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POSITIVE SOLUTIONS TO CAPUTO-HADAMARD FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this article, we study the existence and uniqueness of positive solutions of a Caputo-Hadamard fractional integro-differential equation with integral boundary conditions. The fixed point theorems and the method of upper and lower solutions are used to obtain the desired results. An example illustrating the main results is presented.

1. Introduction

Fractional differential equations arise from a variety of applications including in various fields of science and engineering. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations have received the attention of many authors, see [1]–[24], [26], [27] and the references therein.

In [15], Dhaigude and Bhairat investigated the existence and stability of solutions of the following nonlinear implicit fractional differential equation

$$\begin{cases} \mathfrak{D}_1^\alpha x(t) = f\left(t, x(t), \mathfrak{D}_1^\alpha x(t)\right), \ t \in [1, b], \ b > 1, \\ x^{(k)}(1) = x_k \in \mathbb{R}^n, \ k = 0, 1, ..., m-1, \end{cases}$$

where $\mathfrak{D}_1^\alpha$ is the Caputo-Hadamard fractional derivative of order $m - 1 < \alpha \leq m$. By employing the modified version of contraction principle and the successive approximation method, the authors obtained existence and stability results.

The nonlinear fractional integro-differential equation with nonlinear conditions

$$\begin{cases} {}^C D_{0^+}^\alpha u(t) = f\left(t, u(t)\right) + \int_0^t k(t, s, u(s))ds, \ t \in (0, T], \\ u(0) = u_0 - g(u), \end{cases}$$

has been investigated in [5], where ${}^C D_{0^+}^\alpha$ is the standard Caputo fractional derivative of order $1 < \alpha < 1$, $u_0 \in \mathbb{R}$, g, f and k are given continuous functions. By employing the Krasnoselskii and Banach fixed point theorems, the authors obtained the existence and uniqueness results.

In [2], Abdo, Wahash and Panchat discussed the existence and uniqueness of positive solutions of the following nonlinear fractional differential equation with integral boundary conditions

$$\begin{cases} {}^C D_{0^+}^\alpha x(t) = f\left(t, x(t)\right), \ t \in (0, 1], \\ x(0) = \lambda \int_0^1 x(s)\, ds + d, \end{cases}$$

where $0 < \alpha < 1$, $\lambda \geq 0$, $d > 0$ and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a given continuous function. By using the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the existence and uniqueness of solutions has been established.

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In [21], Lachouri, Ardjouni and Djoudi discussed the existence and uniqueness of the positive solution of the following nonlinear fractional integro-differential equation

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}u(t) = f\left(t, u\left(t\right)\right) + \int_0^t k(t,s,u(s))ds, \ t \in \left(0, T\right], \\ u(0) = \lambda \int_0^T u\left(s\right)ds + d, \end{cases}$$

where $0 < \alpha < 1$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function. By employing the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the authors obtained positivity results.

In this paper, we extend the results in [21] by proving the positivity of solutions for the following nonlinear fractional integro-differential equation with integral boundary conditions

$$\begin{cases} \mathfrak{D}_1^{\alpha}u(t) = f\left(t, u\left(t\right)\right) + \int_1^t k(t,s,u(s))ds, \ t \in \left(1, T\right], \\ u(1) = \lambda \int_1^T u\left(s\right)ds + d, \end{cases} \tag{1}$$

where $0 < \alpha < 1$, $\lambda \geq 0$, $d > 0$, $f : [1,T] \times [0,\infty) \to [0,\infty)$ and $k : [1,T] \times [1,T] \times [0,\infty) \to [0,\infty)$ are given continuous functions, k is non-decreasing on u. To prove the existence and uniqueness of positive solutions, we transform (1) into an equivalent integral equation and then by the method of upper and lower solutions and use the Schauder and Banach fixed point theorems. For details on the Banach and Schauder theorems we refer the reader to [25]. Finally, to illustrate our results, we provide an example.

2. Preliminaries

Let $C\left([1,T]\right)$ be the Banach space endowed with the infinity norm and A a nonempty closed subset of $C\left([1,T]\right)$ defined as

$$A = \left\{u \in C\left([1,T]\right) : u(t) \geq 0, \ t \in [1,T]\right\}.$$

$C^n\left([1,T]\right)$ denotes the class of all real valued functions defined on $[1,T]$ which have a continuous n th order derivative.

Definition 1 ([19]). *The Hadamard fractional integral of order $\alpha > 0$ for a continuous function $x : [1,+\infty) \to \mathbb{R}$ is defined as*

$$\mathfrak{I}_1^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} x(s)\frac{ds}{s}, \ \alpha > 0.$$

Definition 2 ([19]). *The Caputo-Hadamard fractional derivative of order $\alpha > 0$ for a continuous function $x : [1,+\infty) \to \mathbb{R}$ is defined as*

$$\mathfrak{D}_1^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} \delta^n(x)(s)\frac{ds}{s}, \ n-1 < \alpha < n,$$

where $\delta^n = \left(t\frac{d}{dt}\right)^n$, $n \in \mathbb{N}$.

Lemma 1 ([19]). *Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ and $x \in C^n\left([1,T]\right)$. Then*

$$\left(\mathfrak{I}_1^{\alpha}\mathfrak{D}_1^{\alpha}x\right)(t) = x(t) - \sum_{k=0}^{n-1}\frac{x^{(k)}(1)}{\Gamma(k+1)}(\log t)^k.$$

Lemma 2 ([19]). *For all $\mu > 0$ and $\nu > -1$,*

$$\frac{1}{\Gamma(\mu)}\int_1^t \left(\log\frac{t}{s}\right)^{\mu-1} (\log s)^{\nu}\frac{ds}{s} = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(\log t)^{\mu+\nu}.$$

Theorem 1 (Banach's fixed point theorem [25]). *Let Ω be a non-empty closed subset of a Banach space $(S, \|.\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Theorem 2 (Schauder's fixed point theorem [25]). *Let Ω be a nonempty closed bounded convex subset of a Banach space S and $\Phi : \Omega \to \Omega$ be a continuous compact operator. Then Φ has a fixed point in Ω.*

Definition 3. *A function $u \in C^1([1,T])$ is said to be a solution of (1) if u satisfies the equation*

$$\mathfrak{D}_1^{\alpha} u(t) = f(t, u(t)) + \int_1^t k(t,s,u(s))ds, \ t \in (1,T],$$

with integral boundary conditions

$$u(1) = \lambda \int_1^T u(s)\, ds + d.$$

Definition 4. *Let $a, b \in \mathbb{R}^+$ and $b > a$. For any $u \in [a,b]$, we define the upper-control function $U(t,u) = \sup_{a \le \rho \le u} f(t,\rho)$ and lower-control function $L(t,u) = \inf_{u \le \rho \le b} f(t,\rho)$.*

Obviously, $U(t,u)$ and $L(t,u)$ are monotonous non-decreasing on u and

$$L(t,u) \le f(t,u) \le U(t,u).$$

3. Main results

In this section, we shall give the existence and uniqueness results of (1) and prove it. Before starting and proving the main results, we introduce the following lemma.

Lemma 3. *$u \in C^1([1,T])$ is a solution of the boundary value problem (1) if and only if u is a solution of the integral equation*

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left[f(s,x(s)) + \int_1^s k(s,\tau,u(\tau))d\tau\right] \frac{ds}{s} + \lambda \int_1^T u(s)\, ds + d, \ t \in [1,T].$$

Proof. Suppose u satisfies the problem (1), then applying $\mathfrak{I}_1^{\alpha}$ to both sides of (1), we get

$$\mathfrak{I}_1^{\alpha} \mathfrak{D}_1^{\alpha} u(t) = \mathfrak{I}_1^{\alpha} \left(f(t,u(t)) + \int_1^t k(t,s,u(s))ds \right).$$

By using Lemma 1 and the integral boundary condition, we obtain

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left[f(s,x(s)) + \int_1^s k(s,\tau,u(\tau))d\tau\right] \frac{ds}{s} + \lambda \int_1^T u(s)\, ds + d, \ t \in [1,T]. \tag{2}$$

□

To transform (2) to be applicable to Schauder's fixed point, we define the operator $\Phi : A \to A$ by

$$(\Phi u)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left[f(s,x(s)) + \int_1^s k(s,\tau,u(\tau))d\tau\right] \frac{ds}{s} + \lambda \int_1^T u(s)\, ds + d, \ t \in [1,T], \tag{3}$$

where figured fixed point must satisfy the identity operator equation $\Phi u = u$.

We introduce the following assumptions

(H_1) Let $u^*, u_* \in A$ such that $a \leq u_*(t) \leq u^*(t) \leq b$ and

$$\begin{cases} \mathfrak{D}_1^\alpha u^*(t) - \int_1^t k(t,s,u^*(s))ds \geq U(t, u^*(t)), \\ \mathfrak{D}_1^\alpha u_*(t) - \int_1^t k(t,s,u_*(s))ds \leq L(t, u_*(t)), \end{cases}$$

for any $t \in [1,T]$.

(H_2) For $t, s \in [1,T]$ and $u, v \in \mathbb{R}$, there exist two positive constants l_f and l_k such that

$$\begin{aligned} |f(t,u) - f(t,v)| &\leq l_f |u - v|, \\ |k(t,s,u) - k(t,s,v)| &\leq l_k |u - y|. \end{aligned}$$

The functions u^* and u_* are respectively called the pair of upper and lower solutions for problem (1).

The first result is based on the Schauder fixed point theorem.

Theorem 3. *Assume that (H_1) is satisfied, then the problem (1) has at least one positive solution.*

Proof. Let $\Omega = \{u \in A : u_*(t) \leq u(t) \leq u^*(t),\ t \in [1,T]\}$ endowed with the norm $\|u\| = \max_{t\in[1,T]} |u(t)|$, then we have $\|u\| \leq b$. Hence, Ω is convex bounded and closed subset of the Banach space $C([1,T])$. Moreover, the continuity of f and k imply the continuity of the operator Φ on Ω defined by (3). Now, if $u \in \Omega$, there exist two positive constants c_f and c_k such that

$$\max\{f(t,u(t)) : t \in [1,T],\ u(t) \leq b\} \leq c_f,$$

and

$$\max\{k(t,s,u(s)) : t,s \in [1,T],\ u(s) \leq b\} \leq c_k.$$

Then

$$\begin{aligned} |(\Phi u)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left[|f(s,x(s))| + \int_1^s |k(s,\tau,u(\tau))| d\tau\right] \frac{ds}{s} \\ &\quad + \lambda \int_1^T |u(s)| ds + d \\ &\leq \frac{c_f (\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{c_k (\log T)^{\alpha+1}}{\Gamma(\alpha+2)} + \lambda b (T-1) + d. \end{aligned}$$

Thus

$$\|\Phi u\| \leq \frac{c_f (\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{c_k (\log T)^{\alpha+1}}{\Gamma(\alpha+2)} + \lambda b (T-1) + d.$$

Hence, $\Phi(\Omega)$ is uniformly bounded. Next, we prove the equicontinuity of $\Phi(\Omega)$. For each $u \in \Omega$. Then for $t_1, t_2 \in [1, T]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&|(\Phi u)(t_2) - (\Phi u)(t_1)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_1}{s}\right)^{\alpha-1} - \left(\log \frac{t_2}{s}\right)^{\alpha-1} \right] |f(s, u(s))| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} |f(s, u(s))| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_1}{s}\right)^{\alpha-1} - \left(\log \frac{t_2}{s}\right)^{\alpha-1} \right] \left(\int_1^s |k(s, \tau, u(\tau))| \, d\tau \right) \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \left(\int_1^s |k(s, \tau, u(\tau))| \, d\tau \right) \frac{ds}{s} \\
&\leq \frac{c_f}{\Gamma(\alpha+1)} \left[2 \left(\log \frac{t_2}{t_1}\right)^{\alpha} + (\log t_1)^{\alpha} - (\log t_2)^{\alpha} \right] \\
&+ \frac{c_k}{\Gamma(\alpha+2)} \left[2 \left(\log \frac{t_2}{t_1}\right)^{\alpha+1} + (\log t_1)^{\alpha+1} - (\log t_2)^{\alpha+1} \right] \\
&\leq \frac{2c_f}{\Gamma(\alpha+1)} \left(\log \frac{t_2}{t_1}\right)^{\alpha} + \frac{2c_k}{\Gamma(\alpha+2)} \left(\log \frac{t_2}{t_1}\right)^{\alpha+1}. \qquad (4)
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the inequality (4) tends to zero and the convergence is independent of u in Ω, which means that $\Phi(\Omega)$ is equicontinuous. The Arzela-Ascoli theorem implies that $\Phi : \Omega \to A$ is compact. The only thing to apply the Schauder fixed point is to prove that $\Phi(\Omega) \subset \Omega$. For any $u \in \Omega$, then $u_*(t) \leq u(t) \leq u^*(t)$ and by (H_1), we have

$$\begin{aligned}
(\Phi u)(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left[f(s, u(s)) + \int_1^s k(s, \tau, u(\tau)) d\tau \right] \frac{ds}{s} \\
&+ \lambda \int_1^T u(s) \, ds + d \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left[U(s, u(s)) + \int_1^s k(s, \tau, u(\tau)) d\tau \right] \frac{ds}{s} \\
&+ \lambda \int_1^T u(s) \, ds + d \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left[U(s, u^*(s)) + \int_1^s k(s, \tau, u^*(\tau)) d\tau \right] \frac{ds}{s} \\
&+ \lambda \int_1^T u^*(s) \, ds + d \\
&\leq u^*(t),
\end{aligned}$$

and

$$\begin{aligned}
(\Phi u)(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left[f(s,u(s))+\int_1^s k(s,\tau,u(\tau))d\tau\right]\frac{ds}{s} \\
&\quad +\lambda\int_1^T u(s)\,ds+d \\
&\geq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left[L(s,u(s))+\int_1^s k(s,\tau,u(\tau))d\tau\right]\frac{ds}{s} \\
&\quad +\lambda\int_1^T u(s)\,ds+d \\
&\geq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left[L(s,u_*(t))+\int_1^s k(s,\tau,u_*(\tau))d\tau\right]\frac{ds}{s} \\
&\quad +\lambda\int_1^T u_*(s)\,ds+d \\
&\geq u_*(t).
\end{aligned}$$

Hence, $u_*(t)\leq(\Phi u)(t)\leq u^*(t)$, $t\in[1,T]$, that is, $\Phi(\Omega)\subset\Omega$. According to the Schauder fixed point theorem, the operator Φ has at least one fixed point $u\in\Omega$. Therefore, the problem (1) has at least one positive solution. □

The second result is based on the Banach fixed point theorem.

Theorem 4. *Assume that* (H_1) *and* (H_2) *are satisfied and*

$$\frac{l_f(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{l_k(\log T)^{\alpha+1}}{\Gamma(\alpha+2)}+\lambda(T-1)<1. \tag{5}$$

Then the problem (1) has a unique positive solution.

Proof. From Theorem 3, it follows that the problem (1) has at least one positive solution. Hence, we need only to prove that the operator defined in (3) is a contraction in Ω. In fact, for any $u,v\in\Omega$, we have

$$\begin{aligned}
&|(\Phi u)(t)-(\Phi v)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|f(s,u(s))-f(s,v(s))|\frac{ds}{s} \\
&+\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\int_1^s|k(s,\tau,u(\tau))-k(s,\tau,v(\tau))|\,d\tau\right)\frac{ds}{s} \\
&+\lambda\int_1^T|u(s)-v(s)|\,ds \\
&\leq\left(\frac{l_f(\log T)^{\alpha}}{\Gamma(\alpha+1)}+\frac{l_k(\log T)^{\alpha+1}}{\Gamma(\alpha+2)}+\lambda(T-1)\right)\|u-v\|.
\end{aligned}$$

Hence, the operator Φ is a contraction mapping by (5). Therefore, by the Banach fixed point theorem, we conclude that the problem (1) has a unique positive solution. □

Finally, we give an example to illustrate our results.

Example 1. *Consider the fractional integro-differential equation with integral boundary conditions*

$$\begin{cases} \mathfrak{D}_1^{\frac{2}{3}} u(t) = \frac{1}{2}(3+\cos(u(t))) + \frac{1}{6}\int_1^t u(s)\exp\left(-\left(t^2+s^2\right)\right)ds, \ t \in (1,e], \\ u(1) = \frac{1}{6}\int_1^e u(s)\,ds + \frac{3}{2}, \end{cases} \tag{6}$$

where $T = e$, $\alpha = 2/3$, $\lambda = 1/6$, $d = 3/2$, $f(t,u(t)) = \frac{1}{2}(3+\cos(u(t)))$ *and* $k(t,s,u(s)) = \frac{1}{6}u(s)\exp\left(-\left(t^2+s^2\right)\right)$. *Since* f *and* g *are continuous positive functions,* k *is non-decreasing on* u *and*

$$\frac{l_f(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{l_k(\log T)^{\alpha+1}}{\Gamma(\alpha+2)} + \lambda(T-1) \simeq 0.95 < 1,$$

then, by Theorem 4, (6) has a unique positive solution.

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