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A LANDAU'S INEQUALITY FOR SEMIGROUPS REVISITED

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Abstract. H. Kraljevič and S. Kurepa in 1970 ([13]) proved a nice Landau type inequality for semigroups by making use of Taylor's formula with integral remainder for semigroups, which has been a traditional method for proving such inequalities. The author offers another method in the semigroups and reproves the above result by using his earlier result about Ostrowski inequalities for semigroups, see [1], Chapter 16, pp. 259-289. He finds his method easier and convenient. An application is given at the end.

1. Introduction

We state H. Kraljevič, S. Kurepa, 1970, interesting theorem ([13]):

Theorem 1. *Let A be an infinitesimal generator of a strongly continuous semigrou $\{T(t) : t > 0\}$ of bounded linear operators on a Banach space X. Let M, ω be real numbers with the property:*

$$\|T(t)\| \leq Me^{\omega t}, \ \ \forall\ t > 0. \tag{1}$$

Then for any $f \in D(A^2)$ such that $(A - \omega I)^2 f \neq 0$, the inequality

$$\|(A-\omega I) f\|^2 \leq 2M(M+1)\|f\| \left\|(A-\omega I)^2 f\right\| \tag{2}$$

holds.

Other inspirations follow:

Theorem 2. *(R.R. Kallman, G.-C. Rota, (1970), [10]). Let X be a complex Banach space and $t \to T(t)$ $(t \geq 0)$ be a strongly continuous semigroup of linear contractions on X. Let A be its infinitesimal generator. Then, for every $f \in D(A^2)$, it holds*

$$\|Af\|^2 \leq 4\|f\| \left\|A^2 f\right\|. \tag{3}$$

We also mention

Theorem 3. *(E. Hille, 1992, [7]). If A is the infinitesimal generator of a strongly continuous contraction semigroup, if $f \in D(A^n)$ and $1 \leq k < n$, then there exist constants $C_{n,k}$ independent of A, so that*

$$\left\|A^k(f)\right\|^n \leq C_{n,k}^n \|f\|^{n-k} \|A^n(f)\|^k. \tag{4}$$

This type of research started by E. Landau, 1913, ([15]). He proved first that if $f \in C^2([0,1])$, $\|f\|_\infty = 1$, $\|f''\|_\infty = 4$, then $\|f'\|_\infty \leq 4$, with 4 the best constant and the result is not necessarily true for an interval of length < 1. It is customary now to write Landau's inequality as

$$\|f'\|_{p,I} \leq C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \|f''\|_{p,I}^{\frac{1}{2}}, \tag{5}$$

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where $\|\cdot\|_{p,I}$ is the p-norm on the interval I; $p \in [1,\infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$, and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$. $C_p(I) > 0$ is independent of f.

Landau proved the best constants

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2},$$

see also [2].

Using Theorem 2 we can derive a Landau's inequality, see (5), by talking $X = C([0,\infty))$ or $L_p([0,\infty))$ and defining the shift operator

$$T(s) f(t) = f(t+s), \quad s \geq 0, \quad Af(t) = f'(t),$$

see [6], [7].

Of inspiration are also the articles [8], [12], [14], [16].

2. Background

We will be using the following notions. Here all this background comes from [3] (in general see also [5], [17]).

Let X a real or complex Banach space with elements $f, g, ...$ having norm $\|f\|, \|g\|, ...$ and let $\varepsilon(X)$ be the Banach algebra of endomorphisms of X.

If $T \in \varepsilon(X)$, $\|T\|$ denotes the norm of T.

Definition 1. *If $T(t)$ is an operator function on the non-negative real axis $0 \leq t < \infty$ to the Banach algebra $\varepsilon(X)$ satisfying the following conditions:*

$$\begin{cases} \text{(i)} \ T(t_1 + t_2) = T(t_1) T(t_2), \quad (t_1, t_2 \geq 0) \\ \text{(ii)} \ T(0) = I \quad (I = \text{identity operator}), \end{cases} \tag{6}$$

then $\{T(t) : 0 \leq t < \infty\}$ is called a one-parameter semi-group of operators in $\varepsilon(X)$.

The semi-group $\{T(t) : 0 \leq t < \infty\}$ is said to be of class C_0 if it satisfies the further property

$$\text{(iii)} \quad s - \lim_{t \to 0+} T(t) f = f \quad (f \in X) \tag{7}$$

referred to as the strong continuity of $T(t)$ at the origin.

In this paper we shall assume that the family of bounded linear operators $\{T(t) : 0 \leq t < \infty\}$ mapping X to itself is a semi-group of class C_0, thus all three conditions of the above definition are satisifed.

Proposition 1. *(a) $\|T(t)\|$ is bounded on every finite subinterval of $[0,\infty)$.*

(b) For each $f \in X$, the vector-valued function $T(t) f$ on $[0,\infty)$ is strongly continuous, thus vector-Riemann integrable on $[0,a]$, $a > 0$.

Definition 2. *The infinitesimal generator A of the semi-group $\{T(t) : 0 \leq t < \infty\}$ is defined by*

$$Af = s - \lim_{\tau \to 0+} A_\tau f, \quad A_\tau f = \frac{1}{\tau}[T(\tau) - I] f \tag{8}$$

whenever the limit exists; the domain of A, in symbols $D(A)$, is the set of elements f for which the limits exists.

Proposition 2. *(a) $D(A)$ is a linear manifold in X and A is a linear operator.*

(b) If $f \in D(A)$, then $T(t) f \in D(A)$ for each $t \geq 0$ and

$$\frac{d}{dt} T(t) f = AT(t) f = T(t) Af \quad (t \geq 0); \tag{9}$$

furthermore,

$$T(t)f - f = \int_0^t T(u) Af du \quad (t > 0). \tag{10}$$

(c) $D(A)$ is dense in X, i.e. $\overline{D(A)} = X$, and A is a closed operator.

Definition 3. *For $r = 0, 1, 2, ...$ the operator A^r is defined inductively by the relations $A^0 = I$, $A^1 = A$, and*

$$D(A^r) = \left\{ f : f \in D\left(A^{r-1}\right) \text{ and } A^{r-1}f \in D(A) \right\}$$

$$A^r f = A\left(A^{r-1}f\right) = s - \lim_{\tau \to 0+} A_\tau \left(A^{r-1}f\right) \quad (f \in D(A^r)). \tag{11}$$

For the operator A^r and its domain $D(A^r)$ we have the following

Proposition 3. *(a) $D(A^r)$ is a linear subspace in X and A^r is a linear operator.*

(b) If $f \in D(A^r)$, so does $T(t)f$ for each $t \geq 0$ and

$$\frac{d^r}{dt^r} T(t) f = A^r T(t) f = T(t) A^r f. \tag{12}$$

Furthermore

$$T(t)f - \sum_{k=0}^{r-1} \frac{t^k}{k!} A^k f = \frac{1}{(r-1)!} \int_0^t (t-u)^{r-1} T(u) A^r f du, \tag{13}$$

the Taylor's formula for semigroups.

Additionally it holds

$$(T(t) - I)^r f = \int_0^t \int_0^t ... \int_0^t T(u_1 + u_2 + ... + u_r) A^r f du_1 du_2 ... du_r. \tag{14}$$

(c) $D(A^r)$ is dense in X for $r = 1, 2, ...,$; moreover, $\cap_{r=1}^{\infty} D(A^r)$ is dense in X, A^r is a closed operator.

Integrals in (13) and (14) are vector valued Riemann integrals, see [3], [11].

Here we will assume that $f \in D\left(A^2\right)$. Clearly here $\int_0^a T(t) f dt \in X$, where $a > 0$, see [17].

We mention an Ostrowski type inequality for semigroups

Theorem 4. *([1], Chapter 13, p. 219) Let $f \in D(A)$, $a > 0$, and denote*

$$\|\|T(\cdot) Af\|\|_{\infty,[0,a]} := \sup_{u \in [0,a]} \|T(u) Af\|. \tag{15}$$

Then

$$\left\| \frac{1}{a} \int_0^a T(t) f dt - T(t_0) f \right\| \leq \left(\frac{t_0^2 + (a - t_0)^2}{2a} \right) \|\|T(\cdot) Af\|\|_{\infty,[0,a]}, \tag{16}$$

for a fixed $t_0 \in [0, a]$.

We will be using Theorem 10 to reprove Theorem 1 by another method.

3. Main Results

We give the following Landau inequality for semigroups.

Theorem 5. *Let the family of bounded linear operator* $\{T(t) : 0 \leq t < \infty\}$ *mapping* X *into itself be a semigroup of class* C_0. *We assume here that* $\|T(t)\| \leq M$, $\forall\, t \in \mathbb{R}_+$, *where* $M > 0$. *Let* $f \in D\left(A^2\right)$. *Then*

$$\|Af\|^2 \leq 2M(M+1)\|f\|\left\|A^2 f\right\|, \tag{17}$$

where A *is the infinitesimal generator.*

Proof. Let $f \in D(A)$, $t > 0$, and denote $\|\|T(\cdot)Af\|\|_{\infty,[0,t]} := \sup_{u\in[0,t]} \|T(u)Af\|$. By Theorem 4 we have

$$\left\|\frac{1}{t}\int_0^t T(z) f dz - T(z_0) f\right\| \leq \frac{t}{2}\|\|T(\cdot)Af\|\|_{\infty,[0,t]}, \tag{18}$$

for any fixed $z_0 \in [0,t]$.

Here let $f \in D\left(A^2\right)$, and we denote

$$\left\|\left\|T(\cdot)A^2 f\right\|\right\|_{\infty,[0,t]} := \sup_{u\in[0,t]}\left\|T(u)A^2 f\right\|. \tag{19}$$

By (18) we obtain

$$\left\|\frac{1}{t}\int_0^t T(z) Af dz - T(z_0) Af\right\| \leq \frac{t}{2}\left\|\left\|T(\cdot)A^2 f\right\|\right\|_{\infty,[0,t]}, \tag{20}$$

for any fixed $z_0 \in [0,t]$.

Notice here that $\left\|\left\|T(\cdot)A^2 f\right\|\right\|_{\infty,[0,t]} < \infty$. By (10) we get

$$\left\|\frac{1}{t}(T(t) f - f) - T(z_0) Af\right\| \leq \frac{t}{2}\left\|\left\|T(\cdot)A^2 f\right\|\right\|_{\infty,[0,t]}, \tag{21}$$

for any fixed $z_0 \in [0,t]$.

Hence

$$\|T(z_0) Af\| - \frac{\|T(t) f - f\|}{t} \leq \frac{t}{2}\left\|\left\|T(\cdot)A^2 f\right\|\right\|_{\infty,[0,t]}. \tag{22}$$

Furthermore we have

$$\|T(z_0) Af\| \leq \frac{\|T(t) f - f\|}{t} + \frac{t}{2}\left\|\left\|T(\cdot)A^2 f\right\|\right\|_{\infty,[0,t]} \leq$$

$$\frac{(\|T(t) f\| + \|f\|)}{t} + \frac{t}{2}\left\|\left\|T(\cdot)A^2 f\right\|\right\|_{\infty,[0,t]}. \tag{23}$$

By assumption $\|T(t)\| \leq M$, $\forall\, t \geq 0$, so we have that $\|T(t) f\| \leq \|T(t)\|\|f\| \leq M\|f\|$.

Similarly it holds

$$\left\|T(z)A^2 f\right\| \leq M\left\|A^2 f\right\|, \ \ \forall\, z \geq 0.$$

That is

$$\left\|\left\|T(\cdot)A^2 f\right\|\right\|_{\infty,[0,t]} \leq M\left\|A^2 f\right\|, \ \ \forall\, t \geq 0.$$

So we have that

$$\|T(z_0) Af\| \leq \frac{(M+1)}{t}\|f\| + \frac{tM}{2}\left\|A^2 f\right\|, \tag{24}$$

$\forall\, t \in \mathbb{R}_+$, and $\forall\, z_0 \in [0,t]$.

Choosing $z_0 = 0$, we get

$$\|Af\| \leq \frac{(M+1)}{t}\|f\| + \frac{tM}{2}\left\|A^2 f\right\|, \ \ \forall\, t \in \mathbb{R}_+. \tag{25}$$

If $A^2 f = 0$, then $\left\|A^2 f\right\| = 0$, and clearly it holds by (25) that $\|Af\| = 0$, that is $Af = 0$.

So next we will assume that $f \neq 0$ and $A^2 f \neq 0$.

We consider the function

$$y(t) := \frac{(M+1)\|f\|}{t} + t\frac{M\left\|A^2 f\right\|}{2}, \quad \forall\, t \in \mathbb{R}_+. \tag{26}$$

Set

$$A := (M+1)\|f\|, \quad B := \frac{M\left\|A^2 f\right\|}{2}.$$

That is

$$y(t) = \frac{A}{t} + Bt, \ \ \forall\, t \in \mathbb{R}_+. \tag{27}$$

Hence

$$y'(t) = -At^{-2} + B = 0,$$

and the critical number is

$$t_0 = \sqrt{\frac{A}{B}}.$$

Notice that $y''(t) = 2At^{-3}$, and

$$y''(t_0) = y''\left(\sqrt{\frac{A}{B}}\right) = 2A\left(\sqrt{\frac{A}{B}}\right)^{-3} = 2A\left(\frac{A}{B}\right)^{-\frac{3}{2}} = 2A\left(\frac{B}{A}\right)^{\frac{3}{2}} = 2A^{-\frac{1}{2}}B^{\frac{3}{2}} > 0. \tag{28}$$

Hence y has a global minimum which is

$$y(t_0) = y\left(\sqrt{\frac{A}{B}}\right) = 2\sqrt{AB} = \sqrt{2M(M+1)\|f\|\,\|A^2 f\|}. \tag{29}$$

We have proved that

$$\|Af\| \leq \sqrt{2M(M+1)\|f\|\,\|A^2 f\|}, \tag{30}$$

when $A^2 f \neq 0$.

The last inequality is true even when $A^2 f = 0$.

If $f = 0$ then $Af = 0$, so that (30) is trivially again true.

The theorem is proved. □

Next comes a more general Landau inequality for semigroups.

Corollary 1. *Let the family of bounded linear operators $\{T(t) : 0 \leq t < \infty\}$ mapping X into itself be a semigroup of class C_0. Let $M > 0$ and ω be a real number with the property $\|T(t)\| \leq Me^{\omega t}$, $\forall\, t \geq 0$. Then, for any $f \in D\left(A^2\right)$, it holds*

$$\|(A - \omega I) f\|^2 \leq 2M(M+1)\|f\| \left\|(A - \omega I)^2 f\right\|. \tag{31}$$

Proof. We are acting as in the proof of Theorem 1. We have that

$$\left\|e^{-\omega t}T(t)\right\| \leq M, \ \ \forall\, t \geq 0,$$

and $t \to e^{-\omega t}T(t)$ is a semigroup with $A - \omega I$ as the infinitesimal generator. Hence by applying (17) we derive (31). □

Notice that our results are valid without the restriction $(A - \omega I)^2 f \neq 0$, $\omega \in \mathbb{R}$, which is assumed in Theorem 1.

4. Application

Here we follow [4]. It is known that the classical diffusion equation

$$\frac{\partial W}{\partial t}=\frac{\partial^2 W}{\partial x^2}, \quad -\infty<x<\infty, \ \ t>0, \tag{32}$$

with initial condition

$$\lim_{t\to 0+} W(x,t)=f(x), \tag{33}$$

has under general conditions its solution given by

$$W(x,t,f)=(T(t)f)(x)=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(x+u)\,e^{-\frac{u^2}{4t}}du, \tag{34}$$

the so called Gauss-Weierstrass singular integral.

The infinitesimal generator of the semigroup $\{T(t):0\leq t<\infty\}$ is $A=\frac{\partial^2}{\partial x^2}$ ([9], p. 578).

Here we suppose that f belongs to the Banach space $X=UCB(\mathbb{R})$, the space of bounded and uniformly continuous functions from $\mathbb{R}$ into itself, with norm $\|f\|_C := \sup_{x\in\mathbb{R}}|f(x)|$.

We notice that

$$|W(x,t,f)|=\frac{1}{2\sqrt{\pi t}}\left|\int_{-\infty}^{\infty} f(x+u)\,e^{-\frac{u^2}{4t}}du\right|\leq$$

$$\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}|f(x+u)|\,e^{-\frac{u^2}{4t}}du\leq\|f\|_C\left(\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{u^2}{4t}}du\right) \tag{35}$$

$$=\|f\|_C\cdot 1=\|f\|_C<\infty, \quad \forall\ t\in\mathbb{R}_+.$$

That is

$$\|W(\cdot,t,f)\|_C\leq 1\cdot\|f\|_C, \quad \forall\ t\in\mathbb{R}_+. \tag{36}$$

So in Theorem 5, $M=1$.

Let now $f\in D\left(\left(\frac{\partial^2}{\partial x^2}\right)^2\right)=D\left(\frac{\partial^4}{\partial x^4}\right)$. Then, by (17), we derive

$$\left\|\frac{\partial^2 f}{\partial x^2}\right\|_C^2\leq 4\|f\|_C\left\|\frac{\partial^4 f}{\partial x^4}\right\|_C. \tag{37}$$

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