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MATH PROBLEMS - SOLUTIONS, OBSERVATIONS AND COMMENTS

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Abstract. The paper aims to present some theoretical methodological considerations regarding the solving of math problems. Thus, there are presented theoretical aspects regarding the formation of mathematical thought, the activity of education and training through mathematics, types of learning in mathematics and forms of organizing the activities in mathematical education in Romania, reason for which the References that were used are entirely in Romanian. The paper also contains some math problems solved: geometry, algebra or even arithmetic problems, and the solutions are accompanied by observations and possible methodical comments.

1. Introduction

Problem solving has an important role to play in math education activities in Romania, to cultivate and educate creativity and inventiveness. In the primary cycle the elementary basic notions are formed that the man will build his entire system of mathematical knowledge and will use it throughout his life.

The mathematics professor, through the characteristics of his personality, through the attitude manifested in or out of class towards pupil personality and behavior, is the essential factor that provides the favorable climate for the expression of their own ideas.

The notion of problem has broad content and includes concerns and actions in different areas. In general, any matter of a practical or theoretical nature, for which there is no ready-made response and which requires a solution, is called a problem. Particularly, the math problem is a situation that can be solved by a process of thought and computation. The problem of mathematics is actually the transposition of a situation or complex of practical situations into quantitative relationships, these being in some dependence on one of another and of one or more unknown numerical values that are required to be determined.

School mathematical problem solving activity is an optimal framework for cultivating creativity, especially for the development of logical thought. Thought is triggered when we can not cope with a new situation through existing solutions in the experience acquired only by the learned means.

Within the complex of objectives involved in teaching and learning mathematics in primary education, problem solving is an activity of analysis and synthesis. Problem solving challenges the intellectual abilities of pupils, asks them all their mental availability, especially intelligence, and for this reason the primary education mathematical curriculum gives to math problems more attention. In solving math problems, pupils must discover solving ways and also, find the solution, formulate hypotheses, and then verify them, make associations of original ideas and correlations.

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Mathematics contributes to a great extent to the development of logical thought, receptivity, reasoning, etc.

In the first to fourth grades it learns the basic notions, the "tools" with which the pupil will work throughout his life and on which the entire system of mathematical education in Romania are built. If these notions are not appropriated correctly and in time, the pupil will encounter difficulties. These difficulties do not mobilize him for new attempts and lead to lower confidence in his own forces.

As early as in the first grade, the teacher may present problems to pupils (data, condition, question) so that they understand what means to solve the problem, how to solve a simple problem, and later a complex one.

In order for problem-solving activity to materialize formative valences in the direction of the development of logical thought, it is necessary to have a content of problems and to orientate the work of solving them in accordance with this purpose. Thus, the pupils will have the ability to observe problems, to raise a particular problem.

In order to develop their creative thought, pupils must be encouraged in activities, their efforts must be appreciated, and, they must be stimulated even when they give completely erroneous answers. Developing the potential of thought and creativity is achieved through activities that require independence, intelligence, originality.

It is up to the teacher to ensure the concrete conditions that will enable to pupils to understand the content of the problem as well as the learning of the problem solving methods by them.

In solving a problem is mobilized not only thought, but also the whole personality of the person who solves problems in her rational, affective, volitive coordinates.

2. Methodological theoretical considerations on solving math problems

Mathematics plays a very important role in the formation of graduates of pre-university education in Romania. Thus, one of the aims of the study of mathematics in school is to educate the rigorous and objective way of thought, as well as the precise expression.

The last decade has marked a serious marginalization of mathematical education, reaching a critical threshold.

Mathematics is a significant part of this system. Its instrumental importance lies in the development of logical, rigorous thought, but also in its applications in various fields.

2.1. **Mathematical thought.** The formation of mathematical thought is undoubtedly related to the ability to learn in rigorously interconnected stages, which implies a strong concentration of attention. By the end of the nineteenth century, mathematics had been content with the Aristotelian logic, which was legalized more than 2,000 years ago, which is based, besides the notion of logical proposition, on the following three principles of thought:

- the law (or principle) of identity, the first of the three classical laws of thought; it states that each thing is identical with itself. In its symbolic representation, "$a = a$", or "$\forall$ x: $x = x$".

- the law (or principle) of non-contradiction, the second of the three classic laws of thought; it states that contradictory statements cannot both be true in the same sense at the same time, e.g. the two propositions "A is B" and "A is not B" are mutually exclusive.

- the law (or principle) of excluded middle, the third of the three classic laws of thought; it states that for any proposition, either that proposition is true, or its negation is true. The law is also known as the law (or principle) of the excluded third, in Latin principium tertii exclusi.

A series of events at the end of the nineteenth century that took place in mathematics (especially in the study of infinity), in physics (especially in Theory of Relativity and Quantum Mechanics), in biology and other sciences, led to new logic situations, claiming increased attention to the accuracy of reasoning. The appearance of numerous paradoxes led to a series of difficulties, which questioned the validity of the Aristotelian principles. Thus, in the Theory of the crowds, there appeared sentences that were both true and false, defying thus the principle of non-contradiction. In quantum mechanics, the elementary element status contradicted the principle of identity. Equally threatened was the principle of the third party by appearance of conjectures. Thus, mathematics, physics and biology have manifested, even since the beginning of the 20th century, the need for a logic with several values of truth, logic that violates, in one way or another, the principle of of excluded middle.

The highlighting at the beginning of the 20th century of the Axiom of choice, after which, given a family of nonempty and disjoint sets, there is a set that contains one element in every set of family, drew attention to the choice operation because, to admit the existence of an element with certain properties is not one with its actual determination. Thus, it appeared philosophical intuitionism, which does not allow the use of the principle of excluded middle, only in the case of finite sets. However, the reasoning by reducing to absurd, very often used in mathematics, is questioned, or this reasoning is conditioned by the acceptance of the excluded third principle.

It should be noted that Mathematical Logic has re-entered into its rights with the offensive of Informatics, and there has been a need to make a clear distinction between nonconstructive reasoning and constructive, algorithmic reasoning. In this context, Kurt Gödel's clear distinction between a true and a demonstrative sentence has gained a great deal of importance.

Mathematical thought is characterized by rigor and competence. The rigor, in the etymological sense of the word, means harshness, severity, strictness. Moreover, it also means compliance with a certain system of rules which in a certain situation, are very well established. We need to emphasize the conventional character of rigor, and the necessity of using concurrently several degrees of rigor is determined also, by the need to maintain contact with the intuitive sources of concepts, theories, and mathematical problems in the teaching-learning process of mathematics.

In general, according to the dictionary, "competence is someone's ability to pronounce on a thing, based on a deep knowledge of the issue in question." It should be noted that competence, inclusive at mathematics, can only be acquired by fusing innate factors with others aquired factors. An essential role here is the process of creative learning which, building on the innate mechanisms of the brain and appropriately orienting the empirical interaction of stimuli and responses, leads to the formation of superior competencies. Some specialists are of the opinion that the success of learning is conditioned by the suitability of human experience for individual innate particularities. Therefore, acquiring a skill also involves acquiring an appropriate type of rigor.

Today is trying to define competency in relation to performance. The competence of a subject in relation to a certain activity is all the possibilities available to him in this respect. Performance is the actual mode of doing that activity ([2]).

In the teaching-learning process of mathematics, the fundamental role of the teacher is to to form to pupil a mathematical thinking, to make him understand and learn mathematics, because mathematical thought is superior to all other scientific thoughts. The distinction between mathematical thought and other scientific thoughts is that, in the case of unformalized disciplines, the links between the different stages of the thought process are not so strong that a certain stage becomes incomprehensible without knowing the previous stages, as is the case with mathematical thought ([2]).

The gradual formation of a solid mathematical thought at pupils leads to an increase of creativity, which implies the shift to higher competencies and performance.

2.2. **The method - its place and role in the teaching-learning process of mathematics in school.** The fulfillment of the informative and formative objectives of mathematical education impose the use of some specific working methods and procedures, i.e. the use of an appropriate methodology. As any human activity, the mathematical education and training activity is always situated in a determined and concrete context, within which occur some variables (factors, components, conditions) that can be known, other identifiable, kept under control, others may be adapted to the current needs of the teaching process and others need to be accepted frequently, as they are. Therefore, each teaching-learning action of mathematics: - has as a starting point a motivation awareness by the objectives that to be met; - engages the participants - the teachers and pupils, respectively the agents of the training activity; - is carried out in accordance with certain rules; - uses specific methods and means of realization; - is part of a form of work organization; - follows results that will be evaluated. There is a functional interdependence between these components. Transposition the teacher's intentions into concrete didactic actions requires that he detail the task he is about to accomplish in the classroom. Thus, he has all the components that make up the repertoire of teaching abilities, i.e., action and cognitive procedures specific to mathematics (strategies, methods, procedures, teaching techniques), fixed in memory by experience and/or individual study. The teacher's methodological repertoire takes a concrete form in every didactic activity and focuses on actual didactic activities. The realization of these didactic actions results in changes in the structures cognitive, psychomotor, emotional and characteristic of the pupils. Due to their transparency, certain reactions of pupils easily reach at the teacher's perception, such as: mastering a notion verified by answer; an attitude of comprehension or misunderstanding, so a confusion, revealed by the pupil's nonverbal reactions.

2.3. **Teaching-learning methods of mathematics.** The teaching-learning methods of mathematics are the concrete ways in which the teacher-pupil binomial acts to achieve the objectives. These are pedagogical methods, specific methods, traditional methods and modern methods.

The modern didactics advocates an active, directed education. The didactic research has highlighted the efficiency of the active-participative methods, by which the pupil is no longer a simple receiver but also a subject of knowledge and educational action. Many authors have highlighted the fact that the intellectual development of the child takes place through actions, through the use in learning of some procedures that cultivate initiative, imagination, creative thinking, responsibility, ability to cooperate.

A lesson that relies on active-participatory methods associates the learner with his / her own training. The interest in these methods has greatly increased lately as they provide a better learning of the knowledge, the development of the student's cognitive and operative structures, the strengthening of relational and communication behaviors. Based on the idea of combining thought and action, active-participatory methods determine students to explore and discover the knowledge, to process them, and to find solutions to the problems they face. Effective learning implies engaging, engaging learners in the act of learning. Here, the method plays a fundamental role.

2.4. **Specific teaching-learning methods of mathematics in school.** In the activities of mathematical education in Romania, a clear distinction must be made between the pedagogical methods and the specific teaching and learning methods of mathematics.

Pedagogical methods of teaching mathematics in school are those methods that Pedagogy has invented, pedagogical practice has confirmed them, and didactics of mathematics applied them in mathematical didactic act.

From the specific teaching-learning methods of mathematics we enumerate: arithmetic solving methods, algebraic methods for solving arithmetic or geometry problems, the method of undetermined coefficients (i.e., for the determination of coefficients of a polynomial), the method of mathematical induction, the method of reduction to absurdity , method of integration by parts, etc.

Forms of independent work

Among the methods, processes or techniques of activating pupils in the didactic process are those conducted by the pupil without a teacher or guided by him. Each pupil develops skills to work independently only if he is required to do so.

Among the forms of independent work we include: themes for independent work, including the homework, short control work of 5 to 10 minutes, in the homework checking or even in the new knowledge setting phase, the activity with the help of the sheets, which can also take one hour, but even less than an hour, moments of independent work, when putting different questions or problems in the lesson through which to develop the creativity factors.

Any activity of of independent work must end with a confirmation of the correct answer or the correct demonstration.

Independent work can be individual or in groups, and in such an activity, all students have the same task to accomplish or tasks can be differentiated by levels of training.

Differentiated independent work implies respecting pupilss' individuality by adopting training tasks in accordance with the needs and possibilities of each pupil, which in turn implies a profound knowledge of each pupil and a teacher's titanic work, which it can not be achieved with the means and forms of organization of the present. An attempt in this sense would be to establish a differentiated homework, mentioning for heavier problems, "optional" for the weaker pupils and "compulsory" for the better pupils. In this type of activity we can also include scheduled training.

Work on groups. The main form of organizing the teaching activity is the class, understood as a community of pupils. Ensuring the participation of all pupils in the classroom activity is not easy, especially in classes with a large number of pupils and with different learning abilities.

Numerous social psychology research has highlighted the fact that group work has a high efficiency because intelligence and individual effort combine and complement the intelligence and effort of the whole group. The teacher carries out a design activity (materials design and assignment) and a tutorial. The groups can be formed by the teacher or they can self-judge on emotional criteria. The number of pupils in a group depends mainly on the technical endowment of the learning environment, but it has been found out from experience that the most favored group is 4-6 pupils.

In the process of evaluating the activities of pupils by groups, account is taken of the nature of the work and a specific score system or specific grids are used. The teacher may decide to continue or stop working on groups, depending on the findings he or she has on the group's activity.

Work with sheets is another form of independent work (individual or group), the sheets being materials developed by the teacher and which, according to the purpose proposed to be achieved by the work on their basis, can be classified in the following categories: *self-training sheets, development sheets, recovery sheets, exercise sheets (problems) and answer sheets.* These sheets provide the reverse link.

Learning through scheduled training is one of the forms of individual independent activity that is based on materials developed by teachers and called *training programs.*

Learning the mathematics is a process that can be driven by reverse information, mediated by technique, and this idea lies at the basis of scheduled training. With this methodology the subject of the lesson is presented in the form of some programs. Scheduled learning has been applied in many countries and has been the subject of an impressive number of research, studies and publications. The scheduled system takes into account almost all the elementary requirements arising from the psychological laws of learning. This system activates the pupil by mobilizing him to permanently solve various problems, adapts the rhythm and content of learning to the peculiarities of the pupils, provides information about the results obtained, observes the law of effect and exercise.

When developing programs, the teacher must take into account the level of intellectual development of pupils, so that the same content can be developed in program variants. These programs need to be revised so as to create learning conditions for all pupils; to be considered appropriate, an elaborate material must ensure 90% of the correct answers.

Interest in scheduled education was very high in the 1970s. The programmed education offers the learner some autonomy in learning. The pupil passes, at its rhythm, determined contents that have been adapted and organized by the teacher. The good functioning of such a learning technique involves an in-depth analysis and preparation of the subject to be taught. The author of a programmed course must be able to determine the stages that allow the mastery of the knowledge and, at the same time, it take into account the pupils' initial knowledge.

The opponents of this learning mode claim that programming can only refer to a certain type of study materials that can be segmented and dosed in small steps.

Also, fear was expressed that pupils' independence is almost entirely eliminated, and they are guided step by step in towards the appropriation of the knowledge. It is important for the pupil to be accustomed to solving problems, to trouble, to explore the unknown, to make decisions, to think in a personal way.

Today the principles of scheduled education are found in computer-assisted training.

Learning from the manual and from other bibliographic materials. The school manual is the main bibliographic material of the pupil and is a guide to teacher training for the lesson. For the pupil, the school manual contains the amount of knowledge required for the level of compulsory education.

The method of work with the manual is a form of independent work used to assimilate knowledge from mathematical texts. For the same purpose, we use problem collections, monographs, journals.

Learning with the manual involves a pupil's own effort to find a way to demonstrate, compute, understand, memorize, and then apply what he has learned in solving problems In any situation based on independent learning in the manual, the pupil is controlled by the teacher, by checking the mathematical content in the book. If the whole lesson was conducted on the basis of independent work from the manual, the following lesson is necessarily organized as a lesson of fixation and skill training at the theme of the previous lesson, using other methods than independent work.

The method of work with the manual is recommended to be used by teachers in school practice, but should not be used continuously. By using this method one of the fundamental objectives of teaching mathematics is achieved, namely to teach the learner how to learn.

The homework is another way of independent work from the manual or collections. It is given so that the pupil study from the manual or collections the theory and patterns of exercises solved too.

2.5. **Types of math learning.** The teaching-learning process of mathematics at school is unfolding under very different conditions. Despite this variability, the conditions that generate and mediate the learning of this school discipline can be organized and created in a special way. In organizing and conducting the teaching-learning process in mathematics, the teacher has to answer the following questions:

1. What activity does the pupil have to do to acquire knowledge and math skills?
2. What methods and technical means of training can be used for this purpose?
3. How will it be ascertained whether or not the training process has achieved results?

All the answers to these questions make up a theoretical model of the training process by mathematics, a model that seeks to solve the fundamental problem of this education:

"What internal and external activity of the student leads to the goals of learning mathematics?"

From a historical point of view, the first psychological conception of learning mathematics is the association theory - a model of learning focused on conducting the process of accumulation and processing of the sensory experience.

The conditions of math learning, due to associations have been studied well over a hundred years ago. Throughout this period, the boundaries of the association model also emerged. This period begins with the end of the nineteenth century, when the theories of learning based on conditioned reflex, which is nothing more than a model of training based on cognitive stimulation through the orientation and organization of practical activity.

The two above-mentioned concepts have contributed to the foundation and application in mathematical education of some training methods such as: *the story* and *the explanation*, which assures the perception of knowledge, the exercise, a method that leads to the consolidation of the cognitive ties and to their use, to clarify the learning outcomes and to regulate their unfolding: *the verification and the appreciation*.

The boundaries of the associations model result from the fact that not all the facts of the learning activity of mathematics can be explained by associations. The reality of the learning phenomenon in mathematics is much more complex. To understand mathematics, it is not enough to master the mathematical language or the mathematical concepts, but it must acquired the general structures in which these concepts can be found, as well as the possible relationships between concepts and structures. This type of learning is called Semiotic Learning (Tolman) and it consists in acquiring of the type sign-meaning relations. For example, the link between the generalized image of the right-angled triangle and the expression "*right-angled triangle*" can be formed on the basis of associations. But the relationship between the word "*hypotenuse*" and the expression "*sum of the squares of cathetus*" implies their relationship to a cognitive structure. This is an example of a semiotic code link.

The study of the semiotics relations taking place in human learning has led to the shaping of semiotic theories on learning, which means approaching the education as a process of training, at pupils, generalized notional systems and mental processes.

Applying the semiotic theories to learning in Didactics of Mathematics, highlighted new methodological orientations in the education process of mathematics. Thus, the role of mathematical language must be emphasized as an instrument of the teaching-learning activity of mathematics, of interpreting, understanding and acquiring the logical relations between structures, in addition to memorization and representation; it is also important that the teacher's personal experience is included in the sources of mathematics learning, especially in applied mathematics. The subject of communication, mathematical learning, are the general principles, categories and notions , which once acquired, become instruments of mental activity.

Accordingly, new principles are formulated to structure the process of instruction and organizing the mathematical notional content. So:

a) the instruction should begin not only with the particular, but with the general, the whole or the structure;

b) the notional mathematical content will have to be carried out in the order of logical development of the concepts and principles of mathematics science;

c) In knowledge acquisition will gradually have to use the analysis and classification of some concrete objects followed by solving of some classes of problems and then to make a framing of knowledge in generalizing systems.

It follows that the semiotic theories of mathematics learning propose a model of instruction oriented towards the development of the pupil's cognitive-reflective side, but which still does not cover the whole range of acquisitions acquired by the pupil in the process of learning this school discipline.

In mathematics instruction activity, the psychological process of interiorization is closely related to modeling, as well as the principle of learning by action; for interiorization to occur, it is necessary for the pupil to act on the mathematical objects / phenomena or patterns corresponding to them.

Educational methods have a dynamic character in the sense that they maintain what is valuable and eliminate what is morally exploited, thus being open to innovation and perfection - in step with the modern, informational society. The dynamic character , always open-mindedness to the novelties of methodology of Mathematics Didactics, is also emphasized by the relations established between some guiding principles or ideas in the process of increasing the efficiency of methods, including: heuristics, problematization, modeling, algorithmization, etc.

2.6. **The means of learning used in the teaching-learning process of mathematics.** The inventory of traditional techniques of instruction and education in mathematics contained, until recently, in essence: verbal means, chalk, blackboard, pencil and notebook.

The range of these means has expanded considerably, encompassing audio-visual equipment, including the computer, which amplifies the classical teaching and learning possibilities of mathematics. Nowadays, the teaching-learning process of mathematics is being carried out more and more with the help of technical means of instruction, designed to increase its efficiency, to facilitate it. It is not possible to conceive today an improvement of the mathematical education activities without increasing use of the technical means of education.

2.7. **The forms of organizing the activities of mathematical education.** Having stated the objectives, having the support of instruction and education through mathematics in the National Mathematics Curriculum, using some technical means of training, the question arises:

"*How can they be engaged in forms of work appropriate to a particular community?*"

Generally, the history of education has accredited the idea of organizing teaching activities on classes and lessons as a main form of unfolding, a form that has been continually diversifying and improving. As a consequence, the activity of mathematical education has a various forms of organization: lessons, pupil circles, consultations, inter-county competitions, Olympiads, county and/or national mathematical camps.

All these are meant to broaden the area of knowledge and action of pupils' experience in mathematics.

2.8. **The Mathematics Lesson - the main form of activity in mathematical education.** In all forms of work with students in mathematics, the lesson is the main form of activity of the teacher and students. The lesson reveals its efficiency and acquires a dynamic structure depending on a number of factors, such as: the contribution of mathematics as a science, to the presentation of knowledge in programs and textbooks, the level of teacher and class preparation, the level of development of students' motivation to learn mathematics etc.

The mathematics lesson is a fundamental didactic unit, a form of the educational process through which a certain amount of mathematical knowledge is actively perceived and assimilated by students in a given time, through an intentional, systematic, self-regulating activity, provoking in their biopsychic sphere a change in the direction of the desired formation.

According to this characterization we deduce that the mathematics lesson is a didactic program, a system of mathematical knowledge, operational objectives and work procedures able to activate the students.

Types of lessons

We notice that the following events take place during a lesson:
- catching the attention;
- informing the student on the objectives pursued;
- updating the previously acquired knowledge;
- presenting new material and directing learning;
- ensuring the reverse connection (feedback);
- ensuring the possibility of knowledge transfer;
- ensuring the retention (keeping in memory) of new knowledge;
- obtaining performance;
- performance evaluation, which is done throughout the lesson until its end.

Starting from the events that take place in a lesson, we accept the following classification of lessons: the lesson of transmission and assimilation of new knowledge, the lesson of fixing and forming skills and habits, the lesson of recapitulation and systematization, the lesson of verification and appreciation, the lesson in the math lab.

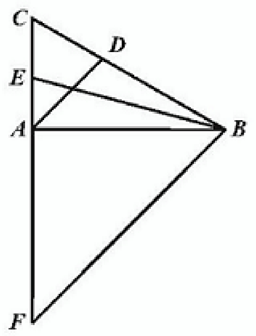
3. Math problems solved and commented

In this section we present several math problems solved. Most of the problems chosen are of geometry, but they are algebraic or even arithmetic, and the solutions are accompanied by observations and possible methodical comments.

Problem 3.1. (Problem E: 14530, Gazeta Matematică no. 6-7-8 / 2013, elementary solution presented by M.Cucoaneş)

Prove that, in a right triangle with an angle of 30°, the length of the bisector of the right angle is half the length of the bisector of the angle of 30°.

Solution. We construct the figure.



Let ABC be the triangle with $m(\angle BAC) = 90°$ and $m(\angle ABC) = 30°$. Let $D \in (BC)$ be the point so that the half-line $(AD$ is the bisector of the angle $\angle BAC$ and let $E \in (AC$ be the point so that the half-line $(BE$ is the bisector of the angle $\angle ABC$.

The problem requires to show that $AD = \frac{BE}{2}$.

We will make the following auxiliary construction: it is draw through point B, the parallel to the right AD. We denote by F the point of intersection of this parallel with the line AC.

Then, $F \in AC$ and $BF \parallel AD$.

Using the hypothesis and the auxiliary construction, we obtain that:

$$m(\angle CAD) = m(\angle DAB) = m(\angle ABF) = m(\angle AFB) = 45°$$

and

$$m(\angle ABE) = m(\angle CBE) = 15°.$$

Since $A \in (EF)$, we deduce that

$$m(\angle EBF) = m(\angle EBA) + m(\angle ABF) = 15° + 45° = 60°.$$

In the triangle ABC we have:

$$m(\angle ACB) = 90° - m(\angle ABC) = 90° - 30° = 60°.$$

Since $E \in (CF)$, it results that

$$m(\angle FBC) = m(\angle FBE) + m(\angle EBC) = 60° + 15° = 75°.$$

In the triangle AEB we have:

$$m(\angle AEB) = m(\angle EAB) - m(\angle ABE) = 90° - 15° = 75°.$$

It is obvious the congruence

$$\angle CFB \equiv \angle EFB \tag{1}$$

because it is exactly the same angle. Also,

$$\angle FCB \equiv \angle FBE \tag{2}$$

because both angles are of $60°$.

From the last two relations it results that $\triangle CFB \sim \triangle BFE$, where we obtain

$$\frac{CF}{BF} = \frac{CB}{BE}$$

and therefore, we have:

$$BE = \frac{BC \cdot BF}{CF}. \tag{3}$$

Since $AD \parallel BF$, we deduce that $\triangle CAD \sim \triangle CFB$, and it results that

$$\frac{AD}{BF} = \frac{AC}{CF}$$

and then

$$AD = \frac{AC \cdot BF}{CF}. \tag{4}$$

From relations (3) and (4) we deduce that

$$AD = \frac{BE \cdot AC}{BC}. \tag{5}$$

In triangle $\triangle ABC$ we have $m(\angle BAC) = 90°$ *and* $m(\angle ABC) = 30°$, which implies

$$\frac{AC}{BC} = \frac{1}{2}. \tag{6}$$

and now, from relations (5) and (6) we deduce that $AD = \frac{BE}{2}$.

Next we will present some geometry problems, published in Gazeta Matematică, problems that were topics at the county and national school competition in 2006, for which solutions were given as simple as possible, as natural as possible.
(http://www.gazetamatematica.net/?q=node/290)

Problem 3.2. (National competition, 2006, 7th grade, solution proposed by Claudiu-Ştefan Popa)

The sharp triangle ABC has the angle C with the measure of 45°. AA_1 and BB_1 are heights in the triangle $\triangle ABC$, and H is the orthocenter of this triangle.

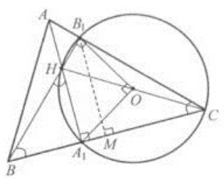
Consider the points D and E located on the segments AA_1 and BC with the property that $A_1D = A_1E = A_1B_1$.

Prove that:

a) $A_1B_1 = \sqrt{\frac{A_1B^2 + A_1C^2}{2}}$;
b) $CH = DE$.

Solution.

a) We construct the figure and the circle circumscribed to the quadrilateral A_1HB_1C, which will be circumscribed also, to the right triangles A_1CH and B_1HC. CH is the diameter of this circle, that we denote by $\mathcal{C}(0; r)$ and $CH = 2r$.



Since $\angle B_1CA_1 = 45°$, it results that $\angle B_1OA_1 = 90°$, $\angle B_1OA_1$ being the angle at the center corresponding to the angle $\angle B_1CA_1$. Since $OA_1 = OB_1 = r$, it follows that

$$A_1B_1^2 = 2r^2 \tag{7}$$

In addition, $\angle BHA_1 = \angle HBA_1 = 45°$, so, $A_1B = HA_1$. In the right triangle $\triangle HA_1C$ we have:

$$4r^2 = CH^2 = HA_1^2 + A_1C^2 = A_1B^2 + A_1C^2 \tag{8}$$

Now, from (7) and (8) it results that $2A_1B_1^2 = 4r^2 = A_1B^2 + A_1C^2$, i.e.

$$A_1B_1 = \sqrt{\frac{A_1B^2 + A_1C^2}{2}}.$$

b) According to the hypothesis, we have:

$$DE^2 = A_1D^2 + A_1E^2 = 2A_1B_1^2 = 4r^2 = CH^2,$$

so, $DE = CH$.

Remark 1. *Since* $\sqrt{\frac{A_1B^2+A_1C^2}{2}}$ *represents the quadratic mean of the lengths* A_1B *and* A_1C*, we have that:*

$$A_1B_1 = \sqrt{\frac{A_1B^2 + A_1C^2}{2}} \geq \frac{A_1B + A_1C}{2} = \frac{BC}{2} = B_1M,$$

where M *is the middle of the side* BC *and* $BM \perp BC$.
In this way, a geometric interpretation of the inequality between the quadratic mean and the arithmetic mean was given.

Remark 2. *The problem is also valid when the triangle has an obtuse angle. This new problem can be a task for students.*

Problem 3.3. (G.M.2/2010)
Let a, b, c be the natural numbers

$$a = 1 + 2 + 3 + 4 + ... + 2009,$$

$$b = 2 + 4 + 6 + 8 + ... + 2010,$$

$$c = 1 + 3 + 5 + 7 + ... + 2009.$$

Show that the numbers a, b+1 and c divided by 2 give the same remainder.

Solution.
For the beginning we calculate the three numbers a, b, c, the three sums, respectively. Then we divide by 2 each number thus obtained.

The number a:
$a = 1 + 2 + 3 + 4 + ... + 2009 = 1004 \times 2009 + 2009 = 2019045.$

Now, we make the following comment:
The number $a = 1 + 2 + 3 + ... + 2007 + 2008 + 2009$ can be calculated by setting the terms of the sum as below:

$$a = 1 + 2 + 3 + ... + 2007 + 2008 + 2009$$

$$a = 2009 + 2008 + 2007 + ... + 3 + 2 + 1$$

and observing that the numbers placed one below the other have the sum 2010. We add the two expressions of a and deduce that there will be 2009 sums that give 2010, i.e.

$$2a = 2009 \times 2010 \implies a = (2009 \times 2010) : 2 = 2019045.$$

Generally, for the calculus of the sum of the first n natural numbers the following relation is used:

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}.$$

Returning, we continue with the calculus of the number b:

$$b=2\times(1+2+3+\cdots+1005)$$

and according to the comment above, we have:

$$b=2\times 1005\times(1005+1):2=1011030.$$

To calculate the number c, we notice that the terms of c are odd numbers.

If we had even numbers too, then we even had the number a. This observation allows us to write c like this:

$$c=1+2+3+4+\cdots+2008+2009-(2+4+\cdots+2008)$$

and it results $c=2019045-1009020=1010025$.

Finally, we obtained:

$$a=2019045;\ b=1011030;\ c=1010025.$$

Now, we divide the numbers a, $b+1$, c by 2 and we obtain:

$$a:2=2019045:2=1009522,\ \textit{remainder } 1$$

$$(b+1):2=1011031:2=505515,\ \textit{remainder } 1$$

$$c:2=1010025:2=505012,\ \textit{remainder } 1,$$

that is, exactly what had to be demonstrated.

4. Instructive and fun math problems

In this section we present some mathematical problems, interesting by their instructive, educational character, with an important role in the logical training, but also in the relaxation of the pupils.

Problem 4.1. Using the mathematical signs $+, -, \times, /$, compose examples with the result 100:

a) five times with the number 1;
b) four times with the number 9;
c) five times with the number 5.

For example, "five times with the number 3": $33\times 3+3/3=100$.

Solution.

a) $111-11=100$;
b) $99+9/9=100$;
c) $5\times 5\times 5-5\times 5=100$.

Problem 4.2. A boy has the same number of sisters as brothers, but each girl has twice as many brothers as sisters. How many children are in this family? How many of them are boys and how many are girls?

Solution. 7 children: 4 boys and 3 girls.

Problem 4.3. How many times per day are the clock indicators perpendicular?

Solution. During any one hour, the minute indicator is perpendicular to the hour indicator exactly 2 times. As a day has 24 hours, it results that in one day the two indicators of the clock will be perpendicular 48 times.

Instead of the End

During one of his lectures, David Hilbert said:

-Many people possesses a certain horizon. When it narrows and becomes infinitely small, he it turns to a point, and then man says, "This is my point of view."

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