TJMM
12 (2020), No. 2, 117-127

CHARACTERIZATIONS OF STRONGLY GENERALIZED CONVEX FUNCTIONS

MUHAMMAD ASLAM NOOR AND KHALIDA INAYAT NOOR

Abstract. In this paper, we define and consider some new concepts of the strongly general convex functions with respect to two arbitrary functions. Some properties of the strongly generalized convex functions are investigated under suitable conditions. It is shown that the optimality conditions of the strongly generalized convex functions are characterized by a class of variational inequalities, which is called the strongly generalized variational inequality. Some special cases also discussed. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

1. Introduction

It is well known that the convex sets and convex functions had played crucial important part in the developments of the pure and applied sciences and are continue to inspire novel and innovative applications. Convex functions have been extended and generalized in various directions in recent years. Strongly convex functions were introduced and studied by Polyak [22]. Karmardian [9] used the strongly convex functions to discuss the unique existence of a solution of the nonlinear complementarity problems. Awan et al [3, 4] have derived Hermite- Hadamard type inequalities for various classes of strongly convex functions, which provide upper and lower estimate for the integrand. For the applications of strongly convex functions in optimization, variational inequalities and other branches of pure and applied sciences, see [1, 2, 6, 9, 12, 13, 19, 21, 22, 25, 27] and the references therein.

It is known that a set may not be a convex set. However, a set can be made convex set with respect to some arbitrary functions. Motivated by this fact, Jian [8] introduced the concept of generalized convexity involving two arbitrary functions, which includes the general convex sets considered by Youness [25] and Noor [17] as special cases. It has been shown [17] that the minimum of a differentiable general convex function on the general convex set can be characterized by the general variational inequalities. Cristescu at al [5] have investigated algebraic and topological properties of the g-convex sets defined by Noor [17] in order to deduce their shape. They are a subclass of star-shaped sets, which have also Youness [25] type convexity. A representation theorem based on extremal points is given for the class of bounded g-convex sets. Examples showing that this convexity is a frequent property in connection with a wild range of applications are given. Noor [17] has shown that the optimality conditions of the differentiable general convex functions can be characterized a class of variational inequalities called general variational inequality, the origin of variational inequalities can be traced back to Stampacchia [23]. For the formulation, applications, numerical methods, sensitivity analysis and other aspects of general variational inequalities, see [10, 11, 14, 14, 15, 16, 17, 18, 19, 23, 24, 26, 27] and

2010 *Mathematics Subject Classification.* 49J40, 26D15, 26D10, 90C23.
Key words and phrases. Convex functions, monotone operators, strongly convex functions.

the references es therein.
These facts and observations motivated us to consider strongly convex function with respect to two arbitrary functions, which is the main motivation of this paper. Several new concepts of monotonicity are introduced. We establish the relationship between these classes and derive some new results under some mild conditions. It is shown that the parallelograms laws for inner product can be obtained from these definitions. We have shown that the minimum of the a higher order strongly convex functions on the general convex set can be characterized by a class of variational inequality. Some various new special cases are discussed, which can be viewed itself an elegant and interesting applications of the strongly generalized convex functions. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2. Preliminary Results

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively. Let $h, g : H \to R$ be two arbitrary functions.

Definition 1. [6, 12] *A set K in H is said to be a convex set, if*

$$u + t(v - u) \in K, \qquad \forall u, v \in K, t \in [0, 1].$$

Definition 2. *A function F is said to be convex function, if*

$$F((1-t)u + tv) \leq (1-t)F(u) + tF(v); \forall u, v \in K, t \in [0; 1]. \tag{1}$$

It is well known that $u \in K$ is a minimum of a differential convex functions F, if and only if, $u \in K$ satisfies the inequality

$$\langle F'(u), v - u \rangle \geq 0, \forall v \in K, \tag{2}$$

which is called the variational inequality, introduced and studied by Stampacchia [23]. Variational inequalities can be regarded as a novel and significant extension of variational principles, the origin of which can be traced back to Euler, Lagrange, Newton and Bernoulli brothers.

We would like to mention that the underlying the set may not be a convex set in many important applications. To overcome this drawback, the set can be made convex set with respect to two arbitrary functions, which is called a generalized or (h, g)-convex set [8].

Definition 3. [8]. *The set $K \subseteq H$is said to be a (h, g)-convex set, if there are two functions h and g such as*

$$(1-t)h(u) + tg(v) \in K; \quad \forall u, v \in K, t \in [0; 1]. \tag{3}$$

We now discuss some special cases of the (h, g)-convex set $K \subseteq D$.

(I). If $g(u) = I(u) = u = h(u)$, the identity operator, then (h, g)-convex set reduces to the classical convex set. Clearly every convex set is a (h, g)-convex set, but the converse is not true.

(II). If $h(u) = I(u) = u$, then the (h, g)-convex set becomes the g-convex set, that is,

Definition 4. *The set K is said to be g-convex set, if*

$$(1-t)u + tg(v) \in K \subseteq D, \quad \forall u, v \in K \subseteq D, t \in [0, 1],$$

which was introduced and studied by Noor [17]. Cristescu et al [5] discussed various applications of the general convex sets related to the necessity of adjusting investment or development projects out of environmental or social reasons. For example, the easiest manner of constructing this kind of convex sets comes from the problem of modernizing the railway transport system. Shape properties of the general convex sets with respect to a projection are investigated.

(III). If $g(u) = I(u) = u$, then the generalized convex set becomes the h-convex set, that is,

Definition 5. *The set K is said to be h-convex set, if*

$$(1-t)h(u) + tv \in K \subseteq D, \qquad \forall u, v \in K \subseteq D, t \in [0,1],$$

which is mainly due to Noor [18].

For the sake of simplicity, we always assume that function $F : D \to R$ and $K \cup h(K) \cup g(K) \subseteq D$. If K is (g,h)-convex set, then this condition becomes $K \subseteq D$.

Definition 6. *A function F is said to be a (h,g)-convex function on the (h,g)-convex set $K \subseteq D$, if there exist two arbitrary non-negative functions h, g such that*

$$F((1-t)h(u) + tg(v)) \leq (1-t)F(h(u)) + tF(g(v)), \quad \forall u, v \in K \subseteq D, t \in [0,1]. \qquad (4)$$

The generalized convex functions were introduced by Noor [18]. Noor [18, 19] proved that the minimum $u \in K \subseteq D$ of a differentiable (h,g)-convex functions F can be characterized by the class of variational inequalities of the type:

$$\langle F'(h(u)), g(v) - h(u)\rangle \geq 0, \quad \forall v \in K \subseteq D, \qquad (5)$$

which is known as the extended general variational inequalities. For the applications of the general variational inequalities in various branches of pure and applied sciences, see [14, 15, 16, 16, 17, 18, 19] and the references therein.

We now introduce some new classes of strongly (h,g)-convex functions and strongly affine (h,g)-convex functions on the (h,g)-convex set $K \subseteq D$.

Definition 7. *A function F on the (h,g)-convex set $K \subseteq D$ is said to be strongly (h,g)-convex with respect to two functions h, g, if there exists a constant $\mu > 0$, such that*

$$F(h(u) + t(g(v) - h(u))) \leq (1-t)F(h(u)) + tF(g(v)) - \mu t(1-t)\|g(v) - h(u)\|^2, \qquad (6)$$
$$\forall u, v \in K \subseteq D, t \in [0,1].$$

A function F is said to strongly (h,g)-concave , if and only if, $-F$ is a strongly (h,g)-convex function.

If $t = \frac{1}{2}$ in Definition 7, then one gets the generalized Jensen type property called strongly $(h,g) - J$-convex function.

We now discuss some special cases.

(IV). If $h(u) = I(u) = u$, then the strongly (h,g)-convex function becomes strongly g-convex functions, that is,

$$F(u+t(g(v)-u)) \leq (1-t)F(u)+tF(g(v))-\mu t(1-t)\|g(v)-u\|^2, \quad \forall u,v \in K \subseteq D, t \in [0,1].$$

For the properties of the strongly convex functions in variational inequalities and equilibrium problems, see Noor [18, 19].

(V). If $g(u) = I(u) = u$, then the strongly (h,g)-convex function becomes strongly h-convex functions, that is,

$$F(h(u)+t(v-h(u)) \leq (1-t)F(h(u))+tF(v)-\mu t(1-t)\|v-h(u)\|^2, \quad \forall u,v \in K \subseteq D, t \in [0,1].$$

Definition 8. *A function F on the (h,g)-convex set $K \subseteq D$ is said to be a strongly (h,g)-quasi-convex, if there exists a constant $\mu > 0$ such that*

$$F(h(u)+t(g(v)-h(u)) \leq \max\{F(h(u)),F(g(v))\}-\mu t(1-t)\|g(v)-h(u)\|^2, \quad \forall u,v \in K \subseteq D, t \in [0,1].$$

Definition 9. *A function F on the (h,g)-convex set $K \subseteq D$ is said to be strongly (h,g)-log-convex, if there exists a constant $\mu > 0$ such that*

$$F(h(u)+t(g(v)-h(u)) \leq (F(h(u)))^{1-t}(F(g(v)))^t-\mu t(1-t)\|g(v)-h(u)\|^2, \quad \forall u,v \in K \subseteq D, t \in [0,1],$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} F(h(u)+t(g(v)-h(u)) &\leq (F(h(u)))^{1-t}(F(g(v)))^t-\mu t(1-t)\|g(v)-h(u)\|^2 \\ &\leq (1-t)F(h(u))+tF(g(v))-\mu t(1-t)\|g(v)-h(u)\|^2 \\ &\leq \max\{F(h(u)),F(g(v))\}-\mu t(1-t)\|g(v)-h(u)\|^2. \end{aligned}$$

This shows that every strongly (h,g)-log-convex function is a strongly (h,g)-convex function and every strongly (h,g)-convex function is a strongly (h,g)-quasi-convex function. However, the converse is not true.

Definition 10. *A function F on the (h,g)-convex set $K \subseteq D$ is said to be a strongly (h,g)-affine, if there exists a constant $\mu > 0$, such that*

$$F(h(u)+t(g(v)-h(u)) = (1-t)F(h(u))+t(F(g(v))-\mu t(1-t)\|g(v)-h(u)\|^2, \quad \forall u,v \in K \subseteq D, t \in [0,1].$$

We would like to remark that, if $t = 1/2$ in Definition 10, then one gets the generalized Jensen type property called strongly $(h,g)-J$-affine function.

For appropriate and suitable choice of the arbitrary functions (h,g), one can obtain several new and known classes of strongly (h,g)-convex functions and their variant forms as special cases of strongly (h,g)-convex functions. This shows that the class of strongly (h,g)-convex functions is quite broad and unifying one.

Definition 11. *Let $K \subseteq D$ be a (h,g)-convex set. An operator $T : K \to H$ is said to be:*

(1) strongly monotone, if and only if, there exists a constant $\alpha > 0$ such that

$$\langle Tu-Tv, h(u)-g(v)\rangle \geq \alpha\|g(v)-h(u)\|^2, \forall u,v \in K \subseteq D.$$

(2) *strongly pseudomonotone, if and only if, there exists a constant* $\nu > 0$ *such that*

$$\langle Tu, g(v) - h(u)\rangle + \nu\|g(v) - h(u)\|^2 \geq 0$$
$$\Rightarrow$$
$$\langle Tv, g(v) - h(u)\rangle \geq 0, \forall u, v \in K \subseteq D.$$

(3) *strongly relaxed pseudomonotone, if and only if, there exists a constant* $\mu > 0$ *such that*

$$\langle Tu, g(v) - h(u)\rangle \geq 0$$
$$\Rightarrow$$
$$-\langle Tv, h(u) - g(v)\rangle + \mu\|g(v) - h(u)\|^2 \geq 0, \forall u, v \in K \subseteq D.$$

Definition 12. *A differentiable function* F *on the* (h, g)*-convex set* $K \subseteq D$ *is said to be strongly* (h, g)*-pseudoconvex function, if and only if, if there exists a constant* $\mu > 0$*, such that*

$$\langle F'(h(u)), g(v) - h(u)\rangle + \mu\|g(v) - h(u)\|^2 \geq 0 \Rightarrow F(g(v)) \geq F(h(u)),$$
$$\forall u, v \in K \subseteq D.$$

3. Main Results

In this section, we consider some basic properties of strongly (h, g)-convex functions.

Theorem 1. *Let* F *be a differentiable function on the* (h, g)*-convex set* $K \subseteq D$*. Then the function* F *is strongly* (h, g)*-convex function, if and only if,*

$$F(g(v)) - F(h(u)) \geq \langle F'(h(u)), g(v) - h(u)\rangle + \mu\|g(v) - h(u)\|^2, \quad (7)$$
$$\forall u, v \in K \subseteq D, t \in [0, 1].$$

Proof. Let F be a strongly (h, g)-convex function. Then

$$F(h(u) + t(g(v) - h(u))) \leq (1 - t)F(h(u)) + tF(g(v)) - \mu t(1 - t)\|g(v) - h(u)\|^2,$$
$$\forall u, v \in K \subseteq D, t \in [0, 1].$$

which can be written as

$$F(g(v)) - F(h(u)) \geq \frac{F(h(u) + t(g(v) - h(u)) - F(h(u))}{t} + (1 - t)\|g(v) - h(u)\|^2.$$

Taking the limit in the above inequality as $t \to 0$, we have

$$F(g(v)) - F(h(u)) \geq \langle F'(h(u)), g(v) - h(u))\rangle + \mu\|g(v) - h(u)\|^2, \forall u, v \in K \subseteq D.$$

which is (7), the required result.

Conversely, let (7) hold. Then, $\forall u, v \in K \subseteq D, t \in [0, 1]$,

$$g(v_t) = h(u) + t(g(v) - h(u)) \in K \subseteq D.$$

It is worth mentioning that, if $t \in [0, 1]$, then two cases may occur, depending on the value of $t : g(v_t)$ and $h(u_t)$.

Thus, from (7), we have

$$\begin{aligned} F(g(v)) - F(g(v_t)) &\geq \langle F'(g(v_t)), g(v) - g(v_t))\rangle + \mu\|g(v) - g(v_t)\|^2 \\ &= (1 - t)F'(g(v_t)), g(v) - h(u)\rangle + \mu(1 - t)^2\|g(v) - h(u)\|^2, \quad (8) \\ &\forall u, v \in K \subseteq D. \end{aligned}$$

In a similar way, we have

$$\begin{aligned} F(h(u)) - F(g(v_t)) &\geq \langle F'(g(v_t)), h(u) - g(v_t))\rangle + \mu\|h(u) - g(v_t))\|^2 \\ &= -tF'(g(v_t)), v - u\rangle + \mu t^2\|g(v) - h(u)\|^2. \end{aligned} \tag{9}$$

Multiplying (8) by t and (9) by $(1-t)$ and adding the resultant, we have

$$F(h(u) + t(g(v) - h(u)) \leq (1-t)F(h(u)) + tF(g(v)) - \mu t(1-t)\|g(v) - h(u)\|^2, \quad \forall u, v \in K \subseteq D,$$

showing that F is a strongly (h, g)-convex function. □

Theorem 2. *Let F be a differentiable strongly (h, g)-convex function on the (h, g)-convex set $K \subseteq D$. Then $F'(.)$ is a strongly monotone operator.*

Proof. Let F be a strongly (h, g)-convex function. Then, from Theorem 1, we have

$$F(g(v)) - F(h(u)) \geq \langle F'(h(u)), g(v) - h(u)\rangle + \mu\|g(v) - h(u)\|^2, \quad \forall u, v \in K \subseteq D. \tag{10}$$

Changing the role of $h(u)$ and $g(v)$ in (10), we have

$$F(h(u)) - F(g(v)) \geq \langle F'(g(v)), h(u) - g(v))\rangle + \mu\|g(v) - h(u)\|^2, \quad \forall u, v \in K \subseteq D. \tag{11}$$

Adding (10) and (11), we have

$$\langle F'(h(u)) - F'(g(v)), h(u) - g(v)\rangle \geq 2\mu\|g(v) - h(u)\|^2, \quad \forall u, v \in K \subseteq D, \tag{12}$$

which shows that $F'(.)$ is a strongly monotone operator.

□

We remark that the converse of Theorem 2 is also true. We have the following result.

Theorem 3. *If the differential operator $F'(.)$ of a differentiable strongly (h, g)-convex function F on the (h, g)-convex set $K \subseteq D$ is strongly monotone operator, then*

$$F(g(v)) - F(h(u)) \geq \langle F'(h(u)), g(v) - h(u)\rangle + 2\mu\frac{1}{2}\|g(v) - h(u)\|^2, \forall u, v \in K \subseteq D. \tag{13}$$

Proof. Let $F'(.)$ be a strongly monotone operator. Then, from (12), we have

$$\langle F'(g(v)), h(u) - g(v)\rangle \geq \langle F'(h(u)), h(u) - g(v))\rangle + 2\mu\|g(v) - h(u)\|^2, \quad \forall u.v \in K \subseteq D. \tag{14}$$

Since $K \subseteq D$ is a (h, g)-convex set, $\forall u, v \in K \subseteq D, t \in [0, 1]$,
$g(v_t) = h(u) + t(g(v) - h(u)) \in K \subseteq D$. Taking $g(v) = g(v_t)$ in (14), we have

$$\begin{aligned} \langle F'(g(v_t)), h(u) - g(v_t)\rangle &\leq \langle F'(h(u)), h(u) - g(v_t)\rangle - 2\mu\|g(v) - h(u)\|^2 \\ &= -t\langle F'(h(u)), g(v) - h(u)\rangle - 2\mu t^2\|g(v) - h(u)\|^2, \end{aligned}$$

which implies that

$$\langle F'(g(v_t)), g(v) - h(u)\rangle \geq \langle F'(h(u)), g(v) - h(u)\rangle + 2\mu t\|g(v) - h(u)\|^2. \tag{15}$$

Consider the auxiliary fixed function ξ defined as

$$\xi(t) = F(h(u) + t(g(v) - h(u))), \quad \forall u, v \in K \subseteq D,$$

from which, we have

$$\xi(1) = F(g(v)), \quad \xi(0) = F(h(u)).$$

Then, from (15), we have

$$\xi'(t) = \langle F'(g(v_t), g(v) - h(u)\rangle \geq \langle F'(h(u)), g(v) - h(u)\rangle + 2\mu t\|g(v) - h(u)\|^2. \tag{16}$$

Integrating (16) between 0 and 1, we have

$$\begin{aligned} \xi(1)-\xi(0) &= \int_0^1 \xi'(t)dt \\ &\geq \langle F'(h(u)), g(v)-h(u)\rangle + \mu\|g(v)-h(u)\|^2. \end{aligned}$$

Thus it follows that

$$F(g(v)) - F(h(u)) \geq \langle F'(h(u)), g(v)-h(u)\rangle + \mu\|g(v)-h(u)\|^2, \forall u, v \in K \subseteq D,$$

which is the required (13). □

We now give a necessary condition for strongly (h, g)-pseudoconvex function.

Theorem 4. *Let $F'(.)$ be a strongly relaxed pseudomonotone operator. Then F is a strongly (h, g)-pseudoconvex function on the (h, g)-convex set $K \subseteq D$.*

Proof. Let $F'(.)$ be a strongly relaxed pseudomonotone operator. Then

$$\langle F'(h(u)), g(v)-h(u)\rangle \geq 0, \forall u, v \in K \subseteq D,$$

implies that

$$\langle F'(g(v)), g(v)-h(u)\rangle \geq \mu\|g(v)-h(u)\|^2, \forall u, v \in K \subseteq D. \tag{17}$$

Since K is a (h, g)-convex set, $\forall u, v \in K \subseteq D, t \in [0, 1]$,
$g(v_t) = h(u) + t(g(v) - h(u)) \in K \subseteq D$. Taking $g(v) = g(v_t)$ in (17), we have

$$\langle F'(g(v_t), g(v)-h(u)\rangle \geq \mu t\|g(v)-h(u)\|^2. \tag{18}$$

Consider the auxiliary fixed function ξ defined as

$$\xi(t) = F(h(u) + t(g(v) - h(u))) = F(g(v_t)), \quad \forall u, v \in K \subseteq D, t \in [0, 1],$$

which is differentiable. Then, using (18), we have

$$\xi'(t) = \langle F'(g(v_t)), g(v) - h(u))\rangle \geq \mu t\|g(v(-h(u)\|^2.$$

Integrating the above relation between 0 to 1, we have

$$\xi(1) - \xi(0) = \int_0^1 \xi'(t)dt \geq \frac{\mu}{2}\|g(v)-h(u)\|^2,$$

that is,

$$F(g(v)) - F(h(u)) \geq \frac{\mu}{2}\|g(v)-h(u)\|^2), \forall u, v \in K \subseteq D,$$

showing that F is a strongly (h, g)-pseudoconvex function. □

Definition 13. *A function F is said to be sharply strongly (h, g)-pseudoconvex on the (h, g)-convex set $K \subseteq D$, if there exists a constant $\mu > 0$ such that*

$$\begin{aligned} &\langle F'(h(u)), g(v)-h(u)\rangle \geq 0 \\ &\Rightarrow \\ &F(g(v)) \geq F(g(v) + t(h(u) - g(v))) + \mu t(1-t)\|g(v)-h(u)\|^2, \forall u, v \in K \subseteq D. \end{aligned}$$

Theorem 5. *Let F be a sharply strongly (h, g)-pseudoconvex function on (h, g)-convex set $K \subseteq D$ with a constant $\mu > 0$. Then*

$$\langle F'(g(v)), g(v)-h(u)\rangle \geq \mu\|g(v)-h(u)\|^2, \forall u, v \in K \subseteq D.$$

Proof. Let F be a sharply strongly (h,g)-pseudoconvex function. Then

$$F(g(v)) \geq F(g(v)+t(h(u)-g(v))) + \mu t(1-t)\|g(v)-h(u)\|^2, \quad \forall u.v \in K \subseteq D, t \in [0,1],$$

from which, we have

$$\frac{F(g(v)+t(h(u)-g(v))-F(g(v))}{t} + \mu t(1-t)\|g(v)-h(u)\|^2 \geq 0.$$

Taking the limit in the above inequality as $t \to 0$, we have

$$\langle F'(g(v)), g(v)-g(u)\rangle \geq \mu\|g(v)-h(u)\|^2, \forall u,v \in K \subseteq D,$$

the required result. □

Theorem 6. *Let $K \subseteq D$ be a (h,g)-convex set. Let f be a strongly (h,g)-affine function. Then F is a strongly (h,g)-convex function, if and only if, $H = F - f$ is a (h,g)-convex function.*

Proof. Let f be a strongly (h,g)-affine function. Then

$$f((1-t)h(u)+tg(v)) = (1-t)f(h(u)) + tf(g(v)) - \mu t(1-t)\|g(v)-h(u)\|^2, \quad \forall u,v \in K \subseteq D. \tag{19}$$

From the strongly (h,g)-convexity of F, we have

$$F((1-t)h(u)+tg(v)) \leq (1-t)F(h(u)) + tF(g(v)) - \mu t(1-t)\|g(v)-h(u)\|^2, \quad \forall u,v \in K \subseteq D. \tag{20}$$

From (19) and (20), we have

$$F((1-t)h(u)+tg(v)) - f((1-t)f(h(u))+tf(g(v)) \leq (1-t)(F(h(u))-f(h(u))) + t(F(g(v))-f(g(v))), \tag{21}$$

from which it follows that

$$\begin{aligned} H((1-t)h(u)+tg(v)) &= F((1-t)h(u)+tg(v)) - f((1-t)h(u)+tg(v)) \\ &\leq (1-t)F(h(u)) + tF(g(v)) - (1-t)f(h(u)(-tf(g(v)) \\ &= (1-t)(F(h(u))-f(h(u))) + t(F(g(v))-f(g(v))), \end{aligned}$$

which shows that $H = F - f$ is a (h,g)-convex function.
The inverse implication is obvious. □

Definition 14. *Let $K \subseteq D$ be a (h,g)-convex set. A function F is said to be a (h,g)-pseudoconvex function with respect to bifunction $B(.,.)$ on $K \subset D$, if*

$$F(g(v)) < F(h(u))$$
$$\Rightarrow$$
$$F(h(u)+(1-t)(g(v),h(u))) < F(h(u)) + t(t-1)B(g(v),h(u)), \forall u,v \in K \subseteq D, t \in [0,1].$$

Theorem 7. *If F is strongly (h,g)-convex function such that $F(g(v)) < F(h(u))$, then the function F is strongly (h,g)-pseudoconvex function.*

Proof. Since $F(g(v)) < F(h(u))$ and F is strongly (h,g)-convex function, then $\forall u, v \in K \subseteq D, \quad t \in [0,1]$, we have

$$\begin{aligned} & F(h(u)+t(g(v)-h(u))) \le F(h(u)) + t(F(g(v)) - F(h(u))) \\ & -\mu t(1-t)\|g(v) - hu)\|^2 \\ < \; & F(h(u)) + t(1-t)(F(g(v)) - F(h(u))) - \mu t(1-t)\|g(v)-h(u)\|^2 \\ = \; & F(h(u)) + t(t-1)(F(h(u)) - F(g(v))) - \mu t(1-t)\|g(v)-h(u)\|^2 \\ < \; & F(h(u)) + t(t-1)B(h(u), g(v)) - \mu t(1-t)\|g(v)-h(u)\|^2, \\ & \forall u, v \in K \subseteq D, \end{aligned}$$

where $B(h(u), g(v)) = F(h(u)) - F(g(v)) > 0$. Hence the function F is a strongly (h,g)-convex function, the required result. □

We now show that inner spaces product can be characterized by the parallelogram laws, which can be obtained from the strongly (h,g)-affine functions.

Setting $F(u) = \|u\|^2$ in Definition 10, we have

$$\|h(u) + t(g(v) - h(u))\|^2 = (1-t)\|h(u)\|^p + t\|g(v)\|^2 - \mu t(1-t)\|g(v) - h(u)\|^2, \quad \forall u, v \in K \subseteq D, t \in [0,1]. \tag{22}$$

Taking $t = \frac{1}{2}$ in (22), we have

$$\|\frac{h(u)+g(v)}{2}\|^2 + \mu\frac{1}{2^2}\|g(v) - h(u)\|^2 = \frac{1}{2}\|h(u)\|^2 + \frac{1}{2}\|g(v)\|^2, \quad \forall u, v \in K \subseteq D. \tag{23}$$

This implies that

$$\|h(u) + g(v)\|^2 + \mu\|g(v) - h(u)\|^2 = 2\{\|h(u)\|^2 + \|g(v)\|^2\}, \forall u, v \in K \subseteq D, \tag{24}$$

which is a new parallelogram law involving two arbitrary functions characterizing the inner product spaces and can be viewed as a novel application of the strongly (h,g)-affine functions.

We now discuss the optimality for the differentiable strongly (h,g)-convex functions, which is the main motivation of our next result.

Theorem 8. *Let F be a differentiable strongly (h,g)-convex function with modulus $\mu > 0$. If $u \in K \subseteq D$ is a minimum of the function F, then*

$$F(g(v)) - F(h(u)) \ge \mu\|g(v) - h(u)\|^2, \quad \forall u, v \in K \subseteq D. \tag{25}$$

Proof. Let $u \in K \subseteq D$ be a minimum of the function F. Then

$$F(h(u)) \le F(g(v)), \forall u \in K \subseteq D. \tag{26}$$

Since $K \subseteq D$ is a (h,g)-convex set, so, $\forall u, v \in K \subseteq D, \quad t \in [0,1]$,

$$g(v_t) = (1-t)h(u) + tg(v) \in K \subseteq D.$$

Taking $g(v) = g(v_t)$ in (26), we have

$$0 \le \lim_{t \to 0} \frac{F(h(u) + t(g(v) - h(u))) - F(h(u))}{t} = \langle F'(h(u)), g(v) - h(u)\rangle. \tag{27}$$

Since F is a differentiable strongly (h,g)-convex function, so

$$\begin{aligned} F(h(u) + t(g(v) - h(u))) \quad \le \quad & F(h(u)) + t(F(g(v)) - F(h(u))) \\ & -\mu t(1-t)\|g(v) - h(u)\|^2, \forall u, v \in K \subseteq D, \end{aligned}$$

from which, using (27), we have

$$\begin{aligned} F(g(v)) - F(h(u)) &\geq \lim_{t\to 0} \frac{F(h(u)+t(g(v)-h(u))) - F(h(u))}{t} + \mu\|g(v)-h(u)\|^2 \\ &= \langle F'(h(u)), g(v)-h(u)\rangle + \mu\|g(v)-h(u)\|^2, \end{aligned}$$

the required result (25). □

Remark 1. *We would like to mention that, if $u \in K \subseteq D$ satisfies the inequality*

$$\langle F'(h(u)), g(v)-h(u)\rangle + \mu\|g(v)-h(u)\|^2 \geq 0, \quad \forall v \in K \subseteq D, \tag{28}$$

then $u \in K \subseteq D$ is the minimum of the function F. The inequality of the type (28) is called the strongly extended general variational inequality and appears to be a new one. It is well known that the inequalities of the type (28) occur, which do not arises as a minimum of the differentiable functions. This motivated us to consider more general problem, which includes the problem (28) as a special case.

For given three operators T, h, g, consider the problem of finding $u \in K \subseteq D$ such that

$$\langle T(h(u)), g(v)-h(u)\rangle + \mu\|g(v)-h(u)\|^2 \geq 0, \quad \forall v \in K \subseteq D, \tag{29}$$

which is called the strongly extended general variational inequality. It is an open problem to discuss the existence of a solution and to develop numerical methods for the problem (29). For more details, see Noor [14, 15, 16, 17, 18, 19] and the references therein.

Conclusion

In this paper, we have introduced and studied a new class of convex functions involving two arbitrary functions, which is called strongly generalized convex function. It is shown that several new classes of strongly convex functions can be obtained as special cases of these strongly generalized convex functions. We have studied the basic properties of these functions. Several important special cases are also discussed, which are obtained from our results. New parallelogram law involving two arbitrary functions characterizing the inner product spaces are obtained, which can be viewed as an interesting application of the strongly (h, g)-affine functions. It is shown that the minimum of the differential generalized convex function can be characterized by the variational inequalities. The interested readers may explore the applications and other properties of the strongly generalized convex functions in various fields of pure and applied sciences. This is an interesting direction of future research.

Acknowledgements

The authors would like to thank the Rector, COMSATS University Islamabad, Islamabad, Pakistan, for providing excellent research and academic environments. The authors wish to express their deepest gratitude to their colleagues, collaborators, friends, referees and the editor, who have direct or indirect contributions in the process of this paper

References

[1] M. Adamek, *On a problem connected with strongly convex functions,* Math. Inequ. Appl., **19(4)**(2016), 1287-1293.

[2] H. Angulo, J. Gimenez, A. M. Moeos and K. Nikodem, *On strongly h-convex functions,* Ann. Funct. Anal. **2(2)**(2011), 85-91.

[3] M. U. Awan, M. A. Noor, M. V. Mihai and K. I. Noor and N. Akhtar, *On approximately harmonic h-convex functions depending on a given function,* Filomat, **33(12)**(2019), 3783-3793.
[4] M. U. Awan, M. A. Noor, T.-S. Du and K. I. Noor, *New refinemnts of fractional Hermite-Hadamard inequality,* RACSAM, **113** (2019), 21-29.
[5] G. Cristescu and M. Găianu, *Shape properties of Noor's convex sets,* Proceed. of the Twelfth Symposium of Mathematics and its Applications, Timisoara, (2009),1-13,
[6] G. Cristescu and L. Lupsa, Non-Connected Convexities and Applications, Kluwer Academic Publisher, Dordrechet, (2002)
[7] F. Glowinski, J. L. Lions and R. Tremileres, Numerical Analysis of Variational Inequalities, NortHolland, Amsterdam, Holland, 1981.
[8] J-B. Jian, *On* (E, F) *generalized convexity,* Internat. J. Math.Sci. **2** (2003) 121–132.
[9] S. Karamardian, *The nonlinear complementarity problems with applications, Part 2,* J. Optim. Theory Appl. **4(3)**(1969), 167-181.
[10] D. Kinderleher and G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press,London, 1980.
[11] J. L. Lions and G. Stampacchia, *Variational inequalities,* Commun. Pure Appl. Math.**20**(1967), 491-512.
[12] C. P. Niculescu and L. E. Persson, Convex Functions and Their Applications, Springer-Verlag, New York, (2018).
[13] K. Nikodem and Z. S. Pales, *Characterizations of inner product spaces by strongly convex functions,* Banach J. Math. Anal., **1**(2011), 83-87.
[14] M. A. Noor, *General variational inequalities,* Appl. Math. Appl. **1**(1988), 119-121.
[15] M. A. Noor, *New apprximations schemes for general variational inequalities,* J. Math. Anal. Appl.**251** (2000), 217-230.
[16] M. A. Noor, *Some developments in general variational inequalities,* Appl. Math. Comput. **152**(2004), 199-277.
[17] M. A. Noor, *Differentiable nonconvex funstions and general variational inequalities,* Appl. Math. Comput, **199(2)**(2008), 623-630.
[18] M. A. Noor, *Extended general variational inequalities,* Appl. Math. Letters, **22** (2009), 182–186
[19] M. A. Noor, *Some aspects of extended general variational inequalities,* Abtract. Appl. Anal. **2012**(2012), ID 303569, pp.16
[20] M. A. Noor and K. I. Noor, *On generalized strongly convex functions involving bifunction,* Appl. Math. Inform. Sci. **13(3)**(2019), 411-416.
[21] J. Pecric, F. Proschan and Y. I. Tong, Convex Functions, Partial Ordering and Statistical Applications, Academic Press, New York, USA.(1992).
[22] B. T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions,* Soviet Math. Dokl., **7**(1966), 2-75.
[23] G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes,* C. R. Acad. Sci. Paris **258**(1964), 4413–4416.
[24] G. Qu and N. Li. *On the exponentially stability of primal-dual gradeint dynamics,* IEEE Control Syst. Letters, **3(1)**(2019). 43-48.
[25] E.A. Youness, *E-convex sets, E-convex functions and E-convex programming,* J. Optim. Theory Appl. **102)** (1999), 439–450.
[26] Y. Zhao and D. Sun, *Alternative theorems for nonlinear projection equations and applications to generalized complementarity problems,* Nonl. Anal. **46** (2001) 853–868.
[27] D. L. Zu and P. Marcotte, *Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities,* SIAM J. Optim., **6(3)**(1996), 714-726.

Department of Mathematics
COMSATS University Islamabad, Islamabad
Pakistan
Email address: noormaslam@gmail.com

Department of Mathematics
COMSATS University Islamabad, Islamabad
Pakistan
Email address: khalidan@gmail.com