

SOME INTEGRAL INEQUALITIES FOR OPERATOR MONOTONE FUNCTIONS

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ABSTRACT. In this paper we prove among others that

$$\int_T p_t f(A_t) A_t d\mu(t) \geq \int_T p_s f(A_s) d\mu(s) \int_T p_t A_t d\mu(t),$$

for an operator monotone function f on $(0, \infty)$, where $(A_t)_{t \in T}$ is a bounded continuous field of positive operators in $\mathcal{B}(H)$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ that satisfy certain conditions while $(p_t)_{t \in T}$ are nonnegative with $\int_T p_t d\mu(t) = \mathbf{1}$. Some particular inequalities of interest are also provided.

1. INTRODUCTION

A real valued continuous function f on an interval I is said to be *operator convex* (*operator concave*) on I if

$$f((1 - \lambda)A + \lambda B) \leq (\geq) (1 - \lambda)f(A) + \lambda f(B) \quad (1)$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I . Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $Sp(A), Sp(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [8] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

Let T be a locally compact Hausdorff space. We say that a field $(A_t)_{t \in T}$ of operators in $\mathcal{B}(H)$ is continuous if the function $t \mapsto A_t$ is norm continuous on T . If in addition μ is a Radon measure on T and the function $t \mapsto \|A_t\|$ is integrable, then we can form the Bochner integral $\int_T A_t d\mu(t)$, which is the unique element in $\mathcal{B}(H)$ such that

$$\varphi \left(\int_T A_t d\mu(t) \right) = \int_T \varphi(A_t) d\mu(t)$$

for every linear functional φ in the norm dual $\mathcal{B}(H)^*$, cf. [11, Section 4.1].

Assume furthermore that there is a field $(\phi_t)_{t \in T}$ of positive linear mappings $\phi_t : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ from $\mathcal{B}(H)$ to another C^* -algebra $\mathcal{B}(K)$, with K a Hilbert space. We

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say that such a field is continuous if the function $t \mapsto \phi_t(A)$ is continuous for every $A \in \mathcal{B}(H)$. If the field $t \mapsto \phi_t(\mathbf{1})$ is integrable with integral $\int_T \phi_t(\mathbf{1}) d\mu(t) = \mathbf{1}$, we say that $(\phi_t)_{t \in T}$ is unital.

The following Jensen's integral inequality has been obtained in [10]:

Theorem 1. *Let $f : I \rightarrow \mathbb{R}$ be an operator convex function defined on an interval I . If $(\phi_t)_{t \in T}$ is a unital field of positive linear mappings $\phi_t : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ , then the inequality*

$$f\left(\int_T \phi_t(A_t) d\mu(t)\right) \leq \int_T \phi_t(f(A_t)) d\mu(t) \quad (2)$$

holds for every bounded continuous field $(A_t)_{t \in T}$ of selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I .

The discrete case is as follows [16]:

$$f\left(\sum_{i=1}^n w_i \phi_i(A_i)\right) \leq \sum_{i=1}^n w_i \phi_i(f(A_i))$$

for operator convex functions f defined on an interval I , where $\phi_i : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$, $i \in \{1, \dots, n\}$ are unital positive linear maps, A_i , $i \in \{1, \dots, n\}$ are selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I and $w_i \geq 0$, $i \in \{1, \dots, n\}$ with $\sum_{i=1}^n w_i = 1$.

Also, if $f : I \rightarrow \mathbb{R}$ is operator convex on I and $U_i \in \mathcal{B}(H)$, $i \in \{1, \dots, n\}$ with $\sum_{i=1}^n U^* U_i = 1$, then [11]

$$f\left(\sum_{i=1}^n U^* A_i U_i\right) \leq \sum_{i=1}^n U^* f(A_i) U_i,$$

where A_i , $i \in \{1, \dots, n\}$ are selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I .

We have the following reverse inequalities in terms of the Fréchet derivative $Df(\cdot)(\cdot)$, see [3]

Theorem 2. *Let $f : I \rightarrow \mathbb{R}$ be an operator convex function of class $C^1(I)$. If $(A_t)_{t \in T}$ is a bounded continuous field $(A_t)_{t \in T}$ of selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I defined on a locally compact Hausdorff space T with a bounded Radon measure μ and such that $\int_T \mathbf{1} d\mu(t) = \mathbf{1}$, then we have the double inequality in terms of the Fréchet derivative $Df(\cdot)(\cdot)$*

$$\begin{aligned} & f(A) - Df(A)(A) + Df(A)\left(\int_T A_t d\mu(t)\right) \\ & \leq \int_T f(A_t) d\mu(t) \\ & \leq f(A) - \int_T Df(A_t)(A) d\mu(t) + \int_T Df(A_t)(A_t) d\mu(t) \end{aligned} \quad (3)$$

for all $A \in \mathcal{SA}_I(H)$.

We have the reverse Jensen's inequality

$$\begin{aligned} 0 & \leq \int_T f(A_t) d\mu(t) - f\left(\int_T A_s d\mu(s)\right) \\ & \leq \int_T Df(A_t)(A_t) d\mu(t) - \int_T Df(A_t)\left(\int_T A_s d\mu(s)\right) d\mu(t). \end{aligned} \quad (4)$$

If $S \in \mathcal{SA}_I(H)$ is an operator satisfying the equality

$$\int_T Df(A_t)(S) d\mu(t) = \int_T Df(A_t)(A_t) d\mu(t), \tag{Sl}$$

then we have the Slater type inequality

$$0 \leq f(S) - \int_T f(A_t) d\mu(t) \leq Df(S)(S) - Df(S)\left(\int_T A_t d\mu(t)\right). \tag{5}$$

Motivated by the above results, in this paper we prove among others that

$$\int_T p_t f(A_t) A_t d\mu(t) \geq \int_T p_s f(A_s) d\mu(s) \int_T p_t A_t d\mu(t),$$

for an operator monotone function f on $(0, \infty)$, where $(A_t)_{t \in T}$ is a bounded continuous field of positive operators in $\mathcal{B}(H)$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ that satisfy certain conditions while $(p_t)_{t \in T}$ are nonnegative with $\int_T p_t d\mu(t) = \mathbf{1}$. Some particular inequalities of interest are also provided.

2. MAIN RESULTS

In 1934, K. Löwner [13] had given a definitive characterization of operator monotone functions as follows, see for instance [2, p. 144-145]:

Lemma 1. *A function $f : (0, \infty) \rightarrow \mathbb{R}$ is operator monotone in $(0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{ts}{t+s} dm(s) \tag{6}$$

where $a \in \mathbb{R}$ and $b \geq 0$ and a positive measure m on $(0, \infty)$ such that

$$\int_0^\infty \frac{s}{1+s} dm(s) < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0, 1]$, [12].

In [6], T. Furuta observed that for $\alpha_j \in [0, 1]$, $j = 1, \dots, n$ the functions

$$g(t) := \left(\sum_{j=1}^n t^{-\alpha_j} \right)^{-1} \quad \text{and} \quad h(t) = \sum_{j=1}^n (1 + t^{-1})^{-\alpha_j}$$

are operator monotone in $(0, \infty)$.

Let $f(t)$ be a continuous function $(0, \infty) \rightarrow (0, \infty)$. It is known that $f(t)$ is operator monotone if and only if $g(t) = t/f(t) =: f^*(t)$ is also operator monotone, see for instance [6] or [14].

Consider the family of functions defined on $(0, \infty)$ and $p \in [-1, 2] \setminus \{0, 1\}$ by

$$f_p(t) := \frac{p-1}{p} \left(\frac{t^p - 1}{t^{p-1} - 1} \right)$$

and

$$f_0(t) := \frac{t}{1-t} \ln t,$$

$$f_1(t) := \frac{t-1}{\ln t} \text{ (logarithmic mean).}$$

We also have the functions of interest

$$f_{-1}(t) = \frac{2t}{1+t} \text{ (harmonic mean), } f_{1/2}(t) = \sqrt{t} \text{ (geometric mean).}$$

We have the representations [4]:

Lemma 2. *Assume that the function $f : (0, \infty) \rightarrow \mathbb{R}$ is operator monotone in $(0, \infty)$ and has the representation (6). Then for all $A, B > 0$ we have*

$$\begin{aligned} & [f(B) - f(A)](B - A) - b(B - A)^2 \\ &= \int_0^\infty s^2 \left[\int_0^1 [((1-t)A + tB + s1_H)^{-1}(B - A)]^2 dt \right] dm(s) \end{aligned} \quad (7)$$

and

$$\begin{aligned} & (B - A)[f(B) - f(A)] - b(B - A)^2 \\ &= \int_0^\infty s^2 \left[\int_0^1 [(B - A)((1-t)A + tB + s1_H)^{-1}]^2 dt \right] dm(s), \end{aligned} \quad (8)$$

where b and m are provided in Lemma 1.

Proof. Since the function $f : (0, \infty) \rightarrow \mathbb{R}$ is operator monotone in $(0, \infty)$, then f can be written as in the equation (2) and for $A, B > 0$ we have the representation

$$\begin{aligned} & f(B) - f(A) \\ &= b(B - A) + \int_0^\infty s \left[B(B + s1_H)^{-1} - A(A + s1_H)^{-1} \right] dm(s). \end{aligned} \quad (9)$$

Observe that for $s > 0$

$$\begin{aligned} & B(B + s1_H)^{-1} - A(A + s1_H)^{-1} \\ &= (B + s1_H - s1_H)(B + s1_H)^{-1} - (A + s1_H - s1_H)(A + s1_H)^{-1} \\ &= (B + s1_H)(B + s1_H)^{-1} - s1_H(B + s1_H)^{-1} \\ &\quad - (A + s1_H)(A + s1_H)^{-1} + s1_H(A + s1_H)^{-1} \\ &= 1_H - s1_H(B + s1_H)^{-1} - 1_H + s1_H(A + s1_H)^{-1} \\ &= s \left[(A + s1_H)^{-1} - (B + s1_H)^{-1} \right]. \end{aligned}$$

Therefore, (9) becomes, see also [7]

$$f(B) - f(A) = b(B - A) + \int_0^\infty s^2 \left[(A + s1_H)^{-1} - (B + s1_H)^{-1} \right] dm(s). \quad (10)$$

Let $T, S > 0$. The function $f(t) = -t^{-1}$ is operator monotonic on $(0, \infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) : \lim_{t \rightarrow 0} \left[\frac{f(T + tS) - f(T)}{t} \right] = T^{-1}ST^{-1} \quad (11)$$

for $T, S > 0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0, 1]$ by

$$f_{C,D}(t) = f((1-t)C + tD), \quad t \in [0, 1].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C, D] := \{(1-t)C + tD, t \in [0, 1]\}$, then we have, by the properties of the Bochner integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt} (f_{C,D}(t)) dt = \int_0^1 \nabla f_{(1-t)C+tD} (D - C) dt. \tag{12}$$

If we write this equality for the function $f(t) = -t^{-1}$ and $C, D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1} (D - C) ((1-t)C + tD)^{-1} dt. \tag{13}$$

Now, if we replace in (13) $C = A + s1_H$ and $D = B + s1_H$ for $s > 0$, then

$$\begin{aligned} & (A + s1_H)^{-1} - (B + s1_H)^{-1} \\ &= \int_0^1 ((1-t)A + tB + s1_H)^{-1} (B - A) ((1-t)A + tB + s1_H)^{-1} dt. \end{aligned} \tag{14}$$

By the representation (10), we derive the following identity of interest

$$\begin{aligned} f(B) - f(A) &= b(B - A) \\ &+ \int_0^\infty s^2 \left[\int_0^1 ((1-t)A + tB + s1_H)^{-1} \right. \\ &\quad \left. \times (B - A) ((1-t)A + tB + s1_H)^{-1} dt \right] dm(s) \end{aligned} \tag{15}$$

for $A, B > 0$.

If we multiply this identity at the right with $B - A$ we get

$$\begin{aligned} & [f(B) - f(A)](B - A) \\ &= b(B - A)^2 \\ &+ \int_0^\infty s^2 \left[\int_0^1 [((1-t)A + tB + s1_H)^{-1} (B - A)]^2 dt \right] dm(s) \end{aligned}$$

for $A, B > 0$, which is (7).

If we multiply (15) at the left with $B - A$ we get

$$\begin{aligned} & (B - A)[f(B) - f(A)] \\ &= b(B - A)^2 \\ &+ \int_0^\infty s^2 \left[\int_0^1 [(B - A)((1-t)A + tB + s1_H)^{-1}]^2 dt \right] dm(s) \end{aligned}$$

for $A, B > 0$, which is (8).

The positivity in both (7) and (8) is obvious. □

Some necessary and sufficient conditions for the operators $A, B > 0$ such that the inequality $(f(B) - f(A))(B - A) \geq 0$ holds for any operator monotone function f on $(0, \infty)$ are included in the following result, see [4]:

Lemma 3. *Let $A, B > 0$. The following statements are equivalent:*

(i) *For all $s \geq 0$,*

$$(A + s)^{-1}(B + s) + (B + s)^{-1}(A + s) \geq 2. \tag{16}$$

(ii) For all $s \geq 0$,

$$\int_0^1 \left[((1-t)A + tB + s)^{-1} (B - A) \right]^2 dt \geq 0.$$

(iii) For all $s \geq 0$,

$$(\ell_s(B) - \ell_s(A))(B - A) \geq 0,$$

where $\ell_s(t) = -(t+s)^{-1}$, $t > 0$.

(iv) For all operator monotone function f on $(0, \infty)$,

$$(f(B) - f(A))(B - A) \geq 0. \quad (17)$$

(v) For all operator monotone function f on $(0, \infty)$,

$$(B - A)(f(B) - f(A)) \geq 0. \quad (18)$$

We remark that if A, B are as in Lemma 2, then obviously

$$(f(B) - f(A))(B - A) = (B - A)(f(B) - f(A)),$$

namely

$$f(A)B + f(B)A = Bf(A) + Af(B).$$

We also have, see [4]:

Lemma 4. Let $A, B > 0$, then the statements (i) and

(i') Operator $A(B+s)^{-1} + B(A+s)^{-1}$ is selfadjoint for all $s \geq 0$, are equivalent.

We define the class of operators

$$\mathfrak{C}\mathfrak{I}_{(0,\infty)}(H) := \{(A, B) \mid A, B > 0 \text{ and satisfy condition (i')}\}.$$

We observe that if $(A, B) \in \mathfrak{C}\mathfrak{I}_{(0,\infty)}(H)$ then $(B, A) \in \mathfrak{C}\mathfrak{I}_{(0,\infty)}(H)$.

Also if $AB = BA$, $A, B > 0$, then $U_s := (A+s)^{-1}(B+s)$ and $U_s^{-1} = (B+s)^{-1}(A+s)$ are selfadjoint and since $U_s + U_s^{-1} \geq 2$, $s \geq 0$ we derive that $(A, B) \in \mathfrak{C}\mathfrak{I}_{(0,\infty)}(H)$. Therefore, if $\mathfrak{C}\mathfrak{O}_{(0,\infty)}(H)$ is the class of all pairs of commutative operators $A, B > 0$, then we have

$$\emptyset \neq \mathfrak{C}\mathfrak{O}_{(0,\infty)}(H) \subset \mathfrak{C}\mathfrak{I}_{(0,\infty)}(H). \quad (19)$$

We have the following operator inequalities:

Theorem 3. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be an operator monotone function on $(0, \infty)$ defined by (6). If $(A_t)_{t \in T}$ is a bounded continuous field of positive operators defined on a locally compact Hausdorff space T with a bounded Radon measure μ and such that $(A_t, A_s) \in \mathfrak{C}\mathfrak{I}_{(0,\infty)}(H)$ for all $t, s \in T$ while $(p_t)_{t \in T}$ are nonnegative with $\int_T p_t d\mu(t) = \mathbf{1}$, then

$$\begin{aligned} & \int_T p_t f(A_t) A_t d\mu(t) - \int_T p_s f(A_s) d\mu(s) \int_T p_t A_t d\mu(t) \\ & \geq b \left(\int_T p_t A_t^2 d\mu(t) - \left(\int_T p_t A_t d\mu(t) \right)^2 \right) \geq 0. \end{aligned} \quad (20)$$

Proof. From (7) we get

$$[f(A_t) - f(A_s)](A_t - A_s) \geq b(A_t - A_s)^2 \geq 0$$

for all $t, s \in T$.

This inequality is equivalent to

$$\begin{aligned} & f(A_t)A_t - f(A_s)A_t - f(A_t)A_s + f(A_s)A_s \\ & \geq b(A_t^2 - A_tA_s - A_sA_t + A_s^2) \geq 0 \end{aligned} \quad (21)$$

for all $t, s \in T$.

If we multiply (21) by $p_t p_s \geq 0$ for all $t, s \in T$ we get

$$\begin{aligned} & p_t p_s f(A_t)A_t - p_t p_s f(A_s)A_t - p_t p_s f(A_t)A_s + p_t p_s f(A_s)A_s \\ & \geq b(p_t p_s A_t^2 - p_t p_s A_t A_s - p_t p_s A_s A_t + p_t p_s A_s^2) \geq 0 \end{aligned} \quad (22)$$

for all $t, s \in T$.

If we take the integral over $t \in T$ in (22) and observe that $\int_T p_t d\mu(t) = 1$, then we get

$$\begin{aligned} & p_s \int_T p_t f(A_t)A_t d\mu(t) - p_s f(A_s) \int_T p_t A_t d\mu(t) \\ & - \left(\int_T p_t f(A_t) d\mu(t) \right) p_s A_s + p_s f(A_s) A_s \\ & \geq b \left(p_s \int_T p_t A_t^2 d\mu(t) - \left(\int_T p_t A_t d\mu(t) \right) p_s A_s \right. \\ & \quad \left. - p_s A_s \int_T p_t A_t d\mu(t) + p_s A_s^2 \right) \\ & \geq 0 \end{aligned} \quad (23)$$

for all $s \in T$.

If we take the integral over $s \in T$ in (23) and observe that $\int_T p_s d\mu(s) = 1$, then we get

$$\begin{aligned} & \int_T p_t f(A_t)A_t d\mu(t) - \int_T p_s f(A_s) d\mu(s) \int_T p_t A_t d\mu(t) \\ & - \left(\int_T p_t f(A_t) d\mu(t) \right) \int_T p_s A_s d\mu(s) + \int_T p_s f(A_s) A_s d\mu(s) \\ & \geq b \left(\int_T p_t A_t^2 d\mu(t) - \left(\int_T p_t A_t d\mu(t) \right) \int_T p_s A_s d\mu(s) \right. \\ & \quad \left. - \int_T p_s A_s d\mu(s) \int_T p_t A_t d\mu(t) + \int_T p_s A_s^2 d\mu(s) \right) \\ & \geq 0, \end{aligned}$$

namely

$$\begin{aligned} & 2 \int_T p_t f(A_t)A_t d\mu(t) - 2 \int_T p_s f(A_s) d\mu(s) \int_T p_t A_t d\mu(t) \\ & \geq 2b \left(\int_T p_t A_t^2 d\mu(t) - \left(\int_T p_t A_t d\mu(t) \right)^2 \right) \geq 0, \end{aligned}$$

which is equivalent to (20). \square

Remark 1. We observe that for an operator monotone function f on $(0, \infty)$, we have

$$\int_T p_t f(A_t)A_t d\mu(t) \geq \int_T p_t f(A_t) d\mu(t) \int_T p_t A_t d\mu(t), \quad (24)$$

where $(A_t)_{t \in T}$ is as in Theorem 3 and $(p_t)_{t \in T}$ are nonnegative with $\int_T p_t d\mu(t) = \mathbf{1}$.

Corollary 1. Let $g : (0, \infty) \rightarrow (0, \infty)$ be an operator monotone function on $(0, \infty)$. If $(A_t)_{t \in T}$ is as in Theorem 3 and $(p_t)_{t \in T}$ are nonnegative with $\int_T p_t d\mu(t) = \mathbf{1}$, then

$$\int_T p_t [g(A_t)]^{-1} A_t^2 d\mu(t) \geq \int_T p_t [g(A_t)]^{-1} A_t d\mu(t) \int_T p_t A_t d\mu(t), \quad (25)$$

Since $g : (0, \infty) \rightarrow (0, \infty)$ is an operator monotone function on $(0, \infty)$, then $f(t) = t/g(t)$, $t > 0$ is operator monotone and by (24) we get (25).

The discrete case is as follows:

Corollary 2. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be an operator monotone function on $(0, \infty)$ defined by (6). If A_k , $k = 1, \dots, n$ is an n -tuple of positive operators such that $(A_k, A_j) \in \mathfrak{C}\mathfrak{I}_{(0, \infty)}(H)$ for all $k, j = 1, \dots, n$ and p_k are nonnegative with $\sum_{k=1}^n p_k = 1$, then

$$\begin{aligned} & \sum_{k=1}^n p_k f(A_k) A_k - \sum_{k=1}^n p_k f(A_k) \sum_{k=1}^n p_k A_k \\ & \geq b \left[\sum_{k=1}^n p_k A_k^2 - \left(\sum_{k=1}^n p_k A_k \right)^2 \right] \geq 0. \end{aligned} \quad (26)$$

For other recent inequalities for operator monotone functions, see [1] and [15]-[17].

3. RELATED RESULTS

A related result is as follows:

Theorem 4. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be an operator monotone function on $(0, \infty)$ defined by (6) and $g : (0, \infty) \rightarrow (0, \infty)$ a continuous function. If $(A_t)_{t \in T}$ is a bounded continuous field of positive operators defined on a locally compact Hausdorff space T with a bounded Radon measure μ such that $(g(A_t), g(\int_T p_s A_s d\mu(s))) \in \mathfrak{C}\mathfrak{I}_{(0, \infty)}(H)$ for $t \in T$ and $(p_t)_{t \in T}$ are nonnegative with $\int_T p_t d\mu(t) = \mathbf{1}$, then

$$\begin{aligned} & \int_T p_t (f \circ g)(A_t) g(A_t) d\mu(t) \\ & + (f \circ g) \left(\int_T p_s A_s d\mu(s) \right) g \left(\int_T p_s A_s d\mu(s) \right) \\ & - (f \circ g) \left(\int_T p_s A_s d\mu(s) \right) \int_T p_t g(A_t) d\mu(t) \\ & - \int_T p_t (f \circ g)(A_t) d\mu(t) g \left(\int_T p_s A_s d\mu(s) \right) \\ & \geq b \left[\int_T p_t g^2(A_t) d\mu(t) + g^2 \left(\int_T p_s A_s d\mu(s) \right) \right. \\ & \left. - \int_T p_t g(A_t) d\mu(t) g \left(\int_T p_s A_s d\mu(s) \right) \right. \\ & \left. - g \left(\int_T p_s A_s d\mu(s) \right) \int_T p_t g(A_t) d\mu(t) \right] \\ & \geq 0. \end{aligned} \quad (27)$$

Proof. We have from (7) for $A = g(A_t)$ and $B = g\left(\int_T p_s A_s d\mu(s)\right)$ that

$$\begin{aligned} & \left[f(g(A_t)) - f\left(g\left(\int_T p_s A_s d\mu(s)\right)\right) \right] \left(g(A_t) - g\left(\int_T p_s A_s d\mu(s)\right) \right) \\ & \geq b \left(g(A_t) - g\left(\int_T p_s A_s d\mu(s)\right) \right)^2 \geq 0 \end{aligned} \quad (28)$$

for all $t \in T$.

If we multiply with $p_t \geq 0$ and integrate over $t \in T$ in (28), then we get

$$\begin{aligned} & \int_T p_t \left[f(g(A_t)) - f\left(g\left(\int_T p_s A_s d\mu(s)\right)\right) \right] \\ & \times \left(g(A_t) - g\left(\int_T p_s A_s d\mu(s)\right) \right) d\mu(t) \\ & \geq b \int_T p_t \left(g(A_t) - g\left(\int_T p_s A_s d\mu(s)\right) \right)^2 d\mu(t) \geq 0. \end{aligned} \quad (29)$$

Observe that

$$\begin{aligned} & \int_T p_t \left[f(g(A_t)) - f\left(g\left(\int_T p_s A_s d\mu(s)\right)\right) \right] \\ & \times \left(g(A_t) - g\left(\int_T p_s A_s d\mu(s)\right) \right) d\mu(t) \\ & = \int_T p_t f(g(A_t)) g(A_t) \mu(t) - f\left(g\left(\int_T p_s A_s d\mu(s)\right)\right) \int_T p_t g(A_t) d\mu(t) \\ & - \int_T p_t f(g(A_t)) d\mu(t) g\left(\int_T p_s A_s d\mu(s)\right) \\ & + \int_T p_t d\mu(t) f\left(g\left(\int_T p_s A_s d\mu(s)\right)\right) g\left(\int_T p_s A_s d\mu(s)\right) \\ & = \int_T p_t f(g(A_t)) g(A_t) d\mu(t) + f\left(g\left(\int_T p_s A_s d\mu(s)\right)\right) g\left(\int_T p_s A_s d\mu(s)\right) \\ & - f\left(g\left(\int_T p_s A_s d\mu(s)\right)\right) \int_T p_t g(A_t) d\mu(t) \\ & - \int_T p_t f(g(A_t)) d\mu(t) g\left(\int_T p_s A_s d\mu(s)\right) \end{aligned}$$

and

$$\begin{aligned} & \int_T p_t \left(g(A_t) - g\left(\int_T p_s A_s d\mu(s)\right) \right)^2 d\mu(t) \\ & = \int_T p_t \left[g^2(A_t) - g(A_t) g\left(\int_T p_s A_s d\mu(s)\right) \right. \\ & \left. - g\left(\int_T p_s A_s d\mu(s)\right) g(A_t) + g^2\left(\int_T p_s A_s d\mu(s)\right) \right] d\mu(t) \end{aligned}$$

$$\begin{aligned}
&= \int_T p_t g^2(A_t) d\mu(t) + g^2\left(\int_T p_s A_s d\mu(s)\right) \\
&\quad - \int_T p_t g(A_t) d\mu(t) g\left(\int_T p_s A_s d\mu(s)\right) \\
&\quad - g\left(\int_T p_s A_s d\mu(s)\right) \int_T p_t g(A_t) d\mu(t).
\end{aligned}$$

By making use of (29), we derive (27). \square

Corollary 3. *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be an operator monotone function on $(0, \infty)$ defined by (6) and $g : (0, \infty) \rightarrow (0, \infty)$ a continuous function. If A_k , $k = 1, \dots, n$ is an n -tuple of positive operators and such that $(g(A_k), g(\sum_{k=1}^n p_k A_k)) \in \mathfrak{C}\mathfrak{I}_{(0, \infty)}(H)$ for $k = 1, \dots, n$ and p_k are nonnegative with $\sum_{k=1}^n p_k = 1$, then*

$$\begin{aligned}
&\sum_{k=1}^n p_k (f \circ g)(A_k) g(A_k) + (f \circ g)\left(\sum_{k=1}^n p_k A_k\right) g\left(\sum_{k=1}^n p_k A_k\right) \tag{30} \\
&\quad - (f \circ g)\left(\sum_{k=1}^n p_k A_k\right) \sum_{k=1}^n p_k g(A_k) - \sum_{k=1}^n p_k (f \circ g)(A_k) g\left(\sum_{k=1}^n p_k A_k\right) \\
&\geq b \left[\sum_{k=1}^n p_k g^2(A_k) + g^2\left(\sum_{k=1}^n p_k A_k\right) \right. \\
&\quad \left. - \sum_{k=1}^n p_k g(A_k) g\left(\sum_{k=1}^n p_k A_k\right) - g\left(\sum_{k=1}^n p_k A_k\right) \sum_{k=1}^n p_k g(A_k) \right] \\
&\geq 0.
\end{aligned}$$

4. SOME EXAMPLES

Let $A, B > 0$, $p : [0, 1] \rightarrow [0, \infty)$ be a continuous function and $d\mu(t) = dt$ the usual Lebesgue measure on $[0, 1]$. In the following we consider $A_t := (1-t)A + tB$, with $(A_t, A_s) \in \mathfrak{C}\mathfrak{I}_{(0, \infty)}(H)$ for all $t, s \in T$, $p_t = p(t)$, $t \in [0, 1]$ and assume that $\int_0^1 p(t) dt = 1$. We observe that for an operator monotone function f on $(0, \infty)$, we have by (24) that

$$\begin{aligned}
&\int_0^1 p(t) ((1-t)A + tB) f((1-t)A + tB) dt \tag{31} \\
&\geq \int_0^1 p(t) f((1-t)A + tB) dt \int_0^1 p(t) ((1-t)A + tB) dt.
\end{aligned}$$

We observe that, if p is symmetric on $[0, 1]$ namely $p(1-t) = p(t)$, for all $t \in [0, 1]$, then

$$\begin{aligned}
\int_0^1 p(t) ((1-t)A + tB) dt &= \left(\int_0^1 p(t)(1-t) dt\right) A + \left(\int_0^1 p(t) t dt\right) B \\
&= \left(\int_0^1 p(1-t)(1-t) dt\right) A + \left(\int_0^1 p(t) t dt\right) B \\
&= \left(\int_0^1 p(s)(s) ds\right) A + \left(\int_0^1 p(t) t dt\right) B \\
&= \left(\int_0^1 p(t) t dt\right) (A + B).
\end{aligned}$$

Then by (31) we get

$$\begin{aligned} & \int_0^1 p(t) ((1-t)A + tB) f((1-t)A + tB) dt \\ & \geq \left(\int_0^1 p(t) t dt \right) \left(\int_0^1 p(t) f((1-t)A + tB) dt \right) (A+B). \end{aligned} \quad (32)$$

Observe that

$$\int_0^1 p(t) t dt = \int_0^1 p(1-t) (1-t) dt = \int_0^1 p(t) (1-t) dt.$$

Then

$$\int_0^1 p(t) t dt + \int_0^1 p(t) (1-t) dt = \int_0^1 p(t) dt = 1,$$

which implies that $\int_0^1 p(t) t dt = 1/2$.

Then by (32) we get

$$\begin{aligned} & \int_0^1 p(t) ((1-t)A + tB) f((1-t)A + tB) dt \\ & \geq \left(\int_0^1 p(t) f((1-t)A + tB) dt \right) \left(\frac{A+B}{2} \right). \end{aligned} \quad (33)$$

From (33) we derive the inequalities

$$\int_0^1 p(t) ((1-t)A + tB)^{r+1} dt \geq \int_0^1 p(t) ((1-t)A + tB)^r dt \left(\frac{A+B}{2} \right), \quad (34)$$

for $r \in (0, 1)$ and

$$\begin{aligned} & \int_0^1 p(t) ((1-t)A + tB) \ln((1-t)A + tB) dt \\ & \geq \int_0^1 p(t) \ln((1-t)A + tB) dt \left(\frac{A+B}{2} \right). \end{aligned} \quad (35)$$

If $A_t := (1-t)A + tB$ is defined on $[0, 1]$ such that $\left(g(A_t), g\left(\int_0^1 p(t) A_s ds\right) \right) \in \mathfrak{C}_{(0, \infty)}(H)$ for $t \in T$, while $g : (0, \infty) \rightarrow (0, \infty)$ is a continuous function, then from (27) we obtain

$$\begin{aligned} & \int_0^1 p(t) (f \circ g)((1-t)A + tB) g((1-t)A + tB) dt \\ & + (f \circ g) \left(\int_0^1 p(t) ((1-t)A + tB) dt \right) g \left(\int_0^1 p(t) ((1-t)A + tB) dt \right) \\ & \geq (f \circ g) \left(\int_0^1 p(t) ((1-t)A + tB) dt \right) \int_0^1 p(t) g((1-t)A + tB) dt \\ & + \int_0^1 p(t) (f \circ g)((1-t)A + tB) dt g \left(\int_0^1 p(t) ((1-t)A + tB) dt \right), \end{aligned} \quad (36)$$

where $f : (0, \infty) \rightarrow \mathbb{R}$ is an operator monotone function on $(0, \infty)$.

If p is symmetric on $[0, 1]$, then

$$\int_0^1 p(t) ((1-t)A + tB) dt = \left(\int_0^1 p(t) t dt \right) (A+B) = \frac{A+B}{2}$$

and by (36) we get

$$\begin{aligned} & \int_0^1 p(t) (f \circ g) ((1-t)A + tB) g((1-t)A + tB) dt \\ & + (f \circ g) \left(\frac{A+B}{2} \right) g \left(\frac{A+B}{2} \right) \\ & \geq (f \circ g) \left(\frac{A+B}{2} \right) \int_0^1 p(t) g((1-t)A + tB) dt \\ & + \left(\int_0^1 p(t) (f \circ g) ((1-t)A + tB) dt \right) g \left(\frac{A+B}{2} \right), \end{aligned} \quad (37)$$

where $f : (0, \infty) \rightarrow \mathbb{R}$ is an operator monotone function on $(0, \infty)$ while $g : (0, \infty) \rightarrow (0, \infty)$ is a continuous function.

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