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ON THE GENERALIZED OSTROWSKI TYPE INEQUALITIES VIA TEMPERED FRACTIONAL INTEGRALS

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Abstract. In this paper, we have found a new version of the Ostrowski type inequality for tempered Riemann-Liouville fractional integrals. In addition, some relevant results have been obtained.

1. Introduction

Fractional analysis is concerned with the study of integral and differential operators of the non-integral order. Many mathematicians, such as Liouville, Riemann and Weyl, have made great contributions to the theory of fractional analysis. The subject of the fractional analysis (integrals and derivatives) has gained considerable popularity and importance during the past there decades or so, due mainly to its demostrated applications in nemerous seemingly diverse and widespread fields of science and engineering. The fractional integral indeed provides several potentially useful tools for a variety of problems, including specific functions of mathematical science, as well as their extensions and generalizations in one or more variables. We have seen that studies on inequalities for fractional integrals have an important place lately. Many authors make new studies for different types of convex functions, and these studies attract the attention of many readers. This subject is still being studied extensively by many authors, see for instance ([2], [5]-[7], [9]-[19]). One of these inequalities is the Ostrowski inequality, and the fractional integral version of this inequality was first given by Set in [13] as follows:

Theorem 1. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative with $a<b$ such that $f'\in L_1([a,b])$. If $|f'|$ is a convex function on $[a,b]$ and $|f'(x)|\leq M,\ x\in[a,b]$, then the following inequality for fractional integrals holds:*

$$\left|\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)-\Gamma(\alpha+1)\left[J_{x^-}^{\alpha}f(a)+J_{x^+}^{\alpha}f(b)\right]\right| \tag{1}$$

$$\leq M\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+1}$$

with $\alpha>0$.

Later, in [5] Farid proved the inequality (1) by using a different method as follows.

Theorem 2. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative with $a<b$ such that $f'\in L_1([a,b])$. If $|f'(x)|\leq M,\ x\in[a,b]$, then the*

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following inequality for fractional integrals holds:

$$\left|\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)-\Gamma(\alpha+1)\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)\right]\right| \tag{2}$$

$$\leq M\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+1}$$

with $\alpha>0$.

The $J_{a+}^{\alpha}f(x)$ and $J_{b-}^{\alpha}f(x)$ expressions given in the Theorem 2 are the right Riemann-Liouville fractional integral and left Riemann-Liouville fractional integral, respectively are defined by follows ([7] and [9]):

$$J_{a^{+}}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)dt,\quad x>a,$$

$$J_{b^{-}}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)dt,\quad x<b.$$

where the gamma function is defined as

$$\Gamma(\alpha)=\int_{0}^{\infty}t^{\alpha-1}e^{-t}dt,\quad \alpha>0.$$

Incomplete gamma function is defined as

$$\gamma(\alpha,x)=\int_{0}^{x}t^{\alpha-1}e^{-t}dt,\quad \alpha>0$$

and by using the recurrence formula

$$\gamma(\alpha+1,x)=\alpha\gamma(\alpha,x)-x^{-\alpha}e^{-x}. \tag{3}$$

In fact, Ostrowski's inequality plays a vital role in studying the error boundaries of different numerical squaring rules, such as midpoint, trapezoidal, Simpson, and other generalized Riemann type. For recent results, generalizations and new inequalities related to the Ostrowski type fractional integral inequalities see, [1]-[6], [11]-[13], [15], [18], [19].

We will introduce the definitions and new notations of tempered fractional operators as follows(see [10], [16], [17]):

Definition 1. *Let* $[a,b]$ *be a real interval and* $\alpha>0,\ \lambda\geq 0$. *Then*

1) The left tempered fractional integral of a function $f\in L^{1}[a,b]$ *is defided by*

$$J_{a^{+}}^{(\alpha,\lambda)_{T}}f(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}e^{-\lambda(x-t)}f(t)dt,\quad x\in[a,b],$$

2) The right tempered fractional integral of a function $f\in L^{1}[a,b]$ *is defided by*

$$J_{b^{-}}^{(\alpha,\lambda)_{T}}f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}e^{-\lambda(t-x)}f(t)dt,\quad x\in[a,b].$$

This is the fractional power of the modified differentiation operator $D=\left(\frac{d}{dx}+\lambda\right)$.

The purpose of this paper is established new Ostrowski type fractional integral inequalities. Using functions whose first derivatives are bounded, we obtained new inequalities that are connected with the celebrated Ostrowski type which cover the previously puplished results.

2. Main Results

In this section, we give an identity which use to assist us is proving our results as follows:

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\left[\frac{\gamma(\alpha,\lambda(x-a))+\gamma(\alpha,\lambda(b-x))}{\lambda^{\alpha}}\right]f(x)-\Gamma(\alpha)\left[J_{a^{+}}^{(\alpha,\lambda)_T}f(x)+J_{b^{-}}^{(\alpha,\lambda)_T}f(x)\right] \tag{4}$$
$$= (x-a)^{\alpha+1}\int_0^1\left(\int_t^1 s^{\alpha-1}e^{-\lambda(x-a)s}ds\right)f'(ta+(1-t)x)\,dt$$
$$-(b-x)^{\alpha+1}\int_0^1\left(\int_t^1 s^{\alpha-1}e^{-\lambda(b-x)s}ds\right)f'(tb+(1-t)x)\,dt$$

for $\alpha>0,\ \lambda\geq 0$.

Proof. Here, we apply integration by parts in integrals of right part of (4), then we have

$$\begin{aligned}\Delta_1 &= \int_0^1\left(\int_t^1 s^{\alpha-1}e^{-\lambda(x-a)s}ds\right)f'(ta+(1-t)x)\,dt\\
&= \frac{1}{(x-a)}\left(\int_0^1 s^{\alpha-1}e^{-\lambda(x-a)s}ds\right)f(x)-\frac{1}{(x-a)}\int_0^1 t^{\alpha-1}e^{-\lambda(x-a)t}f(ta+(1-t)x)dt\\
&= \frac{f(x)}{\lambda^{\alpha}(x-a)^{\alpha+1}}\gamma(\alpha,\lambda(x-a))-\frac{1}{(x-a)^{\alpha+1}}\int_a^x (x-u)^{\alpha-1}e^{-\lambda(x-u)}f(u)du.\end{aligned}$$

And similarly, we obtain

$$\begin{aligned}\Delta_2 &= \int_0^1\left(\int_t^1 s^{\alpha-1}e^{-\lambda(b-x)s}ds\right)f'(tb+(1-t)x)\,dt\\
&= -\left(\int_0^1 s^{\alpha-1}e^{-\lambda(b-x)s}ds\right)\frac{f(x)}{b-x}+\frac{1}{b-x}\int_0^1 t^{\alpha-1}e^{-\lambda(b-x)t}f(tb+(1-t)x)dt\\
&= -\frac{f(x)}{\lambda^{\alpha}(b-x)^{\alpha+1}}\gamma(\alpha,\lambda(b-x))+\frac{1}{(b-x)^{\alpha+1}}\int_x^b (u-x)^{\alpha-1}e^{-\lambda(b-u)}f(u)du.\end{aligned}$$

If we subtract $(b-x)^{\alpha+1}\Delta_2$ from $(x-a)^{\alpha+1}\Delta_1$, we obtain the proof of (4). □

Remark 1. *We note that if we take $\lambda=0$ in Lemma 1, by L'Hospital Theorem it follows that*

$$\lim_{\lambda\to 0}\frac{\gamma(\alpha,\lambda(x-a))+\gamma(\alpha,\lambda(b-x))}{\lambda^{\alpha}}=\left(\frac{0}{0}\right)=\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{\alpha}. \tag{5}$$

Corollary 1. *If we take $\lambda = 0$ in Lemma 1, by using (5), then the equality (4) becomes the identity*

$$\begin{aligned}&[[(x-a)^{\alpha}+(b-x)^{\alpha}]]\,f(x)-\Gamma(\alpha+1)\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)\right]\\ =\;&(x-a)^{\alpha+1}\int_0^1(1-t^{\alpha})\,f'(ta+(1-t)x)\,dt-(b-x)^{\alpha+1}\int_0^1(1-t^{\alpha})\,f'(tb+(1-t)x)\,dt.\end{aligned}$$

Remark 2. *If we take $\lambda = 0$, $\alpha = 1$, in Lemma 1, then the identity (4) becomes the identity of Lemma 1 is given by Alomari et al. in* [1].

Now, we extend some estimates of Ostrowski type inequality for functions whose first derivatives are bounded as follows:

Theorem 3. *Let $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative with $a < b$ such that $f' \in L_1([a,b])$. If $|f'(x)| \leq M$, $x \in [a,b]$, then the following inequality for tempered fractional integrals holds:*

$$\begin{aligned}&\left|\left[\frac{\gamma(\alpha,\lambda(x-a))+\gamma(\alpha,\lambda(b-x))}{\lambda^{\alpha}}\right]f(x)-\Gamma(\alpha)\left[J_{a+}^{(\alpha,\lambda)_T}f(x)+J_{b-}^{(\alpha,\lambda)_T}f(x)\right]\right|\\ \leq\;& M\left[\frac{\gamma(\alpha+1,\lambda(x-a))}{\lambda^{\alpha+1}}+\frac{\gamma(\alpha+1,\lambda(b-x))}{\lambda^{\alpha+1}}\right].\end{aligned}\tag{6}$$

Proof. Using Lemma 1 and the bounded of f', we find that

$$\begin{aligned}&\left|\left[\frac{\gamma(\alpha,\lambda(x-a))+\gamma(\alpha,\lambda(b-x))}{\lambda^{\alpha}}\right]f(x)-\Gamma(\alpha)\left[J_{a+}^{(\alpha,\lambda)_T}f(x)+J_{b-}^{(\alpha,\lambda)_T}f(x)\right]\right|\\ \leq\;&(x-a)^{\alpha+1}\int_0^1\left(\int_t^1 s^{\alpha-1}e^{-\lambda(x-a)s}ds\right)\left|f'(ta+(1-t)x)\right|dt\\ &+(b-x)^{\alpha+1}\int_0^1\left(\int_t^1 s^{\alpha-1}e^{-\lambda(b-x)s}ds\right)\left|f'(tb+(1-t)x)\right|dt\\ \leq\;& M\left[(x-a)^{\alpha+1}\int_0^1\left(\int_t^1 s^{\alpha-1}e^{-\lambda(x-a)s}ds\right)dt+(b-x)^{\alpha+1}\int_0^1\left(\int_t^1 s^{\alpha-1}e^{-\lambda(b-x)s}ds\right)dt\right].\end{aligned}$$

With the change of the integral order, we have

$$\begin{aligned}&\left|\left[\frac{\gamma(\alpha,\lambda(x-a))+\gamma(\alpha,\lambda(b-x))}{\lambda^{\alpha}}\right]f(x)-\Gamma(\alpha)\left[J_{a+}^{(\alpha,\lambda)_T}f(x)+J_{b-}^{(\alpha,\lambda)_T}f(x)\right]\right|\\ \leq\;& M\left[(x-a)^{\alpha+1}\int_0^1 s^{\alpha}e^{-\lambda(x-a)s}ds+(b-x)^{\alpha+1}\int_0^1 s^{\alpha}e^{-\lambda(b-x)s}ds\right]\\ =\;& M\left[\frac{\gamma(\alpha,\lambda(x-a))}{\lambda^{\alpha+1}}+\frac{\gamma(\alpha,\lambda(b-x))}{\lambda^{\alpha+1}}\right]\end{aligned}$$

which this completes the proof of the (6). □

Remark 3. *If in Theorem 3, we get* $\lambda = 0,\ \alpha = 1$, *then, the inequality (6) becomes the classical Ostrowski inequality in* [3], [14].

Remark 4. *If in Theorem 3, we get* $\lambda = 0$, *then the inequality (6) becomes the inequality (2) in Theorem 2.*

Theorem 4. *Let* $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *whose derivative with* $a < b$ *such that* $f' \in L_1([a,b])$. *If* $|f'(x)| \leq M$, $p > 1$, $x \in [a,b]$, *then the following inequality for tempered fractional integrals holds:*

$$\left|\left[\frac{\gamma(\alpha,\lambda(x-a))+\gamma(\alpha,\lambda(b-x))}{\lambda^{\alpha}}\right]f(x)-\Gamma(\alpha)\left[J_{a^+}^{(\alpha,\lambda)_T}f(x)+J_{b^-}^{(\alpha,\lambda)_T}f(x)\right]\right| \tag{7}$$
$$\leq M\left[(x-a)^{\alpha+1}L_1^{(\alpha,\lambda)}(a,x)+(b-x)^{\alpha+1}L_2^{(\alpha,\lambda)}(b,x)\right]$$

where

$$L_1^{(\alpha,\lambda)}(a,x)=\left(\int_0^1\left(\int_t^1 s^{\alpha-1}e^{-\lambda(x-a)s}ds\right)^p dt\right)^{\frac{1}{p}}$$
$$L_2^{(\alpha,\lambda)}(b,x)=\left(\int_0^1\left(\int_t^1 s^{\alpha-1}e^{-\lambda(b-x)s}ds\right)^p dt\right)^{\frac{1}{p}}.$$

Proof. Using Lemma 1, Hölder's inequality and the bounded of f', we find that

$$\left|\left[\gamma(\alpha,\lambda(x-a))+\gamma(\alpha,\lambda(b-x))\right]f(x)-\Gamma(\alpha)\left[J_{a^+}^{(\alpha,\lambda)_T}f(x)+J_{b^-}^{(\alpha,\lambda)_T}f(x)\right]\right|$$
$$\leq (x-a)^{\alpha+1}\left(\int_0^1\left(\int_t^1 s^{\alpha-1}e^{-\lambda(x-a)s}ds\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}$$
$$+(b-x)^{\alpha+1}\left(\int_0^1\left(\int_t^1 s^{\alpha-1}e^{-\lambda(b-x)s}ds\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$
$$\leq M(x-a)^{\alpha+1}\left(\int_0^1\left(\int_t^1 s^{\alpha-1}e^{-\lambda(x-a)s}ds\right)^p dt\right)^{\frac{1}{p}}$$
$$+M(b-x)^{\alpha+1}\left(\int_0^1\left(\int_t^1 s^{\alpha-1}e^{-\lambda(b-x)s}ds\right)^p dt\right)^{\frac{1}{p}}$$

which this completes the proof of the (7). □

Corollary 2. *Under the assumptions of Theorem 4 with* $\lambda = 0$, *then the inequality (7) becomes.*

$$\left|\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)-\Gamma(\alpha+1)\left[J_{a^+}^{\alpha}f(x)+J_{b^-}^{\alpha}f(x)\right]\right|$$
$$\leq M\left(1-\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right].$$

Proof. By using the inequality $A \geq B > 0$ and $p > 1$, $(A-B)^p \leq A^p - B^p$, we get

$$\left(\int_0^1 \left(\int_t^1 s^{\alpha-1} ds \right)^p dt \right)^{\frac{1}{p}} = \frac{1}{\alpha} \left(\int_0^1 (1-t^\alpha)^p \, dt \right)^{\frac{1}{p}} \leq \frac{1}{\alpha} \left(1 - \frac{1}{\alpha p + 1} \right)^{\frac{1}{p}}$$

which this completes the proof. □

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