TJMM
12 (2020), No. 2, 69-75

NEW DIFFERENTIAL OPERATOR INVOLVING THE q-RUSCHEWEYH DERIVATIVE AND THE SYMMETRIC DIFFERENTIAL

[1]KHALID ALSHAMMARI AND [2,*]MASLINA DARUS

Abstract. In this paper, we define a new differential operator involving q- calculus by using a familiar technique which is a Hadamard product. We use convolution between q-analogue of the Ruscheweyh derivative by Aldweby and Darus [1] and the symmetric Salagean differential operator by Ibrahim and Darus [2]. We study some properties of the new operator such as the negative coefficients estimate, growth and the radii of starlikeness and convexity.

1. Introduction

Let $f \in A$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{k=2}^{\infty} a_k z^k, \quad \text{where} \quad z \in U \tag{1}$$

which are analytic in the open unit disc $U = [z : |z| < 1]$. The class $S \subset A$ consists of univalent functions. A function $f \in A$ is said to be starlike of order α if it fulfills the condition

$$\mathcal{R}e\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad 0 \leqslant \alpha < 1.$$

Also, the class $C \subset A$ consists of univalent functions. A function $f \in A$ is said to be convex of order α if it fulfills the condition

$$\mathcal{R}e\left(\frac{(zf'(z))'}{f'(z)}\right) > \alpha, \quad 0 \leqslant \alpha < 1,$$

where $f \in C(\alpha)$, if and only if, $zf' \in S^*(\alpha)$.

Let $f_1, f_2 \in A$. If there exists a Schwartz function $\Phi(z)$ which is analytic in U with $\Phi(0) = 0$ and $|\Phi(z)| < 1$ such that $f_1(z) = f_2(\Phi(z))$, then we say that $f_1(z)$ is subordinate to $f_2(z)$ and write $f_1(z) \prec f_2(z)$, where $\prec$ denote subordination symbol. The Hadamard product of two functions $f(z)$ of the form (1) and $g(z)$ is of the form

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad \textit{where} \quad z \in U$$

as

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Differential operator; q-calculus; Starlikeness; convextiy; Hadamard product.

Many researchers have attracted to study q-calculus in the area of geometric function theory. According to Jackson [5] and Aral [6], we give some necessary definitions about q-calculus, which we shall use throughout.

$$\partial_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}, \qquad (z \neq 0, \ \ z \in (0,1))$$

and

$$\int_0^z f(t)\partial_q t = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k), \qquad q \in (0,1).$$

Now when $k = 1, 2, 3...$ and $z \in U$ we can see that

$$\partial_q \left\{ \sum_{k=1}^{\infty} a_k z^k \right\} = \sum_{k=1}^{\infty} [k]_q a_k z^{k-1}$$

where

$$[k]_q = 1 + \sum_{m=1}^{k-1} q^m = \frac{1-q^k}{1-q}, \qquad [0]_q = 0.$$

For any non-negative integer k, the q-number shift factorial is defined by

$$[k]_q! = \begin{cases} 1 & k = 0 \\ [1]_q[2]_q[3]_q \dots [k]_q & k = 1, 2, 3, \dots \end{cases}$$

Also, the q-generalized Pochhammer symbol for $x > 0$ is given as

$$[k, q]_k = \begin{cases} 1 & k = 0 \\ [x]_q[x+1] \dots [x+k-1]_q & k = 1, 2, 3, \dots \end{cases}$$

Recently, Aldweby and Darus [1] defined the q-Ruscheweyh derivative as follows

$$f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k z^k$$

and Ibrahim and Darus [2] introduced the following symmetric Salagean differential given as

$$g(z) = z + \sum_{k=2}^{\infty} [k(\lambda - (1-\lambda)(-1)^k)]^m a_k z^k.$$

Then by convolution technique we derive the following:

$$\Upsilon_{(q,\lambda,m)} f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!} a_k z^k, \tag{2}$$

which will be used in our main results.

2. Main results

Remark 1. *From the definition in (2), we can see that:*

- When $m = 0$, $\Upsilon_{(q,\lambda,m)}f(z)$ becomes the q-Ruscheweyh operator.
- When $m = 0$ and $q \to 1$, $\Upsilon_{(q,\lambda,m)}f(z)$ becomes the Ruscheweyh operator.
- When $m = 0$ and $\lambda = 1$, $\Upsilon_{(q,\lambda,m)}f(z)$ becomes q-Salagean operator.
- When $m = 0$, $\lambda = 1$, and $q \to 1$, $\Upsilon_{(q,\lambda,m)}f(z)$ becomes Salagean operator.

The following two definitions according to [4]

Definition 1. *Let $f \in A$ of the form (1), then $f \in S_q^*(\alpha)$ if and only if*

$$\mathcal{R}e\left(\frac{zD_q(f(z))}{f(z)}\right) > \alpha \quad 0 \leqslant \alpha < 1, \quad 0 < q < 1, \quad z \in U.$$

Definition 2. *Let $f \in A$ of the form (1), then $f \in C_q(\alpha)$ if and only if*

$$\mathcal{R}e\left(\frac{D_q(zD_q(f(z))}{D_q(f(z))}\right) > \alpha \quad 0 \leq \alpha < 1, \quad 0 < q < 1, \quad z \in U.$$

Now we introduce a new class as follows:

Definition 3. *Let $f \in A$ of the form (1), then $f \in S^*_{q,\lambda,m}(\alpha)$ if and only if*

$$\mathcal{R}e\left(\frac{zD_q(\Upsilon_{(q,\lambda,m)}f(z))}{\Upsilon_{(q,\lambda,m)}f(z)}\right) > \alpha \quad 0 \leq \alpha < 1, \quad 0 < q < 1, \quad z \in U$$

where $\Upsilon_{(q,\lambda,m)}f(z)$ is given by (2). Also, $f \in C_{q,\lambda,m}(\alpha)$ if and only if

$$\mathcal{R}e\left(1 + \frac{D_q(zD_q(\Upsilon_{(q,\lambda,m)}f(z)))}{D_q(\Upsilon_{(q,\lambda,m)}f(z))}\right) > \alpha \quad 0 \leq \alpha < 1, \quad 0 < q < 1, \quad z \in U.$$

We further let $S\tau^*_{q,\lambda,m}(\alpha) = S^*_{q,\lambda,m}(\alpha) \cap \tau$ where,

$$\tau = z - \sum_{k=2}^{\infty} a_k z^k. \tag{3}$$

Theorem 1. *Let $f \in A$ of the form (1). If*

$$\sum_{k=2}^{\infty} ([k]_q - \alpha)\, \frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!} |a_k| \leq 1-\alpha, \qquad 0 \leq \alpha < 1, \tag{4}$$

then, $f \in S^*_{q,\lambda,m}(\alpha)$

Proof. It suffices to show that the value for $\left(\frac{zD_q(\Upsilon_{(q,\lambda,m)}f(z))}{\Upsilon_{(q,\lambda,m)}f(z)}\right)$ lies in a circle centered at $w = 1$ whose radius is $1-\alpha$, so we have

$$\begin{aligned}\left|\frac{zD_q(\Upsilon_{(q,\lambda,m)}f(z))}{\Upsilon_{(q,\lambda,m)}f(z)} - 1\right| &= \left|\frac{\sum_{k=2}^{\infty} ([k]_q - \alpha)\, \frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!} a_k z^k}{z - \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!} a_k z^k}\right| \\ &\leq \frac{\sum_{k=2}^{\infty} ([k]_q - \alpha)\, \frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!} |a_k||z^{k-1}|}{1 - \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!} |a_k||z^{k-1}|} \\ &\leq \frac{\sum_{k=2}^{\infty} ([k]_q - \alpha)\, \frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!} |a_k|}{1 - \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!} |a_k|}.\end{aligned}$$

The last expression is bounded above by $1-\alpha$ if

$$\sum_{k=2}^{\infty}([k]_q-1)\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}|a_k|$$
$$\leq 1-\alpha\left(1-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}|a_k|\right),$$

then the proof is complete. □

Theorem 2. *A necessary and sufficient condition for f of the form (3) to be in the class $S\tau^*_{q,\lambda,m}(\alpha)$ is that*

$$\sum_{k=2}^{\infty}([k]_q-\alpha)\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}a_k\leq 1-\alpha, \qquad a_k\geq 0$$

Proof. If $f\in S\tau^*_{q,\lambda,m}(\alpha)$ and z is real, then

$$\mathcal{R}e\left(\frac{zD_q(\Upsilon_{(q,\lambda,m)}f(z))}{\Upsilon_{(q,\lambda,m)}f(z)}\right)=\mathcal{R}e\left(\frac{z-\sum_{k=2}^{\infty}[k]_q\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}a_kz^k}{z-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}a_kz^k}\right)>\alpha.$$

Since the above inequality is true for all $z\in U$, choose values of z on the real axis so that $zD_q(\Upsilon_{(q,\lambda,m)}f(z))/\Upsilon_{(q,\lambda,m)}f(z)$ is real. Upon clearing the denominator in the last expression and letting $z\to 1$ through real values, we obtain

$$1-\sum_{k=2}^{\infty}[k]_q\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}a_k\geq\alpha\left(1-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}a_k.\right)$$

Now we can get

$$\sum_{k=2}^{\infty}([k]_q-\alpha)\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}a_k\leq 1-\alpha.$$

□

Corollary 1. *If the function f of the form (3) is in the class $S\tau^*_{q,\lambda,m}(\alpha)$, then*

$$a_k\leq\frac{(1-\alpha)[\lambda]_q![k-1]_q!}{([k]_q-\alpha)[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}, \qquad k\geq 2.$$

The result is sharp for the function

$$f_k(z)=z-\frac{(1-\alpha)[\lambda]_q![k-1]_q!}{([k]_q-\alpha)[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}z^k, \qquad k\geq 2.$$

Corollary 2. *If f of the form (3), then it satisfies the following condition*

$$\sum_{k=2}^{\infty}[k]_q([k-1]_q+(1-\alpha))\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}a_k\leq 1-\alpha, \qquad a_k\geq 0.$$

Theorem 3. *If an analytic function f is in the class $S\tau^*_{q,\lambda,m}(\alpha)$ and satisfies the condition (3), then*

$$|z|-\frac{1-\alpha}{(1+q-\alpha)(4\lambda-2)^m[1+\lambda]_q}|z|^2\leq|f(z)|\leq|z|+\frac{1-\alpha}{(1+q-\alpha)(4\lambda-2)^m[1+\lambda]_q}|z|^2.$$

The bounds are sharp since the equality is attained by the function

$$f(z) = z - \frac{1-\alpha}{(1+q-\alpha)(4\lambda-2)^m[1+\lambda]_q} z^2.$$

Proof. Since,

$$\sum_{k=2}^{\infty} ([k]_q - \alpha) \frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!} a_k \leq 1-\alpha,$$

and

$$(1+q-\alpha)(4\lambda-2)^m[1+\lambda]_q \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} ([k]_q - \alpha) \frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!} |a_k| \leq 1-\alpha,$$

we have

$$\sum_{k=2}^{\infty} a_k \leq \frac{1-\alpha}{(1+q-\alpha)(4\lambda-2)^m[1+\lambda]_q}.$$

Thus, for f in $S\tau^*_{q,\lambda,m}(\alpha)$ we obtain

$$\begin{aligned} |f(z)| &= |z - \sum_{k=2}^{\infty} a_k z^k| \\ &\leq |z| + \sum_{k=2}^{\infty} |a_k||z|^2 \\ &\leq |z| + \frac{1-\alpha}{(1+q-\alpha)(4\lambda-2)^m[1+\lambda]_q}|z|^2 \end{aligned}$$

which is the right side.

The left side can be shown as follows:

$$\begin{aligned} |f(z)| &= |z - \sum_{k=2}^{\infty} a_k z^k| \\ &\geq |z| - \sum_{k=2}^{\infty} |a_k||z|^2 \\ &\geq |z| - \frac{1-\alpha}{(1+q-\alpha)(4\lambda-2)^m[1+\lambda]_q}|z|^2. \end{aligned}$$

□

Theorem 4. *If the function f of the form (3). Then $\Upsilon_{(q,\lambda,m)}f(z)$ is starlike of order $0 \leq \delta \leq 1$ in $|z| < R_1$ where*

$$R_1 = \inf_k \left[\frac{(1-\delta)([k]_q - \alpha)}{([k]_q - \delta)(1-\alpha)} \right]^{\frac{1}{k-1}}.$$

Proof. By following the same work of Darus[3], it is suffices to prove

$$\left|\frac{zD_q(\Upsilon_{(q,\lambda,m)}f(z))}{\Upsilon_{(q,\lambda,m)}f(z)}-1\right| = \left|\frac{\sum_{k=2}^{\infty}([k]_q-1)\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}a_kz^k}{z-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}a_kz^k}\right|$$

$$\leq \frac{\sum_{k=2}^{\infty}([k]_q-1)\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}|a_k||z^{k-1}|}{1-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}|a_k||z^{k-1}|}$$

and the last expression is less than $(1-\delta)$.

Thus,

$$\frac{\sum_{k=2}^{\infty}([k]_q-1)\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}|a_k||z^{k-1}|}{1-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}|a_k||z^{k-1}|} < (1-\delta),$$

$$\sum_{k=2}^{\infty}([k]_q-1)\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}|a_k||z^{k-1}| <$$
$$(1-\delta)(1-\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}|a_k||z^{k-1}|),$$

where

$$\sum_{k=2}^{\infty}([k]_q-1+1-\delta)\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}|a_k||z^{k-1}| < (1-\delta)$$

and

$$([k]_q-\delta)\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}|a_k||z^{k-1}|$$
$$<\sum_{k=2}^{\infty}([k]_q-\delta)\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}|a_k||z^{k-1}| < (1-\delta).$$

Thus,

$$([k]_q-\delta)\frac{[k+\lambda-1]_q![k(\lambda-(1-\lambda)(-1)^k)]^m}{[\lambda]_q![k-1]_q!}|a_k||z^{k-1}| < (1-\delta)$$

and finally

$$|z^{k-1}| < \frac{(1-\delta)([k]_q-\alpha)}{([k]_q-\delta)(1-\alpha)}.$$

Therefore, the proof is complete. □

Theorem 5. *If the function f of the form (3). Then $\Upsilon_{(q,\lambda,m)}f(z)$ is convex of order $0\leq\delta\leq1$ in $|z|<R_2$ where*

$$R_2 = \inf_k\left[\frac{(1-\delta)([k]_q-\alpha)}{[k]_q([k-1]_q+(1-\delta))(1-\alpha)}\right]^{\frac{1}{k-1}}.$$

We can prove the theorem by using the same method of the previous theorem.

Aknowledgement: The work here is supported by MOHE grant: FRGS/1/2019/STG06/UKM/01/1.

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Universiti Kebangsaan Malaysia
Department of Mathematical Sciences, Faculty of Science and Technology
43600, Bangi, Selangor, Malaysia
Email address: [1]`khmo.alshammari@uoh.edu.sa`

Universiti Kebangsaan Malaysia
Department of Mathematical Sciences, Faculty of Science and Technology
43600, Bangi, Selangor, Malaysia
Email address: [2,*]`maslina@ukm.edu.my`