \Gamma(\mu)J_{a+}^{\mu} f(x),$$

$$\int_a^x v_x(t)dt = \Gamma(\nu)J_{a+}^{\nu} g(x),$$

The required result (7) is obtained after dividing inequality (9) by $\Gamma(\alpha)$ and using these identities.
The proof of inequality (8) is similar. □

Remark 1. *If* $x = b$ *and* $y = a$ *then inequalities* (7) *and* (8) *coincide.*

Remark 2. *If* $\alpha = \mu = \nu = 1$ *then inequalities* (7) *and* (8) *become inequality* (2) *on the corresponding interval.*

Remark 3. *Inequalities* (7) *and* (8) *are sharp. For example, if either* $f \equiv 0$ *or* $g \equiv 0$ *then* (7) *and* (8) *are identities.*

The inequalities derived above are useful in estimating errors in approximation processes involving Riemann-Liouville fractional integration.

3. Grüss type inequalities in the framework using Erdélyi-Kober fractional integral operator

The Arthur Erdélyi [9] and Hermann Kober [11] fractional integral operator is an extension of the Riemann-Liouville fractional integral with a power weight. It is a particular case in the class of fractional integral operators introduced by Virginia Kiryakova [13].

Definition 2. *Let $f:[0,+\infty)\to\mathbb{R}$ be a continuous function. Consider two real numbers $\gamma>0$ and $\delta>0$. The Erdély-Kober integral operator of order α (see [13]) is defined by :*

$$I^{(\gamma,\delta)}(f)(x)=\frac{x^{-\gamma-\delta}}{\Gamma(\delta)}\int_0^x(x-t)^{\delta-1}t^{\gamma}f(t)dt, \tag{10}$$

$$K^{(\gamma,\delta)}(f)(x)=\frac{x^{\gamma}}{\Gamma(\delta)}\int_x^{\infty}(t-x)^{\delta-1}t^{-\gamma-\delta}f(t)dt. \tag{11}$$

Let us consider two functions $f:[0,\infty)\to\mathbb{R}$ and $g:[0,\infty)\to\mathbb{R}$ and suppose that $\gamma>0$ and $\delta>0$. For every $x\in[0,\infty)$, and numbers $c>0$, $d>0$, $s>0$ and $n>0$ we define the following pair of functions: (u_x,v_x), $u_x:[0,x]\to\mathbb{R}$, $v_x:[0,x]\to\mathbb{R}$, as follows:

$$u_x(t)=t^n(x-t)^{c-1}f(t),$$
$$v_x(t)=t^s(x-t)^{d-1}g(t).$$

The following result contains a Grüss type inequality using the Erdélyi-Kober fractional integral operator, deduced by means of inequality (2) using a splitting reasoning similar to that from the previous section.

Theorem 3. *Suppose that functions f and g are continuous on $[0,\infty)$, $\gamma>0$, $\delta>0$ and $x\in[0,\infty)$. For every pairs of numbers (c,d) and (s,n), $c>0$, $d>0$, $s>0$ and $n>0$ chosen such as $c+d-1=\delta$ and $s+n=\gamma$, we define the number $TI^{(\gamma,\delta)}(f,g)(x)$ as follows:*

$$TI^{(\gamma,\delta)}(f,g)(x)=I^{(\gamma,\delta)}(fg)(x)-\frac{\Gamma(c)\Gamma(d)}{x^{\gamma+\delta-1}\Gamma(\delta)}\left[I^{(n,c)}(f)(x)\right]\left[I^{(s,d)}(g)(x)\right]. \tag{12}$$

Suppose that there are the non-negative numbers $M_{u_x},m_{u_x},M_{v_x},m_{v_x}$ such as

$$m_{u_x}\le u_x(t)\le M_{u_x},$$
$$m_{v_x}\le v_x(t)\le M_{v_x},$$

for all $t\in[0,x]$. Then

$$\left|TI^{(\gamma,\delta)}(f,g)(x)\right|\le\frac{x^{1-\gamma-\delta}}{4\Gamma(\delta)}(M_{u_x}-m_{u_x})(M_{v_x}-m_{v_x}). \tag{13}$$

Proof. First of all, let us compute

$$\int_0^x t^{\gamma}(x-t)^{\delta-1}f(t)g(t)dt=\int_0^x(x-t)^{c-1+d-1}f(t)g(t)dt$$

$$=\int_0^x\left[(x-t)^{c-1}t^nf(t)\right]\left[(x-t)^{d-1}t^sg(t)\right]dt=\int_0^x u_x(t)v_x(t)dt.$$

The functions u_x and v_x defined above are integrable and, by using (2) on $[0,x]$, one gets

$$\left|\frac{1}{x}\int_0^x u_x(t)v_x(t)dt - \left(\frac{1}{x}\int_0^x u_x(t)dt\right)\left(\frac{1}{x}\int_0^x v_x(t)dt\right)\right|$$

$$\leq \frac{1}{4}(M_{u_x} - m_{u_x})(M_{v_x} - m_{v_x}).$$

Now, we remark that

$$I^{(\gamma,\delta)}(fg)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x u_x(t)v_x(t)dt,$$

$$\int_0^x u_x(t)dt = \int_0^x (x-t)^{c-1}t^n f(t)dt = \frac{\Gamma(c)}{x^{-n-c}}I^{(n,c)}f(x),$$

$$\int_0^x v_x(t)dt = \int_0^x (x-t)^{d-1}t^s f(t)dt = \frac{\Gamma(d)}{x^{-s-d}}I^{(s,d)}g(x),$$

and introduce these results in the previous inequality. The required inequality (13) is obtained after dividing the resulting inequality by $\frac{x^{-\gamma-\delta+1}}{\Gamma(\delta)}$ and using these identities. □

Remark 4. *Inequality* (13) *may be proved using the corresponding Grüss type inequality* (7) *for the left-sided Riemann-Liouville fractional integral by the same type of splitting reasoning.*

Corollary 1. *Suppose that functions f and g are continuous on $[0,\infty)$, $\gamma > 0$, $\delta > 0$ and $x \in [0,\infty)$. For every pairs of numbers (c,d) and (s,n), $c > 0$, $d > 0$, $s > 0$ and $n > 0$ chosen such as $c + d - 1 = \delta$ and $s + n = \gamma$, we define pair of functions (F,G), $F,G : [0,x] \to \mathbb{R}$, $F(t) = t^n f(t)$, $G(t) = t^s g(t)$, $t \in [0,x]$, and the number $T_{IJ}I^{(\gamma,\delta)}(f,g)(x)$ as follows:*

$$T_{IJ}I^{(\gamma,\delta)}(f,g)(x) = \frac{1}{x}I^{(\gamma,\delta)}(fg)(x) - \frac{\Gamma(c)\Gamma(d)}{x^{-\gamma-\delta}\Gamma(\delta)}\left[\frac{1}{x}J^c_{0+}(x^n f(x))\right]\left[\frac{1}{x}J^d_{0+}(x^s g(x))\right]. \tag{14}$$

Suppose that there are the non-negative numbers M_F, m_F, M_G, m_G such as

$$m_F \leq F(t) \leq M_F,$$

$$m_G \leq G(t) \leq M_G,$$

for all $t \in [0,x]$. Then

$$\left|T_{IJ}I^{(\gamma,\delta)}(f,g)(x)\right| \leq \frac{x^{-\gamma-\delta}}{4\Gamma(\delta)}(M_F - m_F)(M_G - m_G). \tag{15}$$

The proof follows the same reasoning as the previous theorem, but inequality (7) is used.

Remark 5. *Inequalities* (13) *and* (15) *are sharp. Indeed, if either $f \equiv 0$ or $g \equiv 0$ then* (13) *and* (15) *become identities.*

4. Grüss type inequalities in the framework of Katugampola fractional integration

Katugampola's operators are generalizations of the Riemann-Liouville fractional integral operators. Udita Katugampola considered the following iterative process in 2011 ([11]):

$$\int_a^x t_1^\rho dt_1 \int_a^{t_1} t_2^\rho dt_2 ... \int_a^{t_{n-1}} t_n^\rho f(t_n) dt_n = \frac{(\rho+1)^{1-n}}{(n-1)!} \int_a^x (t^{\rho+1} - \tau^{\rho+1})^{n-1} \tau^\rho f(\tau) d\tau$$

for $n \in \mathbb{N}^*$. This generates Katugampola's concept of fractional integral, defined in [11] and also in [12].

Definition 3. Let $f \in L[a,b]$.

(1) The left-sided Katugampola fractional integral ${}^\rho I_a^\alpha f$ of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ is defined by

$$ {}^\rho I_{a+}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1} f(t)}{(x^\rho - t^\rho)^{1-\alpha}} dt, \quad x > a. \tag{16}$$

(2) The right-sided Katugampola fractional integral ${}^\rho I_{b-}^\alpha f$ of order $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$ is defined by

$$ {}^\rho I_{b-}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1} f(t)}{(t^\rho - x^\rho)^{1-\alpha}} dt, \quad x < b. \tag{17}$$

Remark 6. If $\rho = 1$ and $\alpha \in [0, +\infty)$ then the Katugampola fractional integrals become Riemann-Liouville fractional integrals.

In this section we consider two functions $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ and suppose that $\rho > 0$ and $\alpha \in [0, +\infty)$. For every $x \in [a,b]$, $s > 0$ and $n > 0$ we define the following pairs of functions: $(u_{x,i}, v_{x,i})$, $u_{x,i} : [a,x] \to \mathbb{R}$, $v_{x,i} : [a,x] \to \mathbb{R}$, $i \in \{1,2\}$, as follows:

$$u_{x,1}(t) = t^{\rho-1}(x^\rho - t^\rho)^{s-1} f(t), \qquad v_{x,1}(t) = (x^\rho - t^\rho)^{n-1} g(t),$$
$$u_{x,2}(t) = (x^\rho - t^\rho)^{s-1} f(t), \qquad v_{x,2}(t) = t^{\rho-1}(x^\rho - t^\rho)^{n-1} g(t).$$

In the same hypothesis, we define the pair of functions $(p_{x,i}, q_{x,i})$, $p_{x,i} : [x,b] \to \mathbb{R}$, $q_{x,i} : [x,b] \to \mathbb{R}$, $i \in \{1,2\}$, as follows:

$$p_{x,1}(t) = t^{\rho-1}(t^\rho - x^\rho)^{s-1} f(t), \qquad q_{x,1}(t) = (t^\rho - x^\rho)^{n-1} g(t),$$
$$p_{x,2}(t) = (t^\rho - x^\rho)^{s-1} f(t), \qquad q_{x,2}(t) = t^{\rho-1}(t^\rho - x^\rho)^{n-1} g(t).$$

Theorem 4. *Suppose that functions f and g are continuous on $[a,b]$, $\rho > 0$, $\alpha > 0$ and $x \in (a,b)$. For every pair of numbers (s,n), $s > 0$ and $n > 0$ chosen such as $s+n-1 = \alpha$, we define the numbers $T_i\, {}^\rho I_{a+}^\alpha (f,g)(x)$ and $T_1\, {}^\rho I_{b-}^\alpha (f,g)(x)$, $i \in \{1,2\}$ as follows:*

$$T_1\, {}^\rho I_{a+}^\alpha (f,g)(x) = \frac{1}{x-a}\, {}^\rho I_{a+}^\alpha (fg)(x) - \tag{18}$$
$$\frac{\Gamma(s)\Gamma(n)}{\Gamma(\alpha)} \left(\frac{1}{x-a}\, {}^\rho I_{a+}^s (f)(x) \right) \left(\frac{1}{x-a}\, {}^\rho I_{a+}^n \left(\frac{g(x)}{x^{\rho-1}} \right) \right),$$

$$T_2\, {}^\rho I_{a+}^\alpha (f,g)(x) = \frac{1}{x-a}\, {}^\rho I_{a+}^\alpha (fg)(x) - \tag{19}$$

$$\frac{\Gamma(s)\Gamma(n)}{\Gamma(\alpha)}\left(\frac{1}{x-a}\,{}^{\rho}I^{s}_{a+}\left(\frac{f(x)}{x^{\rho-1}}\right)\right)\left(\frac{1}{x-a}\,{}^{\rho}I^{n}_{a+}(g)(x)\right),$$

Suppose that there are the non-negative numbers $M_{u_{xi}}, m_{u_{xi}}, M_{v_{xi}}, m_{v_{xi}}$ *such as*

$$m_{u_{xi}} \le u_{xi}(t) \le M_{u_{xi}},$$

$$m_{v_{xi}} \le v_{xi}(t) \le M_{v_{xi}},$$

for all $t \in [a, x]$. *Then*

$$\left|T_i\,{}^{\rho}I^{\alpha}_{a+}(f,g)(x)\right| \le \frac{\rho^{1-\alpha}}{4\Gamma(\alpha)}(M_{u_{xi}} - m_{u_{xi}})(M_{v_{xi}} - m_{v_{xi}}), \tag{20}$$

for $i \in \{1, 2\}$.

Proof. We begin by computing the left-sided Katugampola integral of the product of the two functions, as follows:

$${}^{\rho}I^{\alpha}_{a+}(fg)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{t^{\rho-1}f(t)g(t)}{(x^{\rho}-t^{\rho})^{1-\alpha}}dt$$

$$= \frac{\rho^{2-s-n}}{\Gamma(\alpha)}\int_a^x \frac{t^{\rho-1}f(t)g(t)}{(x^{\rho}-t^{\rho})^{2-s-n}}dt$$

$$= \frac{\rho^{1-s}\rho^{1-n}}{\Gamma(\alpha)}\int_a^x \frac{t^{\rho-1}f(t)}{(x^{\rho}-t^{\rho})^{1-s}}\frac{g(t)}{(x^{\rho}-t^{\rho})^{1-n}}dt$$

$$= \frac{\rho^{1-s}\rho^{1-n}}{\Gamma(\alpha)}\int_a^x u_{x1}(t)v_{x1}(t)dt.$$

We use Grüss inequality (2) to evaluate this integral and get

$$\left|\frac{1}{x-a}\int_a^x u_{x1}(t)v_{x1}(t)dt - \left(\frac{1}{x-a}\int_a^x u_{x1}(t)dt\right)\left(\frac{1}{x-a}\int_a^x v_{x1}(t)dt\right)\right|$$

$$\le \frac{1}{4}(M_{u_{xi}} - m_{u_{xi}})(M_{v_{xi}} - m_{v_{xi}}).$$

But we must remark that

$$\int_a^x u_{x1}(t)dt = \int_a^x t^{\rho-1}(x^{\rho}-t^{\rho})^{s-1}f(t)dt = \frac{\Gamma(s)}{\rho^{1-s}}\,{}^{\rho}I^{s}_{a+}(f)(x),$$

and also, in case of the second integral from the product,

$$\int_a^x v_{x1}(t)dt = \int_a^x t^{\rho-1}(x^{\rho}-t^{\rho})^{n-1}\frac{g(t)}{t^{\rho-1}}dt = \frac{\Gamma(n)}{\rho^{1-n}}\,{}^{\rho}I^{s}_{a+}\left(\frac{g(x)}{\rho^{1-n}}\right).$$

Now, we replace these results in the previous inequality and divide the result by $\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}$, obtaining the required result (18). The proof of (19) is similar. □

Theorem 5. *Suppose that functions f and g are continuous on $[a,b]$, $\rho > 0$, $\alpha > 0$ and $x \in (a,b)$. For every pair of numbers (s,n), $s > 0$ and $n > 0$ chosen such as $s+n-1=\alpha$, we define the numbers $T_i\,{}^\rho I^\alpha_{a+}(f,g)(x)$ and $T_1\,{}^\rho I^\alpha_{b-}(f,g)(x)$, $i \in \{1,2\}$ as follows:*

$$T_1\,{}^\rho I^\alpha_{b-}(f,g)(x) = \frac{1}{b-x}\,{}^\rho I^\alpha_{b-}(fg)(x) - \tag{21}$$

$$\frac{\Gamma(s)\Gamma(n)}{\Gamma(\alpha)}\left(\frac{1}{b-x}\,{}^\rho I^s_{b-}(f)(x)\right)\left(\frac{1}{b-x}\,{}^\rho I^n_{b-}\left(\frac{g(x)}{x^{\rho-1}}\right)\right),$$

$$T_2\,{}^\rho I^\alpha_{b-}(f,g)(x) = \frac{1}{b-x}\,{}^\rho I^\alpha_{b-}(fg)(x) - \tag{22}$$

$$\frac{\Gamma(s)\Gamma(n)}{\Gamma(\alpha)}\left(\frac{1}{b-x}\,{}^\rho I^s_{b-}\left(\frac{f(x)}{x^{\rho-1}}\right)\right)\left(\frac{1}{b-x}\,{}^\rho I^n_{b-}(g)(x)\right).$$

Suppose that there are the non-negative numbers $M_{p_{xi}}, m_{p_{xi}}, M_{q_{xi}}, m_{q_{xi}}$ such as

$$m_{p_{xi}} \le p_{xi}(t) \le M_{p_{xi}},$$

$$m_{q_{xi}} \le q_{xi}(t) \le M_{q_{xi}},$$

for all $t \in [a,x]$. Then

$$\left|T_i\,{}^\rho I^\alpha_{b-}(f,g)(x)\right| \le \frac{\rho^{1-\alpha}}{4\Gamma(\alpha)}(M_{p_{xi}} - m_{p_{xi}})(M_{q_{xi}} - m_{q_{xi}}), \tag{23}$$

for $i \in \{1,2\}$.

Proof. The same type of reasoning as in case of the left-sided Katugampola fractional integral is used to prove the bounds in case of the corresponding right-sided operator. □

Remark 7. *If $x = b$ and $y = a$ then inequalities* (18) *and* (21) *coincide. Also, inequalities* (19) *and* (22) *coincide in this case.*

Remark 8. *If $\alpha = \rho = 1$ then inequalities* (18), (21), (19) *and* (22) *become inequalities* (7)*), respectively* (8)*, on the corresponding intervals.*

Remark 9. *Inequalities* (18), (21), (19) *and* (22) *are sharp. For example, if either $f \equiv 0$ or $g \equiv 0$ then all these inequalities become identities.*

5. Conclusions

Grüss type inequalities are derived replacing the Riemann integral by various types of fractional integrals. We consider the framework provided by the Riemann-Liouville fractional integration, Erdélyi-Kober fractional integral on bounded interval and Katugampola fractional integral. The proofs relate all the new inequalities to the classical Grüss inequality by means of a splitting procedure, used at the level of the integration argument. These inequalities are useful in many procedures of assessing the size of the error in approximation of pairs of functions with known mutual behaviour. The reliability polynomials of two dual minimal two-terminal networks are examples of functions of this kind, because they have known complementarity properties with respect to integral operators.

References

[1] Bohner, M., Matthews, T., Tuna, A., Diamond-alpha Grüss type inequalities on time scales, *Int. J. Dyn. Syst. Diff. Eq.*, **3** (2011), 1/2, 234-247.

[2] Cristescu, G., Lupşa, L., *Non-connected Convexities and Applications*, Kluwer Academic Publishers, Dordrecht, Holland, 2002.

[3] Cristescu, G., *New Gruüss type inequalities in fractional integration*, ISREIE 2018, Mathematics and Computer Science, Proceedings, Arad, 2018, 25-29.

[4] Cristescu, G., Drăgoi, V.F., Cubic spline approximation of the reliability polynomials of two dual hammock networks, *Transylvanian Journal of Mathematics and Mechanics*, **11** (2019), 1-2, 77-90.

[5] Cristescu, G., Drăgoi, V.F., Efficient approximation of two-terminal networks reliability polynomials using cubic splines, *IEEE Transactions on Reliability*, **70** (2021), 3, 1193-1203.

[6] Dahmani, Z., Tabharit, L., On weighted inequalities Grüss type inequalities via fractional integrals, *J. Advanc. Research Sci. Comput.*, **2** (2010), 1, 40-45.

[7] Dahmani, Z., Benzidane, A., New Weighted Grüss Type Inequalities Via (α, β) Fractional q-Integral Inequalities, *Int. J. of Innovation and Applied Studies*, **1** (2012), 1, 76-83.

[8] Dragomir, S.S., Wang, S., An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Computers Math. Applic.*, **33** (1997), 11, 16-20.

[9] Erdélyi, A., On fractional integration and its application to the theory of Hankel transforms, *The Quarterly Journal of Mathematics*, Second Series, **11** (1940), 293-303.

[10] Grüss, G., Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$, *Math. Z.*, **39** (1934), 1, 215-226.

[11] Katugampola, U.N., New approach to a generalized fractional integral, *Applied Math. Comp.*, **218** (2011), 860-865.

[12] Katugampola, U.N. , New approach to generalized fractional derivatives, *B. Math. Anal. Appl.*, **6** (2014), 4, 1-15.

[13] Kiryakova, V.S., *Generalized Fractional Calculus and Applications*, Longman, Harlow, UK, 1993.

[14] Kober, H., On fractional integrals and derivatives, *The Quarterly Journal of Mathematics (Oxford Series)*, **11** (1940), 1, 193-211.

[15] Mitrinović, D.S., Pečarić, J.E., Fink, A.M., *Classical and new Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, Holland, 1993.

[16] Pachpatte, B.G., New Inequalities of Ostrowski-Grüss type, *Fasciculi Mathematici*, 2007, 38, 97-104.

Aurel Vlaicu University of Arad
Department of Mathematics and Computer Sciences
Str. Revoluţiei, No. 77, RO-310130 Arad, România
Email address: gabriela.cristescu@gmail.com

Polytechnic University of Timişoara
Department of Computers and Information Technology
Bd. Vasile Pârvan, No. 2, RO-300223 Timişoara, România
Email address: hoarasorin@gmail.com