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AN INTEGRAL EQUATION FROM PHYSICS - A SYNTHESIS SURVEY - PART III

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Abstract. This is the third part of the synthesis survey on the study of the integral equation from physics:

$$x(t) = \int_a^b K(t, s, x(s), x(a), x(b))ds + f(t),\ t \in [a, b],$$

and contains the results concerning the approximation of its solution by applying numerical methods. This part ends with an example.

1. Introduction

The results obtained in the study of the integral equation from physics:

$$x(t) = \int_a^b K(t, s, x(s), x(a), x(b))ds + f(t),\ t \in [a, b], \tag{1}$$

where $K : [a, b] \times [a, b] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ or $K : [a, b] \times [a, b] \times J^3 \longrightarrow \mathbb{R}$, $J \subset \mathbb{R}$ closed interval, and $f : [a, b] \longrightarrow \mathbb{R}$. were organized in a three-part presentation.

In the first part of this synthesis survey, the results on the existence and uniqueness of the solution of this integral equation were presented. Some properties of the solution of this integral equation formulated as integral inequalities, were also presented.

In the second part of the synthesis survey, the results on the continuous data dependence, the differentiability with respect to a and b, and the differentiability with respect to a parameter of the solution of this integral equation were presented.

This last part of this synthesis survey is dedicated to the presentation of the obtained results regarding the approximation of the solution of the integral equation (1), which were published in the papers [1], ([5]-[9]), [11] and [15].

Section 2 contains the notations and the existing results that were used to obtain the algorithms of approximating the solution of integral equation (1).

In section 3 we present a synthesis of the study of approximating the solution of the integral equation (1) containing the results obtained by using the method of successive approximations. For approximate calculus of the integrals that appear in terms of the successive approximations sequence, the following quadrature formulas were used: the trapezoids' rule, the Simpson's rule and the rectangles' quadrature formula.

Finally, section 4 concludes the article with some examples.

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2. Existing results

To study the approximation of the solution of integral equation (1), we used, as in the first two parts, the Banach space $C[a,b]$,

$$C[a,b] = \{x : [a,b] \longrightarrow \mathbf{R} | x \ continuous \ function\},$$

endowed with the Chebyshev norm

$$\|x\|_C := \max_{t\in[a,b]} |x(t)| \,, \quad for \ all \ x \in C[a,b]. \tag{2}$$

In order to establish an algorithm for approximating the solution of the integral equation (1), the existence of a unique solution of this equation is required. Therefore, the conditions in the theorems of existence and uniqueness of the solution of the integral equation (1), presented in the first part of this survey, must be satisfied. We present these theorems below.

Theorem 1 (Ambro M., [1]). *If*

(i) $K \in C([a,b] \times [a,b] \times \mathbb{R}^3)$, $f \in C[a,b]$;

(ii) there exists $L > 0$ *such that*

$$|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)| \leq L(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|),$$

for all $t,s \in [a,b]$, $u_i, v_i \in \mathbb{R}$, $i = 1,2,3$;

(iii) $3L(b-a) < 1$ *(contraction condition),*

then the integral equation (1) has a unique solution $x^* \in C[a,b]$, *which can be obtained by the successive approximations method, starting at any element* $x_0 \in C[a,b]$. *In addition, if* x_n *is the n-th successive approximation, then the following estimation is hold:*

$$\|x^* - x_n\|_{C[a,b]} \leq \frac{[3L(b-a)]^n}{1 - 3L(b-a)} \|x_0 - x_1\|_{C[a,b]} \,. \tag{3}$$

Theorem 2 (Ambro M., [1]). *Suppose that the following conditions:*

(i) $K \in C([a,b] \times [a,b] \times J^3)$, $J \in \mathbb{R}$ *closed interval,* $f \in C[a,b]$;

(ii) there exists $L > 0$ *such that*

$$|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)| \leq L(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|),$$

for all $t,s \in [a,b]$, $u_i, v_i \in J$, $i = 1,2,3$;

(iii) $3L(b-a) < 1$ *(contraction condition),*

are satisfied.

If $r > 0$ *is a positive real number such that*

$$x \in \overline{B}(f;r) \implies x(t) \in J \subset \mathbb{R}, \quad and \quad M(b-a) \leq r,$$

where M *is a positive constant, such that*

$$|K(t,s,u_1,u_2,u_3)| \leq M, \ for \ all \ t,s \in [a,b], u_1,u_2,u_3 \in J,$$

then the integral equation (1) has a unique solution $x^* \in \overline{B}(f;r) \subset C[a,b]$, *which can be obtained by the successive approximations method, starting at any element* $x_0 \in \overline{B}(f;r) \subset C[a,b]$. *Moreover, if we denote by* x_n *is the n-th successive approximation, then the estimation (3) is satisfied.*

These theorems was proved by applying *The Banach Contraction Principle.*

In the process of determining an algorithm for approximating the solution of the integral equation (1), the method of successive approximations was used. For the approximate calculus of the integrals that appear in the terms of the successive approximations sequence, trapezoids' formula, rectangles' formula and Simpson's formula were used; for the basic results that we used see references [2], [3], [4], [15], [16] and [19]; for the results obtained as applications see references [1], [5]-[9] and [11].

In what follows, we recall these methods for calculating the approximate value of the integral of a function f.

2.1. **Trapezoids' formula.**

The trapezoids' method for approximate integration of a function f consists of approximating the function f with a polygonal function, i.e. to approximate a given function f with a polygonal line with vertices on the graph.

Let $f \in C^2[a,b]$ be a function. A formula of the approximate calculus of the integral $\int_a^b f(t)dt$ is:

$$\int_a^b f(t)dt = \frac{b-a}{2}\big[f(a) + f(b)\big] + R(f), \tag{4}$$

where $R(f)$ represent the remainder of this formula. Due to the geometrical interpretation of (4), this relation is called *trapezoids' formula* or *trapezoids' rule.*

To get a better result, it is considered a division Δ of the interval $[a,b]$ into n equal parts by the points $a = t_0 < t_1 < \cdots < t_n = b$ and we apply the trapezoids' formula (4), to each subinterval $[t_{i-1}, t_i]$. Under these conditions it is obtains the following trapezoids' formula (see references [4], [15], [16]):

$$\int_a^b f(t)dt = \frac{b-a}{2n}\left[f(a) + 2\sum_{i=1}^{n-1} f(t_i) + f(b)\right] + R(f), \tag{5}$$

where $R(f) = \sum_{i=1}^{n} R_i(f)$ is the remainder of the formula (5) and for it we have the estimate given by the relation:

$$|R(f)| \le M^T \frac{(b-a)^3}{12n^2}, \tag{6}$$

where was denoted

$$M^T = \max_{t\in[a,b]} |f"(t)|. \tag{7}$$

2.2. **Simpson's formula.**

The approximation method that results using the Simpson's formula, consists in approximation of a given function f on certain intervals with a second degree polynomial, i.e. to approximate the graph of the function f on certain intervals with a parable.

Let $f \in C^4[a,b]$ be a function. The Simpson's formula of the approximate calculus of the integral $\int_a^b f(t)dt$ is:

$$\int_a^b f(t)dt = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] + R(f), \tag{8}$$

where $R(f)$ represent the remainder of this formula.

To get a better result, it is considered a division Δ of the interval $[a,b]$ into n equal parts by the points $a = t_0 < t_1 < \cdots < t_n = b$ and we apply the Simpson's formula (8), to each subinterval $[t_{i-1}, t_i]$. Under these conditions it was obtained the following Simpson's quadrature formula (see references [15], [16]):

$$\int_a^b f(t)dt = \frac{b-a}{6n}\left[f(a) + 2\sum_{i=1}^{n-1} f(t_i) + 4\sum_{i=1}^{n-1} f\left(\frac{t_i + t_{i+1}}{2}\right) + f(b)\right] + R(f), \tag{9}$$

with an estimate of the remainder $R(f) = \sum_{i=1}^{n} R_i(f)$, given by the relation:

$$|R(f)| \leq M^S \frac{(b-a)^5}{2880n^4}, \tag{10}$$

where was denoted

$$M^S = \max_{t\in[a,b]} \left| f^{(4)}(t) \right|. \tag{11}$$

2.3. **Rectangles' formula.**

The rectangles' method for approximate integration of a function f consists in approximating the function f with a constant function on intervals, i.e. the approximation of the graph of function f with a polygonal line with sides parallel to the coordinate axes.

Let $f \in C^1[a,b]$ be a function, Δ a division of the interval $[a,b]$ into n equal parts by the points $a = t_0 < t_1 < \cdots < t_n = b$ and $\sigma_\Delta(f)$ an integral sum corresponding to this division Δ:

$$\sigma_\Delta(f) = \sum_{i=0}^{n-1} f(\xi_i)(t_{i+1} - t_i), \quad t_i \leq \xi_i \leq t_{i+1}.$$

If Δ is a sufficiently fine division, i.e. the norm of division Δ is sufficiently small, then the integral can be approximated by the integral sum, i.e.

$$\int_a^b f(t)dt \approx \sum_{i=0}^{n-1} f(\xi_i)(t_{i+1} - t_i) \tag{12}$$

To simplify the calculus, it was considered that the division Δ of the interval $[a,b]$ is equidistant, i.e.

$$t_{i+1} - t_i = \frac{b-a}{n}.$$

Under these conditions the following two formulas of approximation were obtained (see references [15], [16]):

(a) If we consider the intermediary points of the division Δ of the interval $[a,b]$ on the left end of partial intervals $[t_i, t_{i+1}]$, $\xi_i = t_i$, then the formula below is obtained:

$$\int_a^b = \frac{b-a}{n} \left[f(a) + \sum_{i=1}^{n-1} f(t_i) \right] + R(f); \tag{13}$$

(b) If we consider the intermediary points of the division Δ of the interval $[a,b]$ on the right end of partial intervals $[t_i, t_{i+1}]$, $\xi_i = t_{i+1}$, then it results the formula:

$$\int_a^b = \frac{b-a}{n} \left[\sum_{i=1}^{n-1} f(t_i) + f(b) \right] + R(f), \tag{14}$$

and each of these two formulas is called *rectangles' formula* or *rectangles' rule.*

For the remainder of the formula (13) or (14), we have the estimate given by the relation:

$$|R(f)| \leq M^D \frac{(b-a)^2}{n}, \tag{15}$$

where was denoted

$$M^D = \max_{t\in[a,b]} | \, f'(t)| \, . \tag{16}$$

3. The approximation of the solution

The results regarding the procedure for approximating the solution of the integral equation with modified argument (1), have been published in ([5]-[9]), [11] and [15]. In what follows we present a synthesis of these results.

To obtain these results, the conditions of *Theorem 1*, of existence and uniqueness of the solution of the integral equation (1), stated above, were assumed to be fulfilled.

In order to lay down the ideas we consider the case of the integral equation with a unique solution, $x^* \in C[a,b]$, established by using the successive approximations method, starting at any element $x_0 \in C[a,b]$. Moreover, if x_n is the n-th successive approximation, then the estimation (3) is true.

We have the successive approximations' sequence:

$$\begin{gathered} x_0(t) = f(t),\ \ x_0 \in C[a,b] \\ x_1(t) = \int_a^b K(t,s,x_0(s),x_0(a),x_0(b))ds + f(t) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_m(t) = \int_a^b K(t,s,x_{m-1}(s),x_{m-1}(a),x_{m-1}(b))ds + f(t), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{gathered} \tag{17}$$

To get a better result, it is considered an equidistant division Δ of the interval $[a,b]$ through the points $a = t_0 < t_1 < \cdots < t_n = b$. Then, the successive approximations sequence will be:

$$\begin{gathered} x_0(t_k) = f(t_k), \quad x_0(a) = f(a), \quad x_0(b) = f(b) \\ x_1(t_k) = \int_a^b K(t_k,s,f(s),f(a),f(b))ds + f(t_k) \\ x_1(a) = \int_a^b K(a,s,f(s),f(a),f(b))ds + f(a) \\ x_1(b) = \int_a^b K(b,s,f(s),f(a),f(b))ds + f(b) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_m(t_k) = \int_a^b K(t_k,s,x_{m-1}(s),x_{m-1}(a),x_{m-1}(b))ds + f(t_k), \\ x_m(a) = \int_a^b K(a,s,x_{m-1}(s),x_{m-1}(a),x_{m-1}(b))ds + f(a), \\ x_m(b) = \int_a^b K(b,s,x_{m-1}(s),x_{m-1}(a),x_{m-1}(b))ds + f(b), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{gathered} \tag{18}$$

In the following three paragraphs we present the results concerning the algorithm for approximating the solution of the integral equation (1), obtained by applying the successive approximations method and using the trapezoids' formula, the rectangles' formula and the Simpson's formula for the approximate calculus of the integrals that appear in the terms of the successive approximations sequence.

3.1. **The approximation of the solution using the trapezoids' formula.**

Suppose that the following conditions are fulfilled:

(h_{11}) $K \in C^2([a,b] \times [a,b] \times \mathbb{R}^3)$

(h_{12}) $f \in C^2[a,b]$

and consider an equidistant division Δ of the interval $[a,b]$ through the points $a = t_0 < t_1 < \cdots < t_n = b$. Using the trapezoids' formula (5) for the approximate calculus of the integrals that appear in the terms of the successive approximations sequence (17), with the estimate of the remainder given by (6) and (7), we will approximate the terms of this sequence. Thus, for $x_m(t_k)$ we have:

$$x_m(t_k) = \frac{b-a}{2n}\Bigg[K(t_k, a, x_{m-1}(a), x_{m-1}(a), x_{m-1}(b)) + \tag{19}$$

$$+2\sum_{i=1}^{n-1} K(t_k, t_i, x_{m-1}(t_i), x_{m-1}(a), x_{m-1}(b)) +$$

$$+K(t_k, b, x_{m-1}(b), x_{m-1}(a), x_{m-1}(b))\Bigg] + f(t_k) + R^T_{m,k},\ \ k = \overline{0,n},\ \ m \in \mathbb{N},$$

with the estimate of the remainder:

$$\left|R^T_{m,k}\right| \le \frac{(b-a)^3}{12n^2} \cdot \max_{s\in[a,b]} \left|\frac{\partial^2 K(t_k, s, x_{m-1}(s), x_{m-1}(a), x_{m-1}(b))}{\partial s^2}\right|.$$

According to hypotheses (h_{11}) and (h_{12}), it results that there exists the derivative of the functions K and f from the estimate of the remainder $R^T_{m,k}$, and following its calculus, it was obtained:

$$\left|R^T_{m,k}\right| \le M_0^T \cdot \frac{(b-a)^3}{12n^2}, \quad M_0^T = M_0^T(K, D^\alpha K, f, D^\alpha f), \quad |\alpha| \le 2, \tag{20}$$

where M_0^T doesn't depend on m and k.

Thus, it was obtained a formula for the approximate calculus of the integrals that appear in the terms of the successive approximations sequence. Using the successive approximations method and the formula (19) with the estimation of the remainder resulted from (20), it results an algorithm in order to solve the integral equation (1) approximately. The terms of the successive approximations sequence have been calculated approximately.

In general case for $x_m(t_k)$ it is obtains:

$$x_m(t_k) = \frac{b-a}{2n}\Bigg[K(t_k, a, \tilde{x}_{m-1}(a), \tilde{x}_{m-1}(a), \tilde{x}_{m-1}(b)) +$$

$$+2\sum_{i=1}^{n-1} K(t_k, t_i, \tilde{x}_{m-1}(t_i), \tilde{x}_{m-1}(a), \tilde{x}_{m-1}(b)) +$$

$$+K(t_k, b, \tilde{x}_{m-1}(b), \tilde{x}_{m-1}(a), \tilde{x}_{m-1}(b))\Bigg] + f(t_k) + \tilde{R}^T_{m,k} =$$

$$= \tilde{x}_m(t_k) + \tilde{R}^T_{m,k},\ \ k = \overline{0,n}.$$

Using the contraction condition (Theorem 1) it results the estimate of the remainder:

$$\left|\tilde{R}^T_{m,k}\right| \le \frac{(b-a)^3}{12n^2[1 - 3L(b-a)]} \cdot M_0^T.$$

So, using an equidistant division of the interval $[a,b]$ through the points $a = t_0 < t_1 < \cdots < t_n = b$, it was obtained the sequence $(\tilde{x}_m(t_k))_{m\in\mathbb{N}}$, $k = \overline{0,n}$, that approximates the successive approximations sequence $(x_m(t_k))_{m\in\mathbb{N}}, k=\overline{0,n}$, with the following calculation error:

$$\left|x_m(t_k)-\tilde{x}_m(t_k)\right| \le \frac{(b-a)^3}{12n^2[1-3L(b-a)]}\cdot M_0^T. \tag{21}$$

Finally, using the estimates (3) and (21) it is obtains the following theorem for approximating the solution of the integral equation (1).

Theorem 3. *Suppose that the conditions of Theorem 1 are fulfilled. In addition, we assume that the exact solution x^* of the integral equation (1) is approximated by the sequence $(\tilde{x}_m(t_k))_{m\in\mathbb{N}}$, $k=\overline{0,n}$, on the nodes t_k, $k=\overline{0,n}$, of the equidistant division Δ of the interval $[a,b]$, using the successive approximations method (17) and the trapezoids' method (5)+(6). Under these conditions, the approximation error is given by the evaluation:*

$$\left|x^*(t_k)-\tilde{x}_m(t_k)\right| \le \frac{3^m L^m (b-a)^m}{1-3L(b-a)}|x_1-x_0| + \frac{(b-a)^3}{12n^2[1-3L(b-a)]}\cdot M_0^T. \tag{22}$$

3.2. **The approximation of the solution using the Simpson's formula.**

Suppose that the following conditions are fulfilled:

(h_{21}) $K\in C^4([a,b]\times[a,b]\times\mathbb{R}^3)$

(h_{22}) $f\in C^4[a,b]$

and consider an equidistant division Δ of the interval $[a,b]$ through the points $a=t_0<t_1<\cdots<t_n=b$. Using the Simpson's formula (9) for the approximate calculus of the integrals that appear in the terms of the successive approximations sequence (17), with the estimate of the remainder given by (10) and (11), we will approximate the terms of this sequence. Thus, for $x_m(t_k)$ we have:

$$\begin{aligned} x_m(t_k) = \frac{b-a}{6n}\Bigg[& K(t_k,a,x_{m-1}(a),x_{m-1}(a),x_{m-1}(b)) + \\ & +2\sum_{i=1}^{n-1} K(t_k,t_i,x_{m-1}(t_i),x_{m-1}(a),x_{m-1}(b)) + \\ & +4\sum_{i=1}^{n-1} K\left(t_k,\frac{t_i+t_{i+1}}{2},x_{m-1}\left(\frac{t_i+t_{i+1}}{2}\right),x_{m-1}(a),x_{m-1}(b)\right) + \\ & +K(t_k,b,x_{m-1}(b),x_{m-1}(a),x_{m-1}(b))\Bigg] + f(t_k) + R^S_{m,k},\ k=\overline{0,n},\ m\in\mathbb{N}, \end{aligned} \tag{23}$$

with the estimate of the remainder:

$$\left|R^S_{m,k}\right| \le \frac{(b-a)^5}{2880n^4}\cdot \max_{s\in[a,b]}\left|\frac{\partial^4 K(t_k,s,x_{m-1}(s),x_{m-1}(a),x_{m-1}(b))}{\partial s^4}\right|.$$

According to hypotheses (h_{21}) and (h_{22}), it results that there exists the derivative of the functions K and f from the estimate of the remainder $R^S_{m,k}$, and following its calculation, it was obtained:

$$\left|R^S_{m,k}\right| \le M_0^S\cdot\frac{(b-a)^5}{2880n^4},\quad M_0^S = M_0^S(K,D^\alpha K,f,D^\alpha f),\quad |\alpha|\le 4, \tag{24}$$

where M_0^S doesn't depend on m and k.

Thus, it was obtained a formula for the approximate calculus of the integrals that appear in the terms of the successive approximations sequence. Using the successive approximations method and the formula (23) with an estimation of the remainder resulted from (24), it results an algorithm in order to solve the integral equation (1) approximately. The terms of the successive approximations sequence have been calculated approximately.

In general case for $x_m(t_k)$ it is obtains:

$$x_m(t_k) = \frac{b-a}{6n}\Bigg[K(t_k, a, \tilde{x}_{m-1}(a), \tilde{x}_{m-1}(a), \tilde{x}_{m-1}(b)) +$$

$$+2\sum_{i=1}^{n-1} K(t_k, t_i, \tilde{x}_{m-1}(t_i), \tilde{x}_{m-1}(a), \tilde{x}_{m-1}(b)) +$$

$$+4\sum_{i=1}^{n-1} K\left(t_k, \frac{t_i + t_{i+1}}{2}, \tilde{x}_{m-1}\left(\frac{t_i + t_{i+1}}{2}\right), \tilde{x}_{m-1}(a), \tilde{x}_{m-1}(b)\right) +$$

$$+K(t_k, b, \tilde{x}_{m-1}(b), \tilde{x}_{m-1}(a), \tilde{x}_{m-1}(b))\Bigg] + f(t_k) + \tilde{R}^S_{m,k} =$$

$$= \tilde{x}_m(t_k) + \tilde{R}^S_{m,k},\ k = \overline{0, n}.$$

Using the contraction condition (Theorem 1) it results the estimate of the remainder:

$$\left|\tilde{R}^S_{m,k}\right| \le \frac{(b-a)^5}{2880 n^4 [1 - 3L(b-a)]} \cdot M^S_0.$$

Consequently, using an equidistant division of the interval $[a, b]$ through the points $a = t_0 < t_1 < \cdots < t_n = b$, it was obtained the sequence $(\tilde{x}_m(t_k))_{m\in\mathbb{N}},\ k = \overline{0, n}$, that estimates the successive approximations sequence $(x_m(t_k))_{m\in\mathbb{N}},\ k = \overline{0, n}$, with the following calculation error:

$$\left|x_m(t_k) - \tilde{x}_m(t_k)\right| \le \frac{(b-a)^5}{2880 n^4 [1 - 3L(b-a)]} \cdot M^S_0. \tag{25}$$

Finally, using the estimates (3) and (25) it is obtains a theorem for approximating the solution of the integral equation (1), that we present below.

Theorem 4. *Suppose that the conditions of Theorem 1 are fulfilled. In addition, we assume that the exact solution x^* of the integral equation (1) is approximated by the sequence $(\tilde{x}_m(t_k))_{m\in\mathbb{N}},\ k = \overline{0, n}$, on the nodes $t_k,\ k = \overline{0, n}$ of the equidistant division Δ of the interval $[a, b]$, using the successive approximations method (21) and the Simpson's method (9)+(10). Under these conditions, the approximation error is given by the evaluation:*

$$\left|x^*(t_k) - \tilde{x}_m(t_k)\right| \le \frac{3^m L^m (b-a)^m}{1 - 3L(b-a)} |x_1 - x_0| + \frac{(b-a)^5}{2880 n^4 [1 - 3L(b-a)]} \cdot M^S_0. \tag{26}$$

3.3. **The approximation of the solution using the rectangles' formula.**

Suppose that the following conditions are fulfilled:

(h_{31}) $K \in C^1([a, b] \times [a, b] \times \mathbb{R}^3)$

(h_{32}) $f \in C^1[a, b]$

and consider an equidistant division Δ of the interval $[a, b]$ through the points $a = t_0 < t_1 < \cdots < t_n = b$. Using one of the rectangles' formula (13) or (14) for the approximate calculus of the integrals that appear in the terms of the successive approximations sequence (17), with the estimate of the remainder given by (15) and (16), we will approximate the terms of this sequence. We used the formula (13) with the remainder given by (15) and (16), considering the intermediary points of the division Δ of the interval $[a, b]$ on the left end of the partial intervals $[t_i, t_{i+1}],\ \xi_i = t_i$. For $x_m(t_k)$ we have:

$$x_m(t_k) = \frac{b-a}{n}\Bigg[K(t_k, a, x_{m-1}(a), x_{m-1}(a), x_{m-1}(b)) + \tag{27}$$

$$+\sum_{i=1}^{n-1} K(t_k,t_i,x_{m-1}(t_i),x_{m-1}(a),x_{m-1}(b))\Bigg] + f(t_k) + R^D_{m,k},\ k=\overline{0,n},\ m\in\mathbb{N},$$

with the estimate of the remainder:

$$\left|R^T_{m,k}\right| \le \frac{(b-a)^2}{n}\cdot\max_{s\in[a,b]}\left|\frac{\partial K(t_k,s,x_{m-1}(s),x_{m-1}(a),x_{m-1}(b))}{\partial s}\right|.$$

According to hypotheses (h_{31}) and (h_{32}) it results that there exists the derivative of the functions K and f from the estimate of the remainder $R^D_{m,k}$ and following its calculation, it was obtained:

$$\left|R^D_{m,k}\right| \le M^D_0\cdot\frac{(b-a)^2}{n},\quad M^D_0 = M^D_0(K,D^\alpha K,f,D^\alpha f),\quad |\alpha|=1, \tag{28}$$

where M^D_0 doesn't depend on m and k.

Thus, it was obtained a formula for the approximate calculus of the integrals that appear in the terms of the successive approximations sequence. Using the successive approximations method and the formula (27) with the estimation of the remainder resulted from (28), it results an algorithm in order to solve the integral equation (1) approximately. The terms of the successive approximations sequence have been calculated approximately.

In general case for $x_m(t_k)$ we obtain:

$$x_m(t_k) = \frac{b-a}{n}\Bigg[K(t_k,a,\tilde{x}_{m-1}(a),\tilde{x}_{m-1}(a),\tilde{x}_{m-1}(b))+$$

$$+\sum_{i=1}^{n-1} K(t_k,t_i,\tilde{x}_{m-1}(t_i),\tilde{x}_{m-1}(a),\tilde{x}_{m-1}(b))\Bigg] + f(t_k) + \tilde{R}^D_{m,k} =$$

$$= \tilde{x}_m(t_k) + \tilde{R}^D_{m,k},\ k=\overline{0,n}$$

Using the contraction condition (Theorem 1) it results the estimate of the remainder:

$$\left|\tilde{R}^D_{m,k}\right| \le \frac{(b-a)^2}{n[1-3L(b-a)]}\cdot M^D_0.$$

So, using an equidistant division of interval $[a,b]$ through the points $a=t_0<t_1<\cdots<t_n=b$, we obtain the sequence $(\tilde{x}_m(t_k))_{m\in\mathbb{N}},\ k=\overline{0,n}$, that estimates the successive approximations sequence $(x_m(t_k))_{m\in\mathbb{N}},\ k=\overline{0,n}$, with the following calculation error:

$$\left|x_m(t_k)-\tilde{x}_m(t_k)\right| \le \frac{(b-a)^2}{n[1-3L(b-a)]}\cdot M^D_0. \tag{29}$$

Finally, using the estimates (3) and (29) it is obtains the following theorem for approximating the solution of the integral equation (1).

Theorem 5. *Suppose that the conditions of Theorem 1 are fulfilled. In addition, we assume that the exact solution x^* of the integral equation (1) is approximated by the sequence $(\tilde{x}_m(t_k))_{m\in\mathbb{N}},\ k=\overline{0,n}$, on the nodes $t_k,\ k=\overline{0,n}$ of the equidistant division Δ of the interval $[a,b]$, using the successive approximations method (17) and the rectangles' method (13)+(15). Under these conditions, the approximation error is given by the evaluation:*

$$\left|x^*(t_k)-\tilde{x}_m(t_k)\right| \le \frac{3^mL^m(b-a)^m}{1-3L(b-a)}|x_1-x_0| + \frac{(b-a)^2}{n[1-3L(b-a)]}\cdot M^D_0. \tag{30}$$

4. Example

We consider the integral equation with modified argument:

$$x(t) = \int_0^1 \left[\frac{2\sin(x(s))}{7} + \frac{x(0) + x(1)}{5}\right] ds + \cos t,\ t \in [0,1] \tag{31}$$

where $K \in C([0,1] \times [0,1] \times \mathbb{R}^3)$, $K(t,s,u_1,u_2,u_3) = \frac{2\sin(u_1)}{7} + \frac{u_2+u_3}{5}$, $f \in C[0,1]$, $f(t) = \cos t$, $x \in C[0,1]$

The existence and uniqueness of the solution of this integral equation was presented in the first part of this synthesis survey.

The conditions of Theorem 1 and Theorem 2 are fulfilled and therefore, we establish under what conditions the integral equation (31) has a unique solution in the space $C[0,1]$ and in the sphere $\overline{B}(\cos t; r) \subset C[0,1]$, respectively.

Consider the case when the conditions of Theorem 2 are fulfilled. So the integral equation (31) has a unique solution $x^* \in \overline{B}(\cos t; r) \subset C[0,1]$. From the contraction condition $3L(b-a) = 3L = \frac{24}{35}$, it results that $L = \frac{8}{35}$.

To determine x^*, we apply the successive approximations method, starting at any element $x_0 \in \overline{B}(\cos t; r) \subset C[0,1]$, and if x_n is the n-th successive approximation, then the following estimation is true:

$$|x_n - x^*| \le \frac{24^n}{35^{n-1} \cdot 11} |x_1 - x_0|. \tag{32}$$

To calculate the integrals that appear in the terms of the successive approximations sequence, there have been used the following quadrature formulas: the trapezoids' formula, the rectangles' formula and the Simpson's formula, respectively.

It is observed that the functions K and f fulfill the conditions:

- (h_{11}), (h_{12}), necessary to apply the trapezoids' formula;
- (h_{21}), (h_{22}), necessary to apply the Simpson's formula;
- (h_{31}), (h_{32}), necessary to apply the rectangles' formula.

Also, to get a better approximation of the solution, an equidistant division of the interval $[0,1]$ through the points $0 = t_0 < t_1 < t_1 < \cdots < t_n = 1$ was considered.

The approximate value of the integral that arise in the general term of the successive approximations sequence:

$$x_m(t_k) = \int_0^1 \left[\frac{2\sin(x_{m-1}(s))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5}\right] ds + \cos_{t_k}$$

was calculated as it follows:

a) when we used the trapezoids' formula, we have the relation:

$$\int_0^1 \left[\frac{2\sin(x_{m-1}(s))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5}\right] ds =$$

$$= \frac{1}{2n}\left[\frac{2\sin(x_{m-1}(0))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5} + \right.$$

$$+2\sum_{i=1}^{n-1}\left(\frac{2\sin(x_{m-1}(t_i))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5}\right) +$$

$$\left. + \frac{2\sin(x_{m-1}(1))}{7} + \frac{x_{m-1}(0) + x_{m-1}(1)}{5}\right] + R^T_{m,k}, \quad k = \overline{0,n},\ m \in \mathbb{N}$$

with the estimate of the remainder:

$$|R^T_{m,k}| \leq \frac{1}{12n^2} \cdot \max_{s\in[0,1]} \left| \frac{\partial^2 K(t,s,x(s),x(0),x(1))}{\partial s^2} \right|.$$

$b)$ when we used the Simpson's formula, we have the relation:

$$\int_0^1 \left[\frac{2\sin(x_{m-1}(s))}{7} + \frac{x_{m-1}(0)+x_{m-1}(1)}{5} \right] ds =$$
$$= \frac{1}{6n} \left[\frac{2\sin(x_{m-1}(0))}{7} + \frac{x_{m-1}(0)+x_{m-1}(1)}{5} + \right.$$
$$+2\sum_{i=1}^{n-1} \left(\frac{2\sin(x_{m-1}(t_i))}{7} + \frac{x_{m-1}(0)+x_{m-1}(1)}{5} \right) +$$
$$+4\sum_{i=1}^{n-1} \left(\frac{2\sin(x_{m-1}(\frac{t_i+t_{i+1}}{2}))}{7} + \frac{x_{m-1}(0)+x_{m-1}(1)}{5} \right) +$$
$$\left. + \frac{2\sin(x_{m-1}(1))}{7} + \frac{x_{m-1}(0)+x_{m-1}(1)}{5} \right] + R^S_{m,k}, \quad k=\overline{0,n},\ m\in\mathbb{N}$$

with the estimate of the remainder:

$$|R^S_{m,k}| \leq \frac{1}{2880n^4} \cdot \max_{s\in[0,1]} \left| \frac{\partial^4 K(t,s,x(s),x(0),x(1))}{\partial s^4} \right|.$$

$c)$ when we used the rectangles' formula, we have the relation:

$$\int_0^1 \left[\frac{2\sin(x_{m-1}(s))}{7} + \frac{x_{m-1}(0)+x_{m-1}(1)}{5} \right] ds =$$
$$= \frac{1}{n} \left[\frac{2\sin(x_{m-1}(0))}{7} + \frac{x_{m-1}(0)+x_{m-1}(1)}{5} + \right.$$
$$\left. +\sum_{i=1}^{n-1} \left(\frac{2\sin(x_{m-1}(t_i))}{7} + \frac{x_{m-1}(0)+x_{m-1}(1)}{5} \right) \right] + R^D_{m,k}, \quad k=\overline{0,n},\ m\in\mathbb{N}$$

with the estimate of the remainder:

$$|R^D_{m,k}| \leq \frac{1}{n} \cdot \max_{s\in[0,1]} \left| \frac{\partial K(t,s,x(s),x(0),x(1))}{\partial s} \right|.$$

Thus, using an equidistant division of the interval $[0,1]$ through the points $a = t_0 < t_1 < \cdots < t_n = b$, we obtain the sequence $\left(\tilde{x}_{m-1}(t_k)\right)_{m\in\mathbb{N}}, k=\overline{0,n}$, that estimates the successive approximations sequence $\left(x_{m-1}(t_k)\right)_{m\in\mathbb{N}}, k=\overline{0,n}$, with the following error in calculation:

$a)$ when we used the trapezoids' formula, the error is:

$$|x_m(t_k) - \tilde{x}_m(t_k)| \leq 0,0238095238 \cdot \frac{1}{n^2};$$

$b)$ when we used the Simpson's formula, the error is:

$$|x_m(t_k) - \tilde{x}_m(t_k)| \leq 0,0001984126 \cdot \frac{1}{n^4};$$

$c)$ when we used the rectangles' formula, the error is:

$$|x_m(t_k) - \tilde{x}_m(t_k)| \leq 0,2404202813 \cdot \frac{1}{n}.$$

The calculus of the approximate value of the integral from the expression of the general term of the successive aproximations sequence using the trapezoids' formula, the Simpson's formula and the rectangles' formula respectively, was performed with a software developed in MATLAB.

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