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THREE VARIABLES FRACTIONAL ANALOGUES OF TRAPEZIUM LIKE INEQUALITIES

MUHAMMAD UZAIR AWAN[1,*] MARCELA V. MIHAI[2] MUHAMMAD ZAKRIA JAVED[1]
MUHAMMAD ASLAM NOOR[3] KHALIDA INAYAT NOOR[3]

Abstract. The aim of this paper is to derive some new fractional analogues of trapezium like inequalities essentially using a new three variable extension of Riemann-Liouville fractional integrals. In order to establish the main results of the paper we use the three variable convexity property of the functions.

1. Introduction and Preliminaries

Convexity is very old notion as its history can be traced back to Archimedean. Although it is very simple in nature but has very interesting applications in various areas of pure and applied sciences. During the last century several successful attempts have been made in generalizing the classical concepts of convex sets and convex functions using novel and innovative ideas. Dragomir [4] extended the notion of convexity on coordinates. The class of coordinated convex functions is defined as:

Definition 1. *Consider a rectangle $D := [a, b] \times [c, d]$ in $\mathbb{R}^2$. A function $\mathcal{F} : D \to \mathbb{R}$ is said to be coordinated convex function on D, if*

$$\mathcal{F}(tx + (1-t)y, ru + (1-r)w)$$
$$\leq tr\mathcal{F}(x, u) + t(1-r)\mathcal{F}(x, w) + r(1-t)\mathcal{F}(y, u) + (1-t)(1-r)\mathcal{F}(y, w),$$

whenever $x, y \in [a, b]$, $u, w \in [c, d]$ and $t, r \in [0, 1]$.

After an introduction of new class it is a natural problem to discuss its properties. It is also a known fact that theory of convexity and theory of inequality are closely related to each other. So with the introduction of every new class inequalities experts try to derive new versions of classical inequalities for this new class. Same was done by Dragomir [4], he also derived some new variants of Hermite-Hadamard's inequality involving coordinated convex functions. Since then many authors have shown their keen interest in this direction and consequently, other generalizations of convex functions have been extended on coordinates and simultaneously associated inequalities for these classes have also been obtained, for example, see [1, 3, 6, 7, 10].
Sarikaya et al. [9] used the concepts of fractional calculus and obtained a new fractional analogue of Hermite-Hadamard's inequality. This idea opened a new avenue of research in the field of inequalities involving convex functions and immense research has been done in this direction. Sarikaya [8] extended the notions of fractional calculus on two dimensions and obtained two variable fractional analogues of Hermite-Hadamard's inequality. The main motivation of this paper is to develop some three variable fractional analogues of Hermite-Hadamard's inequality essentially using three variable convexity property of the functions. In order to obtain the main results of the paper, we first extend the classical

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notions from fractional calculus for three variables. We hope that the ideas and techniques of this paper will inspire interested readers working in this field.

2. Three Variable Analogue of Riemann-Liouville Fractional Integrals

In this section, we define the three variable analogue of Riemann-Liouville fractional integrals. The classical versions of left and right Riemann-Liouville fractional integrals are defined as:

Definition 2. *Let $F \in L_1[a,b]$. Then the Riemann -Liouville integrals of order $\alpha > o$ with $a > 0$ are defined as*

$$J_{a^+}^{\alpha} F(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} F(t)\mathrm{d}t, \quad x > a$$

and

$$J_{b^-}^{\alpha} F(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} F(t)\mathrm{d}t, \quad x < b$$

respectively. Here $\Gamma(\alpha)$ Gamma function .

We now extend this notion for three variables.

Definition 3. *Let $F \in L_1([a,b] \times [c,d] \times [e,f])$. Then Riemann-Liouville integrals of order $\alpha, \beta, \gamma > 0$ are defined as*

$$J_{a^+,c^+,e^+}^{\alpha,\beta,\gamma} F(x,y,z) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_a^x \int_c^y \int_e^z (x-t)^{\alpha-1}(y-r)^{\beta-1}(z-s)^{\gamma-1} F(t,r,s)\mathrm{d}s\mathrm{d}r\mathrm{d}t,$$
$$x > a, y > c, z > e,$$

$$J_{a^+,c^+,f^-}^{\alpha,\beta,\gamma} F(x,y,z) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_a^x \int_c^y \int_z^f (x-t)^{\alpha-1}(y-r)^{\beta-1}(s-z)^{\gamma-1} F(t,r,s)\mathrm{d}s\mathrm{d}r\mathrm{d}t,$$
$$x > a, y > c, z < f,$$

$$J_{a^+,d^-,f^-}^{\alpha,\beta,\gamma} F(x,y,z) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_a^x \int_y^d \int_z^f (x-t)^{\alpha-1}(r-y)^{\beta-1}(s-z)^{\gamma-1} F(t,r,s)\mathrm{d}s\mathrm{d}r\mathrm{d}t,$$
$$x > a, y < d, z < f,$$

$$J_{a^+,d^-,e^+}^{\alpha,\beta,\gamma} F(x,y,z) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_a^x \int_y^d \int_e^z (x-t)^{\alpha-1}(r-y)^{\beta-1}(z-s)^{\gamma-1} F(t,r,s)\mathrm{d}s\mathrm{d}r\mathrm{d}t,$$
$$x > a, y < d, z > e,$$

$$J_{b^-,c^+,f^-}^{\alpha,\beta,\gamma} F(x,y,z) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_x^b \int_c^y \int_z^f (t-x)^{\alpha-1}(r-y)^{\beta-1}(s-z)^{\gamma-1} F(t,r,s)\mathrm{d}s\mathrm{d}r\mathrm{d}t,$$
$$x < b, y > c, z < f,$$

$$J^{\alpha,\beta,\gamma}_{b^-,d^-,e^+}F(x,y,z) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_x^b\int_y^d\int_e^z (t-x)^{\alpha-1}(r-y)^{\beta-1}(z-s)^{\gamma-1}F(t,r,s)\mathrm{d}s\mathrm{d}r\mathrm{d}t,$$
$$x<b, y<d, z>e,$$

$$J^{\alpha,\beta,\gamma}_{b^-,c^+,e^+}F(x,y,z) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_x^b\int_c^y\int_e^z (t-x)^{\alpha-1}(r-y)^{\beta-1}(z-s)^{\gamma-1}F(t,r,s)\mathrm{d}s\mathrm{d}r\mathrm{d}t,$$
$$x<b, y>c, z>e,$$

and

$$J^{\alpha,\beta,\gamma}_{b^-,d^-,f^-}F(x,y,z) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_x^b\int_y^d\int_z^f (t-x)^{\alpha-1}(r-y)^{\beta-1}(s-z)^{\gamma-1}F(t,r,s)\mathrm{d}s\mathrm{d}r\mathrm{d}t,$$
$$x<b, y<d, z<f,$$

respectively. Here, Γ *is the Gamma function ,*

$$J^{0,0,0}_{a^+,c^+,e^+}F(x,y,z) = J^{0,0,0}_{a^+,c^+,f^-}F(x,y,z) = J^{0,0,0}_{a^+,d^-,e^+}F(x,y,z) = J^{0,0,0}_{a^+,d^-,f^-}F(x,y,z)$$
$$= J^{0,0,0}_{b^-,d^-,f^-}F(x,y,z) = J^{0,0,0}_{b^-,d^-,e^+}F(x,y,z) = J^{0,0,0}_{b^-,c^+,f^-}F(x,y,z) = J^{0,0,0}_{b^-,c^+,e^+}F(x,y,z)$$
$$= F(x,y,z).$$

Similarly we introduce the following fractional integrals of double order in three variables.

$$J^{\alpha,\beta}_{a^+,c^+}F\left(x,y,\frac{e+f}{2}\right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-t)^{\alpha-1}(y-r)^{\beta-1}F\left(t,r,\frac{e+f}{2}\right)\mathrm{d}r\mathrm{d}t,$$
$$x>a, y>c,$$

$$J^{\alpha,\beta}_{a^+,d^-}F(x,y,\frac{e+f}{2}) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (x-t)^{\alpha-1}(r-y)^{\beta-1}F\left(t,r,\frac{e+f}{2}\right)\mathrm{d}r\mathrm{d}t,$$
$$x>a, y<d,$$

$$J^{\alpha,\beta}_{a^+,e^+}F(x,\frac{c+d}{2},z) = \frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_a^x\int_e^z (x-t)^{\alpha-1}(z-s)^{\gamma-1}F\left(t,\frac{c+d}{2},s\right)\mathrm{d}s\mathrm{d}t,$$
$$x>a, z>e,$$

$$J^{\alpha,\beta}_{a^+,f^-}F\left(x,\frac{c+d}{2},z\right) = \frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_a^x\int_z^f (x-t)^{\alpha-1}(s-z)^{\gamma-1}F\left(t,\frac{c+d}{2},s\right)\mathrm{d}s\mathrm{d}t,$$
$$x>a, z<f,$$

$$J^{\alpha,\beta}_{b^-,c^+}F\left(x,y,\frac{e+f}{2}\right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (t-x)^{\alpha-1}(y-r)^{\beta-1}F\left(t,r,\frac{e+f}{2}\right)\mathrm{d}r\mathrm{d}t,$$
$$x<b, y>c,$$

$$J^{\alpha,\beta}_{b^-,d^-}F\left(x,y,\frac{e+f}{2}\right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(r-y)^{\beta-1}F\left(t,r,\frac{e+f}{2}\right)\mathrm{d}r\mathrm{d}t,$$
$$x<b, y<d,$$

$$J^{\alpha,\beta}_{b^-,e^+}F\left(x,\frac{c+d}{2},z\right) = \frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_x^b\int_e^z (t-x)^{\alpha-1}(z-s)^{\gamma-1}F\left(t,\frac{c+d}{2},s\right)\mathrm{d}s\mathrm{d}t,$$
$$x<b, z>e,$$

$$J^{\alpha,\beta}_{b^-,f^-}F\left(x,\frac{c+d}{2},z\right)=\frac{1}{\Gamma(\alpha)\Gamma(\gamma)}\int_x^b\int_z^f(t-x)^{\alpha-1}(s-z)^{\gamma-1}F\left(t,\frac{c+d}{2},s\right)\mathrm{d}s\mathrm{d}t,$$
$$x<b, z<f,$$

$$J^{\alpha,\beta}_{c^+,e^+}F\left(\frac{a+b}{2},y,z\right)=\frac{1}{\Gamma(\beta)\Gamma(\gamma)}\int_c^y\int_e^z(y-r)^{\beta-1}(z-s)^{\gamma-1}F\left(\frac{a+b}{2},r,s\right)\mathrm{d}r\mathrm{d}s,$$
$$y>c, z>e,$$

$$J^{\alpha,\beta}_{c^+,f^-}F\left(\frac{a+b}{2},y,z\right)=\frac{1}{\Gamma(\beta)\Gamma(\gamma)}\int_c^y\int_z^f(y-r)^{\beta-1}(s-z)^{\gamma-1}F\left(\frac{a+b}{2},r,s\right)\mathrm{d}r\mathrm{d}s,$$
$$y>c, z<f,$$

$$J^{\alpha,\beta}_{d^-,e^+}F\left(\frac{a+b}{2},y,z\right)=\frac{1}{\Gamma(\beta)\Gamma(\gamma)}\int_y^d\int_e^z(r-y)^{\beta-1}(z-s)^{\gamma-1}F\left(\frac{a+b}{2},r,s\right)\mathrm{d}r\mathrm{d}s,$$
$$d>y, z>e,$$

$$J^{\alpha,\beta}_{d^-,f^-}F\left(\frac{a+b}{2},y,z\right)=\frac{1}{\Gamma(\beta)\Gamma(\gamma)}\int_y^d\int_z^f(r-y)^{\beta-1}(s-z)^{\gamma-1}F\left(\frac{a+b}{2},r,s\right)\mathrm{d}r\mathrm{d}s,$$
$$d>y, z<f,$$

Similarly we define the fractional integrals of single order in three variables.

$$J^{\alpha}_{a^+}F\left(x,\frac{c+d}{2},\frac{e+f}{2}\right)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}F\left(t,\frac{c+d}{2},\frac{e+f}{2}\right)\mathrm{d}t\quad x>a,$$

$$J^{\alpha}_{b^-}F\left(x,\frac{c+d}{2},\frac{e+f}{2}\right)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}F(t,\frac{c+d}{2},\frac{e+f}{2})\mathrm{d}t\quad x<b,$$

$$J^{\beta}_{c^+}F\left(\frac{a+b}{2},y,\frac{c+d}{2}\right)=\frac{1}{\Gamma(\beta)}\int_c^y(y-r)^{\beta-1}F\left(\frac{a+b}{2},r,\frac{e+f}{2}\right)\mathrm{d}r,\quad y>c,$$

$$J^{\beta}_{d^-}F\left(\frac{a+b}{2},y,\frac{c+d}{2}\right)=\frac{1}{\Gamma(\beta)}\int_y^d(r-y)^{\beta-1}F\left(\frac{a+b}{2},r,\frac{e+f}{2}\right)\mathrm{d}r,\quad y>c,$$

$$J^{\gamma}_{e^+}F(\frac{a+b}{2},\frac{c+d}{2},z)=\frac{1}{\Gamma(\gamma)}\int_e^z(z-s)^{\gamma-1}F(\frac{a+b}{2},\frac{c+d}{2},s)\mathrm{d}s\quad z>e,$$

$$J^{\gamma}_{f^-}F(\frac{a+b}{2},\frac{c+d}{2},z)=\frac{1}{\Gamma(\gamma)}\int_z^f(s-z)^{\gamma-1}F(\frac{a+b}{2},\frac{c+d}{2},s)\mathrm{d}s\quad z<f.$$

Definition 4. *Consider a rectangular box $V:=[a,b]\times[c,d]\times[e,f]$ in $\mathbb{R}^3$. A mapping $F:V\rightarrow\mathbb{R}$ is said to be co-ordinated convex function on V, if*

$$\begin{aligned}&F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\\&\leq(1-t)(1-r)(1-s)F(a,c,e)+(1-t)(1-r)sF(a,c,f)+(1-t)r(1-s)F(a,d,e)\\&+(1-t)rsF(a,d,f)+t(1-r)(1-s)F(b,c,e)+t(1-r)sF(b,c,f)\\&+tr(1-s)F(b,d,e)+trsF(b,d,f),\end{aligned}$$

holds for all $(a,c,e),(b,d,f)\in V$ and $r,s,t\in[0,1]$.

3. Hermite-Hadamard-type inequalities for fractional integral

Firstly we derive Hermite -Hadamard inequality using co-ordinated convex function for three variable.

Theorem 1. *Let $F : V \subset \mathbb{R}^3 \to \mathbb{R}$ be a co-ordinated convex function on $V = [a,b] \times [c,d] \times [e,f]$ in $\mathbb{R}^3$ with $0 \le a < b, 0 \le c < d, 0 \le e < f$ and $F \in L_1(V)$. Then the following inequality holds:*

$$\begin{aligned}
&F\left(\frac{a+b}{2}+\frac{c+d}{2}+\frac{e+f}{2}\right)\\
&\le \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{8(b-a)^\alpha(d-c)^\beta(f-e)^\gamma}\Big[J^{\alpha,\beta,\gamma}_{a^+,c^+,e^+}F(b,d,f)+J^{\alpha,\beta,\gamma}_{a^+,d^-,e^+}F(b,c,f)\\
&+J^{\alpha,\beta,\gamma}_{a^+,c^+,f^-}F(b,d,e)+J^{\alpha,\beta,\gamma}_{a^+,d^-,f^-}F(b,c,e)+J^{\alpha,\beta,\gamma}_{b^-,c^+,e^+}F(a,d,f)\\
&+J^{\alpha,\beta,\gamma}_{b^-,d^-,e^+}F(a,c,f)+J^{\alpha,\beta,\gamma}_{b^-,c^+,f^-}F(a,d,e)++J^{\alpha,\beta,\gamma}_{b^-,d^-,f^-}F(a,c,e)\Big]\\
&\le \frac{F(a,c,e)+F(a,c,f)+F(a,d,e)+F(a,d,f)+F(b,c,e)+F(b,c,f)+F(b,d,e)+F(b,d,f)}{8}.
\end{aligned}$$

Proof. For $t = r = s = \frac{1}{2}$ we have

$$\begin{aligned}
&F\left(\frac{a+b}{2}+\frac{c+d}{2}+\frac{e+f}{2}\right)\\
&\le \frac{1}{8}\left[F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\right.\\
&+F((1-t)a+tb,(1-r)c+rd,se+(1-s)f)\\
&+F((1-t)a+tb,rc+(1-r)d,(1-s)e+sf)\\
&+F((1-t)a+tb,rc+(1-r)d,se+(1-s)f)\\
&+F(ta+(1-t)b,(1-r)c+rd,(1-s)e+sf)\\
&+F(ta+(1-t)b,(1-r)c+rd,se+(1-s)f)\\
&+F(ta+(1-t)b,rc+(1-r)d,(1-s)e+sf)\\
&\left.+F(ta+(1-t)b,rc+(1-r)d,se+(1-s)f)\right].
\end{aligned}$$

Now multiplying both sides of above inequality by $t^{\alpha-1}r^{\beta-1}s^{\gamma-1}$ and integrating with respect to t, r, s on $[0,1]\times[0,1]\times[0,1]$,

$$\begin{aligned}
&\frac{1}{\alpha\beta\gamma}F\left(\frac{a+b}{2}+\frac{c+d}{2}+\frac{e+f}{2}\right)\\
&\le \frac{1}{8}\int_0^1\int_0^1\int_0^1 t^{\alpha-1}r^{\beta-1}s^{\gamma-1}\left[F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\right.\\
&+F((1-t)a+tb,(1-r)c+rd,se+(1-s)f)\\
&+F((1-t)a+tb,rc+(1-r)d,(1-s)e+sf)\\
&+F((1-t)a+tb,rc+(1-r)d,se+(1-s)f)\\
&+F(ta+(1-t)b,(1-r)c+rd,(1-s)e+sf)\\
&+F(ta+(1-t)b,(1-r)c+rd,se+(1-s)f)\\
&+F(ta+(1-t)b,rc+(1-r)d,(1-s)e+sf)\\
&\left.+F(ta+(1-t)b,rc+(1-r)d,se+(1-s)f)\right]\mathrm{d}s\mathrm{d}r\mathrm{d}t
\end{aligned}$$

$$\leq \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{8(b-a)^{\alpha}(d-c)^{\beta}(f-e)^{\gamma}}\Big[J^{\alpha,\beta,\gamma}_{a^+,c^+,e^+}F(b,d,f)+J^{\alpha,\beta,\gamma}_{a^+,d^-,e^+}F(b,c,f)$$
$$+J^{\alpha,\beta,\gamma}_{a^+,c^+,f^-}F(b,d,e)+J^{\alpha,\beta,\gamma}_{a^+,d^-,f^-}F(b,c,e)+J^{\alpha,\beta,\gamma}_{b^-,c^+,e^+}F(a,d,f)$$
$$+J^{\alpha,\beta,\gamma}_{b^-,d^-,e^+}F(a,c,f)+J^{\alpha,\beta,\gamma}_{b^-,c^+,f^-}F(a,d,e)++J^{\alpha,\beta,\gamma}_{b^-,d^-,f^-}F(a,c,e)\Big]$$

Multipling the above inequality by $\alpha\beta\gamma$ we get,

$$F\left(\frac{a+b}{2}+\frac{c+d}{2}+\frac{e+f}{2}\right)$$
$$\leq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)^{\alpha}(d-c)^{\beta}(f-e)^{\gamma}}\Big[J^{\alpha,\beta,\gamma}_{a^+,c^+,e^+}F(b,d,f)+J^{\alpha,\beta,\gamma}_{a^+,d^-,e^+}F(b,c,f)$$
$$+J^{\alpha,\beta,\gamma}_{a^+,c^+,f^-}F(b,d,e)+J^{\alpha,\beta,\gamma}_{a^+,d^-,f^-}F(b,c,e)+J^{\alpha,\beta,\gamma}_{b^-,c^+,e^+}F(a,d,f)$$
$$+J^{\alpha,\beta,\gamma}_{b^-,d^-,e^+}F(a,c,f)+J^{\alpha,\beta,\gamma}_{b^-,c^+,f^-}F(a,d,e)++J^{\alpha,\beta,\gamma}_{b^-,d^-,f^-}F(a,c,e)\Big] \tag{1}$$

To prove our second inequality we use co-ordinated convexity

$$F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)$$
$$\leq (1-t)(1-r)(1-s)F(a,c,e)+(1-t)(1-r)sF(a,c,f)+(1-t)r(1-s)F(a,d,e)$$
$$+(1-t)rsF(a,d,f)+t(1-r)(1-s)F(b,c,e)+t(1-r)sF(b,c,f)+tr(1-s)F(b,d,e)$$
$$+trsF(b,d,f). \tag{2}$$

$$F((1-t)a+tb,(1-r)c+rd,se+(1-s)f)$$
$$\leq (1-t)(1-r)sF(a,c,e)+(1-t)(1-r)(1-s)F(a,c,f)+(1-t)rsF(a,d,e)$$
$$+(1-t)r(1-s)F(a,d,f)+t(1-r)sF(b,c,e)+t(1-r)(1-s)F(b,c,f)+trsF(b,d,e)$$
$$+tr(1-s)F(b,d,f). \tag{3}$$

$$F((1-t)a+tb,rc+(1-r)d,(1-s)e+sf)$$
$$\leq (1-t)r(1-s)F(a,c,e)+(1-t)rsF(a,c,f)+(1-t)(1-r)(1-s)F(a,d,e)$$
$$+(1-t)(1-r)sF(a,d,f)+tr(1-s)F(b,c,e)+trsF(b,c,f)+t(1-r)(1-s)F(b,d,e)$$
$$+t(1-r)sF(b,d,f). \tag{4}$$

$$F((1-t)a+tb,rc+(1-r)d,se+(1-s)f)$$
$$\leq (1-t)rs)F(a,c,e)+(1-t)r(1-s)sF(a,c,f)+(1-t)(1-r)sF(a,d,e)$$
$$+(1-t)(1-r)(1-s)F(a,d,f)+trsF(b,c,e)+tr(1-s)F(b,c,f)+t(1-r)sF(b,d,e)$$
$$+t(1-r)(1-s)F(b,d,f). \tag{5}$$

$$F(ta+(1-t)b,(1-r)c+rd,(1-s)e+sf)$$
$$\leq t(1-r)(1-s)F(a,c,e)+t(1-r)sF(a,c,f)+tr(1-s)F(a,d,e)$$
$$+trsF(a,d,f)+(1-t)(1-r)(1-s)F(b,c,e)+(1-t)(1-r)sF(b,c,f)$$
$$+(1-t)r(1-s)F(b,d,e)+(1-t)rsF(b,d,f). \tag{6}$$

$$F(ta+(1-t)b,(1-r)c+rd,se+(1-s)f)$$
$$\leq t(1-r)sF(a,c,e)+t(1-r)(1-s)F(a,c,f)+trsF(a,d,e)$$
$$+tr(1-s)F(a,d,f)+(1-t)(1-r)sF(b,c,e)+(1-t)(1-r)(1-s)F(b,c,f)$$
$$+(1-)trsF(b,d,e)+(1-t)r(1-s)F(b,d,f). \tag{7}$$

$$\begin{aligned}
&F(ta+(1-t)b, rc+(1-r)d, (1-s)e+sf) \\
&\leq tr(1-s)F(a,c,e)+trsF(a,c,f)+t(1-r)(1-s)F(a,d,e) \\
&+t(1-r)sF(a,d,f)+(1-t)r(1-s)F(b,c,e)+(11-t)rsF(b,c,f) \\
&+(1-t)(1-r)(1-s)F(b,d,e)+(1-t)(1-r)sF(b,d,f). \qquad (8)
\end{aligned}$$

$$\begin{aligned}
&F(ta+(1-t)b, rc+(1-r)d, se+(1-s)f) \\
&\leq trsF(a,c,e)+tr(1-s)sF(a,c,f)+t(1-r)sF(a,d,e) \\
&+t(1-r)(1-s)F(a,d,f)+(1-t)rsF(b,c,e)+(1-t)r(1-s)F(b,c,f) \\
&+(1-t)(1-r)sF(b,d,e)+(1-t)(1-r)(1-s)F(b,d,f). \qquad (9)
\end{aligned}$$

Another inequalty formed by adding $(2)-(9)$ and now multiplying the both sides of newly formed inequality by $t^{\alpha-1}r^{\beta-1}s^{\gamma-1}$ and integrating with repect to t, r, s on $[0,1]\times[0,1]\times[0,1]$ we get

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{8(b-a)^{\alpha}(d-c)^{\beta}(f-e)^{\gamma}}\Big[J^{\alpha,\beta,\gamma}_{a^+,c^+,e^+}F(b,d,f)+J^{\alpha,\beta,\gamma}_{a^+,d^-,e^+}F(b,c,f) \\
&+J^{\alpha,\beta,\gamma}_{a^+,c^+,f^-}F(b,d,e)+J^{\alpha,\beta,\gamma}_{a^+,d^-,f^-}F(b,c,e)+J^{\alpha,\beta,\gamma}_{b^-,c^+,e^+}F(a,d,f) \\
&+J^{\alpha,\beta,\gamma}_{b^-,d^-,e^+}F(a,c,f)+J^{\alpha,\beta,\gamma}_{b^-,c^+,f^-}F(a,d,e)++J^{\alpha,\beta,\gamma}_{b^-,d^-,f^-}F(a,c,e)\Big] \\
&\leq\frac{F(a,c,e)+F(a,c,f)+F(a,d,e)+F(a,d,f)+F(b,c,e)+F(b,c,f)+F(b,d,e)+F(b,d,f)}{8}. \qquad (10)
\end{aligned}$$

By comparing the (1) and (10) we get the required result. This completes the proof. □

Lemma 1. *Le $F: V\subset\mathbb{R}^3\to\mathbb{R}$ be a partial differentiable mapping on $V =: [0,1]\times[0,1]\times[0,1]$ In $\mathbb{R}^3$ with $0\leq a<,b, 0\leq c<d, 0\leq e<f$. If $\frac{\partial^3 F}{\partial t\partial r\partial s}\in L_1(V)$, then the following inequality holds:*

$$\begin{aligned}
&\frac{F(a,c,e)+F(a,c,f)+F(a,d,e)+F(a,d,f}{8} \\
&+\frac{F(b,c,e)+F(b,c,f)+F(b,d,e)+F(b,d,f)}{8} \\
&-\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{8(b-a)^{\alpha}(d-c)^{\beta}(f-e)^{\gamma}}\Big[J^{\alpha,\beta,\gamma}_{a^+,c^+,e^+}F(b,d,f)+J^{\alpha,\beta,\gamma}_{a^+,d^-,e^+}F(b,c,f) \\
&+J^{\alpha,\beta,\gamma}_{a^+,c^+,f^-}F(b,d,e)+J^{\alpha,\beta,\gamma}_{a^+,d^-,f^-}F(b,c,e)+J^{\alpha,\beta,\gamma}_{b^-,c^+,e^+}F(a,d,f) \\
&+J^{\alpha,\beta,\gamma}_{b^-,d^-,e^+}F(a,c,f)+J^{\alpha,\beta,\gamma}_{b^-,c^+,f^-}F(a,d,e)++J^{\alpha,\beta,\gamma}_{b^-,d^-,f^-}F(a,c,e)\Big]+A-B
\end{aligned}$$

$$= \frac{(b-a)(d-c)(f-e)}{8}$$
$$\Big[\int_0^1\int_0^1\int_0^1 t^\alpha r^\beta s^\gamma \frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\mathrm{d}s\mathrm{d}r\mathrm{d}t$$
$$-\int_0^1\int_0^1\int_0^1 t^\alpha r^\beta (1-s)^\gamma \frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\mathrm{d}s\mathrm{d}r\mathrm{d}t$$
$$-\int_0^1\int_0^1\int_0^1 t^\alpha (1-r)^\beta s^\gamma \frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\mathrm{d}s\mathrm{d}r\mathrm{d}t$$
$$+\int_0^1\int_0^1\int_0^1 t^\alpha (1-r)^\beta (1-s)^\gamma \frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\mathrm{d}s\mathrm{d}r\mathrm{d}t$$
$$-\int_0^1\int_0^1\int_0^1 (1-t)^\alpha r^\beta s^\gamma \frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\mathrm{d}s\mathrm{d}r\mathrm{d}t$$
$$+\int_0^1\int_0^1\int_0^1 (1-t)^\alpha r^\beta (1-s)^\gamma \frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\mathrm{d}s\mathrm{d}r\mathrm{d}t$$
$$+\int_0^1\int_0^1\int_0^1 (1-t)^\alpha (1-r)^\beta s^\gamma \frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\mathrm{d}s\mathrm{d}r\mathrm{d}t$$
$$-\int_0^1\int_0^1\int_0^1 (1-t)^\alpha (1-r)^\beta (1-s)^\gamma \frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)$$
$$\mathrm{d}s\mathrm{d}r\mathrm{d}t\Big].$$

Where

$$A = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}\Big[J^{\alpha,\beta}_{b^-,d^-}F(a,c,f) + J^{\alpha,\beta}_{b^-,d^-}F(a,c,e) + J^{\alpha,\beta}_{b^-,c^+}F(a,d,e)$$
$$+J^{\alpha,\beta}_{b^-,c^+}F(a,d,f) + J^{\alpha,\beta}_{a^+,c^+}F(b,d,e) + J^{\alpha,\beta}_{a^+,c^+}F(b,d,f) + J^{\alpha,\beta}_{a^+,d^-}F(b,c,e) + J^{\alpha,\beta}_{a^+,d^-}F(b,c,f)\Big]$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{(b-a)^\alpha(f-e)^\beta}\Big[J^{\alpha,\gamma}_{b^-,f^-}F(a,d,e) + J^{\alpha,\gamma}_{b^-,f^-}F(a,c,e) + J^{\alpha,\gamma}_{b^-,e^+}F(a,d,f)$$
$$+J^{\alpha,\gamma}_{b^-,e^+}F(a,c,f) + J^{\alpha,\gamma}_{a^+,e^+}F(b,d,f) + J^{\alpha,\gamma}_{a^+,e^+}F(b,c,f) + J^{\alpha,\gamma}_{a^+,f^-}F(b,d,e) + J^{\alpha,\gamma}_{a^+,f^-}F(b,c,e)\Big]$$
$$+\frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{(d-c)^\beta(f-e)^\beta}\Big[J^{\beta,\gamma}_{d^-,f^-}F(b,c,e) + J^{\beta,\gamma}_{d^-,f^-}F(a,c,e) + J^{\beta,\gamma}_{d^-,e^+}F(a,c,f)$$
$$+J^{\beta,\gamma}_{d^-,e^+}F(b,c,f) + J^{\beta,\gamma}_{c^+,e^+}F(a,d,f) + J^{\beta,\gamma}_{c^+,e^+}F(b,d,f) + J^{\beta,\gamma}_{c^+,f^-}F(a,d,e) + J^{\beta,\gamma}_{c^+,f^-}F(b,c,e)\Big].$$

and

$$B = \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{a^+}F(b,d,e) + J^\alpha_{a^+}F(b,c,f) + J^\alpha_{a^+}F(b,c,e) + J^\alpha_{a^+}F(b,d,f)\right.$$
$$\left.+J^\alpha_{b^-}F(a,d,f) + J^\alpha_{b^-}F(a,d,e) + J^\alpha_{b^-}F(a,c,f) + J^\alpha_{b^-}F(a,c,e)\right]$$
$$+\frac{\Gamma(\beta+1)}{(d-c)^\beta}\Big[J^\beta_{c^+}F(a,d,f) + J^\beta_{c^+}F(a,d,e) + J^\beta_{c^+}F(b,d,e) + J^\beta_{c^+}F(b,d,f)$$
$$+J^\beta_{d^-}F(b,c,e) + J^\beta_{d^-}F(b,c,f) + J^\beta_{d^-}F(a,c,f) + J^\beta_{d^-}F(a,c,f)\Big]$$
$$+\frac{\Gamma(\gamma+1)}{(f-e)^\gamma}\Big[J^\gamma_{e^+}F(b,d,f) + J^\gamma_{e^+}F(a,d,f) + J^\gamma_{e^+}F(a,c,f) + J^\gamma_{e^+}F(b,c,f)$$
$$+J^\gamma_{f^-}F(b,d,e) + J^\gamma_{f^-}F(b,c,e) + J^\gamma_{f^-}F(a,c,e) + J^\gamma_{f^-}F(a,d,e)\Big].$$

Proof. Consider the right hand side

$$
\begin{aligned}
I &= \frac{(b-a)(d-c)(f-e)}{8}[I_1 - I_2 - I_3 + I_4 - I_5 + I_6 + I_7 - I_8]\\
I_1 &= \int_0^1 \int_0^1 \int_0^1 t^\alpha r^\beta s^\gamma \frac{\partial^3}{\partial t \partial r \partial s} F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\mathrm{d}s\mathrm{d}r\mathrm{d}t\\
&= \int_0^1 \int_0^1 r^\beta s^\gamma \left[\frac{1}{b-a}\frac{\partial^2}{\partial r \partial s}F(b,(1-r)c+rd,(1-s)e+sf)\right.\\
&\left. - \frac{\alpha}{b-a}\int_0^1 t^{\alpha-1}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\mathrm{d}t\right]\mathrm{d}r\mathrm{d}s\\
&= \int_0^1 s^\gamma \left[\frac{1}{(b-a)(d-c)}\frac{\partial}{\partial s}F(b,d,(1-s)e+sf) - \frac{\beta}{(b-a)(d-c)}r^{\beta-1}\right.\\
&\frac{\partial}{\partial s}F(b,(1-r)c+rd,(1-s)e+sf) - \frac{\alpha\int_0^1 t^{\alpha-1}}{(b-a)(d-c)}\frac{\partial}{\partial s}F((1-t)a+tb,d,(1-s)e+sf)\mathrm{d}t\\
&\left. + \frac{\alpha\beta\int_0^1 t^{\alpha-1}}{(b-a)(d-c)}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\mathrm{d}t\mathrm{d}r\right]\mathrm{d}s\\
&= \frac{1}{(b-a)(d-c)(f-e)}F(b,d,f) - \frac{\gamma}{(b-a)(d-c)(f-e)}\int_0^1 s^{\gamma-1}F(b,d,(1-s)e+sf)\mathrm{d}s\\
&- \frac{\beta}{(b-a)(d-c)(f-e)}\int_0^1 r^{\beta-1}F(b,(1-r)c+rd,f)\mathrm{d}r + \frac{\beta\gamma}{(b-a)(d-c)(f-e)}\\
&\int_0^1\int_0^1 r^{\beta-1}s^{\gamma-1}F(b,(1-r)c+rd,(1-s)e+sf)\mathrm{d}r\mathrm{d}s - \frac{\alpha}{(b-a)(d-c)(f-e)}\\
&\int_0^1 t^{\alpha-1}F((1-t)a+tb,d,f)\mathrm{d}t + \frac{\alpha\gamma}{(b-a)(d-c)(f-e)}\int_0^1\int_0^1 t^{\alpha-1}s^{\gamma-1}\\
&F(b,(1-r)c+rd,(1-s)e+sf)\mathrm{d}s\mathrm{d}t + \frac{\alpha\beta}{(b-a)(d-c)(f-e)}\int_0^1\int_0^1\\
&F((1-ta+tb,(1-r)c+rd,f)\mathrm{d}r\mathrm{d}t - \frac{\alpha\beta\gamma}{(b-a)(d-c)(f-e)}\int_0^1\int_0^1\int_0^1 t^{\alpha-1}r^{\beta-1}s^{\gamma-1}\\
&F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\mathrm{d}s\mathrm{d}r\mathrm{d}t\\
&= \frac{F(b,d,f)}{(b-a)(d-c)(f-e)} - \frac{\Gamma(\gamma+1)}{(b-a)(d-c)(f-e)^{\gamma+1}}J^{\gamma}_{f^-}F(b,d,e)\\
&- \frac{\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}(f-e)}J^{\beta}_{d^-}F(b,c,f) - \frac{\Gamma(\alpha+1)}{(b-a)^{(\alpha+1)}(f-e)}J^{\alpha}_{b^-}F(a,d,f)\\
&+ \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{(\alpha+1)}(d-c)^{(\beta+1)}(f-e)}J^{\alpha,\beta}_{b^-,d^-}F(a,c,f) + \frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)(d-c)^{\beta+1}(f-e)^{(\gamma+1)}}\\
&J^{\beta,\gamma}_{d^-,f^-}F(b,c,e) + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{(\alpha+1)}(d-c)(f-e)^{\gamma+1}}J^{\alpha,\gamma}_{b^-,f^-}F(a,d,e)\\
&- \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}(f-e)^{\gamma+1}}J^{\alpha,\beta,\gamma}_{b^-,d^-,f^-}F(a,c,e).
\end{aligned}
$$

Similarly we have

$$I_2 = -\frac{F(b,d,e)}{(b-a)(d-c)(f-e)} + \frac{\Gamma(\gamma+1)}{(b-a)(d-c)(f-e)^{\gamma+1}} J^{\gamma}_{e^+} F(b,d,f)$$
$$+ \frac{\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}(f-e)} J^{\beta}_{d^-} F(b,c,e) + \frac{\Gamma(\alpha+1)}{(b-a)^{(\alpha+1)}(f-e)} J^{\alpha}_{b^-} F(a,d,e)$$
$$- \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{(\alpha+1)}(d-c)^{(\beta+1)}(f-e)} J^{\alpha,\beta}_{b^-,d^-} F(a,c,e) - \frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)(d-c)^{\beta+1}(f-e)^{(\gamma+1)}}$$
$$J^{\beta,\gamma}_{d^-,e^+} F(b,c,f) - \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{(b-a)^{(\alpha+1)}(d-c)(f-e)^{\gamma+1}} J^{\alpha,\gamma}_{b^-,e^+} F(a,d,f)$$
$$+ \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}(f-e)^{\gamma+1}} J^{\alpha,\beta,\gamma}_{b^-,d^-,e^+} F(a,c,f).$$

$$I_3 = -\frac{F(b,c,f)}{(b-a)(d-c)(f-e)} + \frac{\Gamma(\gamma+1)}{(b-a)(d-c)(f-e)^{\gamma+1}} J^{\gamma}_{f^-} F(b,c,e)$$
$$+ \frac{\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}(f-e)} J^{\beta}_{c^+} F(b,d,f) + \frac{\Gamma(\alpha+1)}{(b-a)^{(\alpha+1)}(f-e)} J^{\alpha}_{b^-} F(a,c,f)$$
$$- \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{(\alpha+1)}(d-c)^{(\beta+1)}(f-e)} J^{\alpha,\beta}_{b^-,c^+} F(a,d,f) - \frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)(d-c)^{\beta+1}(f-e)^{(\gamma+1)}}$$
$$J^{\beta,\gamma}_{c^+,f^-} F(b,d,e) - \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{(b-a)^{(\alpha+1)}(d-c)(f-e)^{\gamma+1}} J^{\alpha,\gamma}_{b^-,f^-} F(a,c,e)$$
$$+ \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}(f-e)^{\gamma+1}} J^{\alpha,\beta,\gamma}_{b^-,c^+-,f^-} F(a,d,e).$$

$$I_4 = \frac{F(b,c,e)}{(b-a)(d-c)(f-e)} - \frac{\Gamma(\gamma+1)}{(b-a)(d-c)(f-e)^{\gamma+1}} J^{\gamma}_{e^+} F(b,c,f)$$
$$- \frac{\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}(f-e)} J^{\beta}_{c^+} F(b,d,e) - \frac{\Gamma(\alpha+1)}{(b-a)^{(\alpha+1)}(f-e)} J^{\alpha}_{b^-} F(a,c,e)$$
$$+ \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{(\alpha+1)}(d-c)^{(\beta+1)}(f-e)} J^{\alpha,\beta}_{b^-,c^+} F(a,d,e) + \frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)(d-c)^{\beta+1}(f-e)^{(\gamma+1)}}$$
$$J^{\beta,\gamma}_{c^+,e^+} F(b,d,f) + \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{(b-a)^{(\alpha+1)}(d-c)(f-e)^{\gamma+1}} J^{\alpha,\gamma}_{b^-,c^+} F(a,d,e)$$
$$- \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}(f-e)^{\gamma+1}} J^{\alpha,\beta,\gamma}_{b^-,c^+,e^+} F(a,d,f).$$

$$I_5 = -\frac{F(a,d,f)}{(b-a)(d-c)(f-e)} + \frac{\Gamma(\gamma+1)}{(b-a)(d-c)(f-e)^{\gamma+1}} J^{\gamma}_{f^-} F(a,d,e)$$
$$+ \frac{\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}(f-e)} J^{\beta}_{d^-} F(a,c,f) + \frac{\Gamma(\alpha+1)}{(b-a)^{(\alpha+1)}(f-e)} J^{\alpha}_{a^+} F(b,d,f)$$
$$- \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{(\alpha+1)}(d-c)^{(\beta+1)}(f-e)} J^{\alpha,\beta}_{a^+,d^-} F(b,c,f) - \frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)(d-c)^{\beta+1}(f-e)^{(\gamma+1)}}$$
$$J^{\beta,\gamma}_{d^-,f^-} F(a,c,e) - \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{(b-a)^{(\alpha+1)}(d-c)(f-e)^{\gamma+1}} J^{\alpha,\gamma}_{a^+,f^-} F(b,d,e)$$
$$+ \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}(f-e)^{\gamma+1}} J^{\alpha,\beta,\gamma}_{a^+,d^-,f^-} F(b,c,e).$$

$$
\begin{aligned}
I_6 &= \frac{F(a,d,e)}{(b-a)(d-c)(f-e)} - \frac{\Gamma(\gamma+1)}{(b-a)(d-c)(f-e)^{\gamma+1}} J^{\gamma}_{e^+} F(a,d,f) \\
&- \frac{\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}(f-e)} J^{\beta}_{d^-} F(a,c,e) - \frac{\Gamma(\alpha+1)}{(b-a)^{(\alpha+1)}(f-e)} J^{\alpha}_{a^+} F(b,d,e) \\
&+ \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{(\alpha+1)}(d-c)^{(\beta+1)}(f-e)} J^{\alpha,\beta}_{a^+,d^-} F(b,c,e) + \frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)(d-c)^{\beta+1}(f-e)^{(\gamma+1)}} \\
& J^{\beta,\gamma}_{d^-,e^+} F(a,c,f) + \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{(b-a)^{(\alpha+1)}(d-c)(f-e)^{\gamma+1}} J^{\alpha,\gamma}_{a^+,e^+} F(b,d,f) \\
&- \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}(f-e)^{\gamma+1}} J^{\alpha,\beta,\gamma}_{a^+,d^-,e^+} F(b,c,f).
\end{aligned}
$$

$$
\begin{aligned}
I_7 &= \frac{F(a,c,f)}{(b-a)(d-c)(f-e)} - \frac{\Gamma(\gamma+1)}{(b-a)(d-c)(f-e)^{\gamma+1}} J^{\gamma}_{f^-} F(a,c,e) \\
&- \frac{\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}(f-e)} J^{\beta}_{c^+} F(a,d,f) - \frac{\Gamma(\alpha+1)}{(b-a)^{(\alpha+1)}(f-e)} J^{\alpha}_{a^+} F(b,c,f) \\
&+ \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{(\alpha+1)}(d-c)^{(\beta+1)}(f-e)} J^{\alpha,\beta}_{a^+,c^+} F(b,d,f) + \frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)(d-c)^{\beta+1}(f-e)^{(\gamma+1)}} \\
& J^{\beta,\gamma}_{c^+,f^-} F(a,d,e) + \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{(b-a)^{(\alpha+1)}(d-c)(f-e)^{\gamma+1}} J^{\alpha,\gamma}_{a^+,f^-} F(b,c,e) \\
&- \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}(f-e)^{\gamma+1}} J^{\alpha,\beta,\gamma}_{a^+,c^+,f^-} F(b,d,e).
\end{aligned}
$$

$$
\begin{aligned}
I_8 &= -\frac{F()}{(b-a)(d-c)(f-e)} + \frac{\Gamma(\gamma+1)}{(b-a)(d-c)(f-e)^{\gamma+1}} J^{\gamma}_{e^+} F(a,c,f) \\
&+ \frac{\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}(f-e)} J^{\beta}_{c^+} F(a,d,e) + \frac{\Gamma(\alpha+1)}{(b-a)^{(\alpha+1)}(f-e)} J^{\alpha}_{a^+} F(b,c,e) \\
&- \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{(\alpha+1)}(d-c)^{(\beta+1)}(f-e)} J^{\alpha,\beta}_{a^+,c^+} F(b,d,e) - \frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)(d-c)^{\beta+1}(f-e)^{(\gamma+1)}} \\
& J^{\beta,\gamma}_{c^+,e^+} F(a,d,f) - \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{(b-a)^{(\alpha+1)}(d-c)(f-e)^{\gamma+1}} J^{\alpha,\gamma}_{a^+,e^+} F(b,c,f) \\
&+ \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}(f-e)^{\gamma+1}} J^{\alpha,\beta,\gamma}_{+,c^+,e^+} F(b,d,f).
\end{aligned}
$$

By substituting the values of $I_1, I_2, I_3, I_4, I_5, I_6, I_7$ and I_8 in I we get the required result. This completes the proof. □

Theorem 2. *Let $F : V \subset \mathbb{R}^3 \to \mathbb{R}$ be partial differentiable mapping on $V := [a,b] \times [c,d] \times [e,f]$ with $a < b, c < d, e < f$. If $\frac{\partial^3 F}{\partial t \partial r \partial s}$ us co-ordinated convex on V then following* *inequality exist*

$$\Bigg|\frac{F(a,c,e)+F(a,c,f)+F(a,d,e)+F(a,d,f}{8}$$
$$+\frac{F(b,c,e)+F(b,c,f)+F(b,d,e)+F(b,d,f)}{8}$$
$$-\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{8(b-a)^{\alpha}(d-c)^{\beta}(f-e)^{\gamma}}\Big[J^{\alpha,\beta,\gamma}_{a^+,c^+,e^+}F(b,d,f)+J^{\alpha,\beta,\gamma}_{a^+,d^-,e^+}F(b,c,f)$$
$$+J^{\alpha,\beta,\gamma}_{a^+,c^+,f^-}F(b,d,e)+J^{\alpha,\beta,\gamma}_{a^+,d^-,f^-}F(b,c,e)+J^{\alpha,\beta,\gamma}_{b^-,c^+,e^+}F(a,d,f)$$
$$+J^{\alpha,\beta,\gamma}_{b^-,d^-,e^+}F(a,c,f)+J^{\alpha,\beta,\gamma}_{b^-,c^+,f^-}F(a,d,e)++J^{\alpha,\beta,\gamma}_{b^-,d^-,f^-}F(a,c,e)\Big]+A-B\Bigg|$$
$$\leq\frac{(b-a)(d-c)(f-e)}{8(\alpha+1)(\beta+1)(\gamma+1)}\left(\left|\frac{\partial^3F(a,c,e)}{\partial t\partial r\partial s}\right|+\left|\frac{\partial^3F(a,c,f)}{\partial t\partial r\partial s}\right|+\left|\frac{\partial^3F(a,d,e)}{\partial t\partial r\partial s}\right|+\left|\frac{\partial^3F(a,d,f)}{\partial t\partial r\partial s}\right|\right.$$
$$\left.+\left|\frac{\partial^3F(b,c,e)}{\partial t\partial r\partial s}\right|+\left|\frac{\partial^3F(b,c,f)}{\partial t\partial r\partial s}\right|+\left|\frac{\partial^3F(b,d,e)}{\partial t\partial r\partial s}\right|+\left|\frac{\partial^3F(b,d,f)}{\partial t\partial r\partial s}\right|\right),$$

where A and B are defined in Lemma 1.

Proof. We use Lemma 1 and convexity of $\frac{\partial^3F}{\partial t\partial r\partial s}$ we have

$$\Bigg|\frac{F(a,c,e)+F(a,c,f)+F(a,d,e)+F(a,d,f)}{8}$$
$$+\frac{F(b,c,e)+F(b,c,f)+F(b,d,e)+F(b,d,f)}{8}$$
$$-\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{8(b-a)^{\alpha}(d-c)^{\beta}(f-e)^{\gamma}}\Big[J^{\alpha,\beta,\gamma}_{a^+,c^+,e^+}F(b,d,f)+J^{\alpha,\beta,\gamma}_{a^+,d^-,e^+}F(b,c,f)$$
$$+J^{\alpha,\beta,\gamma}_{a^+,c^+,f^-}F(b,d,e)+J^{\alpha,\beta,\gamma}_{a^+,d^-,f^-}F(b,c,e)+J^{\alpha,\beta,\gamma}_{b^-,c^+,e^+}F(a,d,f)$$
$$+J^{\alpha,\beta,\gamma}_{b^-,d^-,e^+}F(a,c,f)+J^{\alpha,\beta,\gamma}_{b^-,c^+,f^-}F(a,d,e)++J^{\alpha,\beta,\gamma}_{b^-,d^-,f^-}F(a,c,e)\Big]+A-B\Bigg|$$
$$\leq\frac{(b-a)(d-c)(f-e)}{8}$$
$$\Big[\int_0^1\int_0^1\int_0^1 t^{\alpha}r^{\beta}s^{\gamma}\left|\frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\right|\mathrm{d}s\mathrm{d}r\mathrm{d}t$$
$$-\int_0^1\int_0^1\int_0^1 t^{\alpha}r^{\beta}(1-s)^{\gamma}\left|\frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\right|\mathrm{d}s\mathrm{d}r\mathrm{d}t$$
$$-\int_0^1\int_0^1\int_0^1 t^{\alpha}(1-r)^{\beta}s^{\gamma}\left|\frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\right|\mathrm{d}s\mathrm{d}r\mathrm{d}t$$
$$+\int_0^1\int_0^1\int_0^1 t^{\alpha}(1-r)^{\beta}(1-s)^{\gamma}\left|\frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\right|\mathrm{d}s\mathrm{d}r\mathrm{d}t$$
$$-\int_0^1\int_0^1\int_0^1 (1-t)^{\alpha}r^{\beta}s^{\gamma}\left|\frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\right|\mathrm{d}s\mathrm{d}r\mathrm{d}t$$

$$+\int_0^1\int_0^1\int_0^1 (1-t)^{\alpha} r^{\beta}(1-s)^{\gamma}\left|\frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\right| \mathrm{d}s\mathrm{d}r\mathrm{d}t$$
$$+\int_0^1\int_0^1\int_0^1 (1-t)^{\alpha}(1-r)^{\beta}s^{\gamma}\left|\frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\right| \mathrm{d}s\mathrm{d}r\mathrm{d}t$$
$$-\int_0^1\int_0^1\int_0^1 (1-t)^{\alpha}(1-r)^{\beta}(1-s)^{\gamma}\left|\frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\right|$$
$$\mathrm{d}s\mathrm{d}r\mathrm{d}t]$$

.

$$\leq \frac{(b-a)(d-c)(f-e)}{8}\int_0^1\int_0^1\int_0^1 [(1-t)^{\alpha}(1-r)^{\beta}(1-s)^{\gamma}+(1-t)^{\alpha}(1-r)^{\beta}s^{\gamma}$$
$$+(1-t)^{\alpha}r^{\beta}(1-s)^{\gamma}+(1-t)^{\alpha}r^{\beta}s^{\gamma}+t^{\alpha}(1-r)^{\beta}(1-s)^{\gamma}$$
$$+t^{\alpha}(1-r)^{\beta}s^{\gamma}+t^{\alpha}r^{\beta}(1-s)^{\gamma}+t^{\alpha}r^{\beta}s^{\gamma}]\left[(1-t)(1-r)(1-s)\left|\frac{\partial^3F(a,c,e)}{\partial t\partial r\partial s}\right|\right.$$
$$+(1-t)(1-r)s\left|\frac{\partial^3F(a,c,f)}{\partial t\partial r\partial s}\right|+(1-t)r(1-s)\left|\frac{\partial^3F(a,d,e)}{\partial t\partial r\partial s}\right|+(1-t)rs\left|\frac{\partial^3F(a,d,f)}{\partial t\partial r\partial s}\right|$$
$$+t(1-r)(1-s)\left|\frac{\partial^3F(b,c,e)}{\partial t\partial r\partial s}\right|+t(1-r)s\left|\frac{\partial^3F(b,c,f)}{\partial t\partial r\partial s}\right|+tr(1-s)\left|\frac{\partial^3F(b,d,e)}{\partial t\partial r\partial s}\right|$$
$$\left.+trs\left|\frac{\partial^3F(b,d,f)}{\partial t\partial r\partial s}\right|\right]$$

By integrating the above inequality we get required result .
This completes the proof. □

Theorem 3. *Let $F: V\subset\mathbb{R}^3\to\mathbb{R}$ be partial differentiable mapping on $V:=[a,b]\times[c,d]\times[e,f]$ with $a<b, c<d, e<f$. If $\left|\frac{\partial^3F}{\partial t\partial r\partial s}\right|^q$ is co-ordinated convex on V then following inequality exist*

$$\left|\frac{F(a,c,e)+F(a,c,f)+F(a,d,e)+F(a,d,f}{8}\right.$$
$$+\frac{F(b,c,e)+F(b,c,f)+F(b,d,e)+F(b,d,f)}{8}$$
$$-\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{8(b-a)^{\alpha}(d-c)^{\beta}(f-e)^{\gamma}}\left[J^{\alpha,\beta,\gamma}_{a^+,c^+,e^+}F(b,d,f)+J^{\alpha,\beta,\gamma}_{a^+,d^-,e^+}F(b,c,f)\right.$$
$$+J^{\alpha,\beta,\gamma}_{a^+,c^+,f^-}F(b,d,e)+J^{\alpha,\beta,\gamma}_{a^+,d^-,f^-}F(b,c,e)+J^{\alpha,\beta,\gamma}_{b^-,c^+,e^+}F(a,d,f)$$
$$\left.\left.+J^{\alpha,\beta,\gamma}_{b^-,d^-,e^+}F(a,c,f)+J^{\alpha,\beta,\gamma}_{b^-,c^+,f^-}F(a,d,e)++J^{\alpha,\beta,\gamma}_{b^-,d^-,f^-}F(a,c,e)\right]+A-B\right|$$
$$\leq\frac{(b-a)(d-c)(f-e)}{[(\alpha p+1)(\beta p+1)(\gamma p+1)]^{\frac{1}{p}}}\left(\frac{1}{8}\right)^{\frac{1}{q}}\left(\left|\frac{\partial^3F(a,c,e)}{\partial t\partial r\partial s}\right|^q+\left|\frac{\partial^3F(a,c,f)}{\partial t\partial r\partial s}\right|^q+\left|\frac{\partial^3F(a,d,e)}{\partial t\partial r\partial s}\right|^q\right.$$
$$\left.+\left|\frac{\partial^3F(a,d,f)}{\partial t\partial r\partial s}\right|^q+\left|\frac{\partial^3F(b,c,e)}{\partial t\partial r\partial s}\right|^q+\left|\frac{\partial^3F(b,c,f)}{\partial t\partial r\partial s}\right|^q+\left|\frac{\partial^3F(b,d,e)}{\partial t\partial r\partial s}\right|^q+\left|\frac{\partial^3F(b,d,f)}{\partial t\partial r\partial s}\right|^q\right)^{\frac{1}{q}},$$

where A and B are defined in Lemma 1.

Proof. We use Lemma 1, Holder's inequality and the convexity of $\left|\frac{\partial^3 F}{\partial t\partial r\partial s}\right|^q$

$$\Bigg|\frac{F(a,c,e)+F(a,c,f)+F(a,d,e)+F(a,d,f}{8}$$
$$+\frac{F(b,c,e)+F(b,c,f)+F(b,d,e)+F(b,d,f)}{8}$$
$$-\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{8(b-a)^{\alpha}(d-c)^{\beta}(f-e)^{\gamma}}\Big[J^{\alpha,\beta,\gamma}_{a^+,c^+,e^+}F(b,d,f)+J^{\alpha,\beta,\gamma}_{a^+,d^-,e^+}F(b,c,f)$$
$$+J^{\alpha,\beta,\gamma}_{a^+,c^+,f^-}F(b,d,e)+J^{\alpha,\beta,\gamma}_{a^+,d^-,f^-}F(b,c,e)+J^{\alpha,\beta,\gamma}_{b^-,c^+,e^+}F(a,d,f)$$
$$+J^{\alpha,\beta,\gamma}_{b^-,d^-,e^+}F(a,c,f)+J^{\alpha,\beta,\gamma}_{b^-,c^+,f^-}F(a,d,e)+J^{\alpha,\beta,\gamma}_{b^-,d^-,f^-}F(a,c,e)\Big]+A-B\Bigg|$$
$$\leq\frac{(b-a)(d-c)(f-e)}{8}\Bigg[\left(\int_0^1\int_0^1\int_0^1(1-t)^{\alpha p}(1-r)^{\beta p}(1-s)^{\gamma p}\mathrm{d}s\mathrm{d}r\mathrm{d}t\right)^{\frac{1}{p}}$$
$$+\left(\int_0^1\int_0^1\int_0^1(1-t)^{\alpha p}(1-r)^{\beta p}s^{\gamma p}\mathrm{d}s\mathrm{d}r\mathrm{d}t\right)^{\frac{1}{p}}+\left(\int_0^1\int_0^1\int_0^1(1-t)^{\alpha p}r^{\beta p}(1-s)^{\gamma p}\mathrm{d}s\mathrm{d}r\mathrm{d}t\right)^{\frac{1}{p}}$$
$$+\left(\int_0^1\int_0^1\int_0^1(1-t)^{\alpha p}r^{\beta p}s^{\gamma p}\mathrm{d}s\mathrm{d}r\mathrm{d}t\right)^{\frac{1}{p}}+\left(\int_0^1\int_0^1\int_0^1t^{\alpha p}(1-r)^{\beta p}(1-s)^{\gamma p}\mathrm{d}s\mathrm{d}r\mathrm{d}t\right)^{\frac{1}{p}}$$
$$+\left(\int_0^1\int_0^1\int_0^1t^{\alpha p}(1-r)^{\beta p}s^{\gamma p}\mathrm{d}s\mathrm{d}r\mathrm{d}t\right)^{\frac{1}{p}}+\left(\int_0^1\int_0^1\int_0^1t^{\alpha p}r^{\beta p}(1-s)^{\gamma p}\right)^{\frac{1}{p}}$$
$$+\left(\int_0^1\int_0^1\int_0^1t^{\alpha p}r^{\beta p}s^{\gamma p}\mathrm{d}s\mathrm{d}r\mathrm{d}t\right)^{\frac{1}{p}}\Bigg]$$
$$\left[\int_0^1\int_0^1\int_0^1(1-t)^{\alpha}(1-r)^{\beta}s^{\gamma}\left|\frac{\partial^3}{\partial t\partial r\partial s}F((1-t)a+tb,(1-r)c+rd,(1-s)e+sf)\right|^q\mathrm{d}s\mathrm{d}r\mathrm{d}t\right]^{\frac{1}{q}}$$
$$\leq\frac{(b-a)(d-c)(f-e)}{[(\alpha p+1)(\beta p+1)(\gamma p+1)]^{\frac{1}{p}}}\Bigg[\int_0^1\int_0^1\int_0^1\Bigg((1-t)(1-r)(1-s)\left|\frac{\partial^3F(a,c,e)}{\partial t\partial r\partial s}\right|^q$$
$$+(1-t)(1-r)s\left|\frac{\partial^3F(a,c,f)}{\partial t\partial r\partial s}\right|^q+(1-t)r(1-s)\left|\frac{\partial^3F(a,d,e)}{\partial t\partial r\partial s}\right|^q+(1-t)rs\left|\frac{\partial^3F(a,d,f)}{\partial t\partial r\partial s}\right|^q$$
$$+t(1-r)(1-s)\left|\frac{\partial^3F(b,c,e)}{\partial t\partial r\partial s}\right|^q+t(1-r)s\left|\frac{\partial^3F(b,c,f)}{\partial t\partial r\partial s}\right|^q+tr(1-s)\left|\frac{\partial^3F(b,d,e)}{\partial t\partial r\partial s}\right|^q$$
$$+trs\left|\frac{\partial^3F(b,d,f)}{\partial t\partial r\partial s}\right|^q\Bigg)\mathrm{d}s\mathrm{d}r\mathrm{d}t\Bigg]^{\frac{1}{q}},$$

integrating the above inequality we get the required result. This completes the proof. □

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[1]Department of Mathematics
Government College University
Faisalabad, Pakistan
Email address: awan.uzair@gmail.com

[2]Department scientific-methodical sessions,
Romanian Mathematical Society-Branch Bucharest, Bucharest, Romania
Email address: marcelamiahai58@yahoo.com

[3]Department of Mathematics,
COMSATS University Islamabad, Islamabad, Pakistan
Email address: noormaslam@gmail.com, khalidan@gmail.com