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THE DIRECT PROBLEM AND THE INVERSE PROBLEM OF THE NORMAL DISTRIBUTION. MATHEMATICAL MODELS-ALGORITHMS-PROGRAM

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Abstract. In this article we present two mathematical models, the algorithms that implement them, and the software that solves the direct problem and the inverse problem of the normal distribution. Tandem solving of the two issues provides both the possibility of forming an overview of the normal distribution as well as a set of useful and strictly necessary information to perform an informative analysis of the effectiveness of the used mathematical models and algorithms. The efficiency of mathematical models and, implicitly, of their algorithms is ultimately expressed through the accuracy and precision of the results obtained by using the software encoding these algorithms. The accuracy of the results reflects the extent to which the calculated value approaches the real value and the precision of the results refers to the exact number of digits in the representation of a double-precision mobile floating-point solution. Here, we have also demonstrated, using the concepts of absolute error and relative error, that the scientific approach in this article is rigorously substantiated both from a mathematical point of view and from the point of view of implementing algorithms in a high-level programming language.

1. Introduction

The random variable X follows a normal distribution of parameters $m, \sigma \in \mathbb{R}, \sigma > 0$ if it has the distribution density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}, \quad x \in \mathbb{R}, m \in \mathbb{R}, \sigma > 0 \tag{1}$$

where m is the mean value and σ is the standard deviation.

The fact that the random variable X has a normal distribution of parameters m and σ is marked by $X \sim N(m, \sigma)$.

If $X \sim N(m, \sigma)$, then the following properties are met:

1. $f(x) \geq 0, \forall x \in \mathbb{R}$
2. $\int\limits_{-\infty}^{\infty} f(x)dx = 1.$

The cumulative distribution function (CFD) of the random variable $X \sim N(m, \sigma)$ is

$$F(x) = \int\limits_{-\infty}^{x} f(t)dt = \frac{1}{\sigma\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-\frac{(t-m)^2}{2\sigma^2}} dt, \quad t, x \in \mathbb{R}, m, \sigma \in \mathbb{R}, \sigma > 0 \tag{2}$$

and the value of this function for a given x_α represents the probability that the value of the random variable X is less than x_α, namely that $F(x_\alpha) = P(X < x_\alpha)$. The value x_α

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of a random variable $X \sim N(m, \sigma)$ for which $F(x_\alpha) = \alpha$ is called the quantile of order α or $\alpha - quantile$.

Observation. If $m = 0$ and $\sigma = 1$, then, by convention, the random variable is denoted by Z and is said to follow a standard normal distribution, that is $Z \sim N(0,1)$. In this case, the distribution density is

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad \forall z \in \mathbb{R}. \tag{3}$$

The cumulative distribution function of $Z \sim N(0,1)$ is

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt \tag{4}$$

and is called *Laplace's function.*

Regarding the normal distribution, two types of problems are of general interest:

- **The direct problem** of determining the probability $\alpha = F(x_\alpha)$ when the value of the $\alpha - quantile$ x_α is known;
- **The inverse problem** of determining the $\alpha - quantile$ x_α such that $F(x_\alpha) = \alpha$ when the probability value α, $\alpha \in (0,1)$ is known.

To easily solve the two problems, we notice that between the cumulative distribution function F of the random variable $X \sim N(m, \sigma)$ and the cumulative distribution function Φ of the variable $Z \sim N(0,1)$ one can establish a causal dependence: if in (2) we make the variable change $t - m = \sigma z$, we have that

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-m}{\sigma}} e^{-\frac{z^2}{2}} dz = \Phi\left(\frac{x-m}{\sigma}\right). \tag{5}$$

In this context, whatever the random variable $X \sim N(m, \sigma)$, *solving the direct problem first requires determining the value* $z_\alpha = \frac{x_\alpha - m}{\sigma}$ and then determining the value $\alpha = \Phi(z_\alpha)$ while solving the inverse problem first requires determining the value $z_\alpha = \Phi^{-1}(\alpha)$ from the equation $\Phi(z_\alpha) = \alpha$ and then determining the value $x_\alpha = m + \sigma \cdot z_\alpha$. Therefore, it is sufficient to only solve the direct problem and the inverse problem of the standard normal distribution and this will be discussed in detail in the next sections.

2. Numerical model and algorithm for solving the direct problem

Problem statement. Knowing the value of the $\alpha - quantile$ z_α of the random variable $Z \sim N(0,1)$, the task is to determine the probability $\alpha = \Phi(z_\alpha)$.

Solution. If $z_\alpha > 0$ then $-z_\alpha < 0$, and $\Phi(z_\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_\alpha} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{z^2}{2}} dz + \frac{1}{\sqrt{2\pi}} \int_{0}^{z_\alpha} e^{-\frac{z^2}{2}} dz = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_{0}^{z_\alpha} e^{-\frac{z^2}{2}} dz$ and because $\frac{1}{\sqrt{2\pi}} \int_{-z_\alpha}^{0} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} \int_{0}^{z_\alpha} e^{-\frac{z^2}{2}} dz$, we deduce that $\Phi(-z_\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z_\alpha} e^{-\frac{z^2}{2}} dz = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_{0}^{z_\alpha} e^{-\frac{z^2}{2}} dz$. Therefore, calculation the value of the indefinite integral $\Phi(z_\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_\alpha} e^{-\frac{z^2}{2}} dz$ is reduced to computing the

value of the definite integral:

$$\Phi^\star(z_\alpha) = \frac{1}{\sqrt{2\pi}} \int_0^{z_\alpha} e^{-\frac{z^2}{2}} dz,\ z_\alpha > 0 \tag{6}$$

and the probability value α will be

$$\alpha = \begin{cases} \frac{1}{2} + \Phi^\star(z_\alpha) & \text{if} \quad z_\alpha > 0 \\ \frac{1}{2} & \text{if} \quad z_\alpha = 0 \\ \frac{1}{2} - \Phi^\star(-z_\alpha) & \text{if} \quad z_\alpha < 0. \end{cases} \tag{7}$$

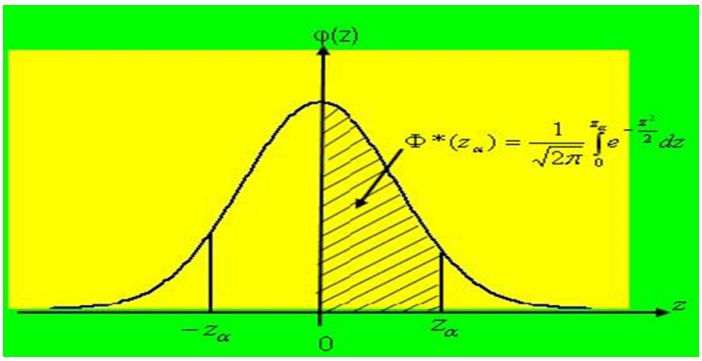

FIGURE 1

We notice that in order to determine the probability $\alpha = \Phi(z_\alpha)$, it is necessary and sufficient to determine $\Phi^\star(|z_\alpha|) = \frac{1}{\sqrt{2\pi}} \int_0^{|z_\alpha|} e^{-\frac{z^2}{2}} dz$. Because it is not possible to analytically determine an antiderivative of $g(z) = e^{-\frac{z^2}{2}}$, in order to calculate the approximate value $\Phi^\star(|z_\alpha|)$ we can use, e.g.[1], the generalized Simpson quadrature formula, that can be summarized as: if $g : [a, b] \to \mathbb{R}$ is an integrable function on the interval $[a, b]$, and $z_0 = a, z_i = a + i \cdot h, i = \overline{1, n-1}, z_n = b$ is a division of the interval $[a, b]$, with $h = \frac{b-a}{n}, n \in \mathbb{N}$, n chosen conveniently, then the approximate value for $I(g) = \int_a^b g(z)dz$ is

$$I(g) \cong \frac{1}{3}\frac{b-a}{n}\left[\frac{g(a)+g(b)}{2} + \sum_{i=1}^{n-1} g(z_i) + 2\sum_{i=0}^{n-1} g(\frac{z_i + z_{i+1}}{2})\right]. \tag{8}$$

In this case, the interval $[a, b] = [0, |z_\alpha|]$ and $\Phi^\star(|z_\alpha|) \cong \frac{1}{\sqrt{2\pi}} \cdot I(g)$. With these specifications, the algorithm that solves the direct problem of the standard normal distribution is structurally composed of the following steps:

Step 1. Read the value z_α and assign $\epsilon = 10^{-18}$.

Step 2. If $|z_\alpha| \leq \epsilon$ then $\alpha = \frac{1}{2}$, otherwise perform the following operations:

- determine the value $I(g)$[2] and then $\Phi^\star(|z_\alpha|) = \frac{1}{\sqrt{2\pi}} \cdot I(g)$;

[1] In order to obtain results with a better precision, we can use the Gauss quadrature formula, which also guarantees much faster convergence.

[2] The value of the integral is determined by the procedure **Simpson(g,n,0,ABS(zalfa),vi)**.

- if $|z_\alpha| > \epsilon$ then the probability α is $\alpha = \frac{1}{2} + \Phi^\star(|z_\alpha|)$;
- if $|z_\alpha| < -\epsilon$ then the probability α is $\alpha = \frac{1}{2} - \Phi^\star(|z_\alpha|)$.

Step 3. Stop.

The subprogram **FUNCTION probability(zalfa:EXTENDED):EXTENDED;** implements this algorithm.

3. Numerical model and algorithm for solving the inverse problem

Problem statement. For the random variable $Z \sim N(0,1)$ the probability value α is known and the task is to determine the value of the $\alpha - qunatile$ such that $\Phi(z_\alpha) = \alpha$, that is, the task is to determine $z_\alpha = \Phi^{-1}(\alpha)$.

Solution. From (7) we deduce that $z_\alpha = 0$ when $\alpha = \frac{1}{2}$, and if $\alpha \neq \frac{1}{2}$ then the equation $\Phi^\star(z_\alpha) = \alpha - \frac{1}{2}$ must be solved when $z_\alpha > 0$, or the equation $\Phi^\star(-z_\alpha) = \frac{1}{2} - \alpha$ must be solved when $z_\alpha < 0$. Because the (positive or negative) nature of z_α is not known a priori, we do not know which of the two equations is to be solved. But, if $z_\alpha > 0$ then $\Phi^\star(z_\alpha) > 0$, i.e., $\alpha > \frac{1}{2}$ and if $z_\alpha < 0$ then $\Phi^\star(-z_\alpha) > 0$, i.e., $\alpha < \frac{1}{2}$. Therefore, the equation to be solved is either $\Phi^\star(z_\alpha) = \alpha - \frac{1}{2}$ if $\alpha > \frac{1}{2}$, in which case $z_\alpha > 0$, or $\Phi^\star(-z_\alpha) = \frac{1}{2} - \alpha$ if $\alpha < \frac{1}{2}$, in which case $z_\alpha < 0$. But if $z_\alpha < 0$ then $-z_\alpha > 0$ and therefore solving the inverse problem of the standard normal distribution is reduced to solving the equation

$$\Phi^\star(|z_\alpha|) = \beta \tag{9}$$

where

$$\beta = \begin{cases} \alpha - \frac{1}{2} & \text{if} \quad \alpha > \frac{1}{2} \\ \frac{1}{2} - \alpha & \text{if} \quad \alpha < \frac{1}{2}. \end{cases} \tag{10}$$

Provided that through an arbitrary procedure $u_\alpha = |z_\alpha| = (\Phi^\star)^{-1}(\beta)$ can be determined, then

$$z_\alpha = \begin{cases} u_\alpha & \text{if} \quad \alpha > \frac{1}{2} \\ -u_\alpha & \text{if} \quad \alpha < \frac{1}{2}. \end{cases} \tag{11}$$

The remaining open problem is: how to determine the value $u_\alpha = |z_\alpha| = (\Phi^\star)^{-1}(\beta)$, *which is the solution of the equation* $\Phi^\star(u_\alpha) = \beta$, i.e., the solution of the equation

$$\frac{1}{\sqrt{2\pi}} \int_0^{u_\alpha} e^{-\frac{z^2}{2}} dz = \beta, \; u_\alpha > 0. \tag{12}$$

Equation (12) is equivalent with the equation

$$\int_0^{u_\alpha} e^{-\frac{z^2}{2}} dz = \gamma \tag{13}$$

where $\gamma = \beta\sqrt{2\pi}$.

Since it is not possible to analytically determine an antiderivative of $g(z) = e^{-\frac{z^2}{2}}$, in order to solve equation (13), we designed a novel numerical approach that we called: **the small steps with return method**.

The mathematical support of this method is the following: the solution u_α of the equation $\int_0^{u_\alpha} g(z)dz = \gamma$ is certainly located inside an interval $[0, b] \in \mathbb{R}_+$. Without restricting

generality, we can assume that $[0,b]=\bigcup_{k=0}^{m} J_k$ is a union of equidistant intervals such that $J_i \bigcap J_j=\emptyset, \forall i\neq j$. The intervals $J_k=[a_k,b_k],\ k=\overline{0,m}$ are progressively constructed as follows: $a_0=0,\ b_0=a_0+p,\ a_{i+1}=b_i,\ b_{i+1}=a_{i+1}+p,\ \forall i=\overline{0,m-1}$ with $p,\ p>0$, being a conveniently chosen step; e.g., $p=0.1$. Under these circumstances, we have that

$$\int_0^b g(z)dz=\sum_{k=0}^{m}\int_{a_k}^{b_k} g(z)dz=\sum_{k=0}^{m} I_k,\quad \text{where}\ \ I_k=\int_{a_k}^{b_k} g(z)dz. \tag{14}$$

In order to determine the value of u_α we proceed as follows: let $n\in\mathbb{N}$ be the first natural number for which $S=\sum_{k=0}^{n} I_k\geq\gamma$. If for any $\epsilon>0$, e.g., $\epsilon=10^{-18}$, we have that $|S-\gamma|\leq\epsilon$, then $u_\alpha=b_n$, and if $|S-\gamma|>\epsilon$, we subtract from S the value of I_n and we proceed to process the interval $[a_n,b_n]$ in a similar manner to the interval $[0,b]$ while reducing the step size p. For example, we can continue with a step size of $p/10$. After a finite number of steps we will have that $|S-\gamma|\leq\epsilon$ and, if we denote by I_q the last integral added to the sum S, then $u_\alpha=b_q$. In order to calculate the value of a given integral $I_k=\int_{a_k}^{b_k} g(z)dz$, we can use, as in the case of the direct problem, the generalized Simpson quadrature formula.

In light of the previous analysis, the algorithm that solves the inverse problem of the standard normal distribution is structurally composed of the following steps:

Step 1. Read the probability value α and assign $\epsilon=10^{-18}$.

Step 2. If $\alpha=\frac{1}{2}$, then assign the value 0 to z_α. Otherwise execute steps 3, 4, 5 and 6.

Step 3. If $\alpha>\frac{1}{2}$, then assign the value $\sqrt{2\pi}(\alpha-\frac{1}{2})$ to γ or if $\alpha<\frac{1}{2}$, then assign the value $\sqrt{2\pi}(\frac{1}{2}-\alpha)$ to γ.

Step 4. Make the initializations: $a\leftarrow 0;\ p\leftarrow 0.1;\ s\leftarrow 0;\ n\leftarrow 5000;\ kod\leftarrow 1$[3].

Step 5. While the condition $kod\neq 0$ is satisfied, perform the following operations:

1. Repeat the following operations:
   - **a)** $b\leftarrow a+p$;
   - **b)** Call the procedure **Simpson(g,n,a,b,vi)** in order to determine the value $vi\leftarrow\int_a^b g(z)dz$;
   - **c)** Make the assignments: $s\leftarrow s+vi;\ a\leftarrow b$;

   until $s>\gamma$.
2. If $|s-\gamma|<\epsilon$, then assign the value 0 to the variable *kod*. Otherwise, make the following assignments: $kod\leftarrow 1;\ a\leftarrow b-p;\ p\leftarrow p/10$.

Step 6. If $\alpha>\frac{1}{2}$, then assign the value b to z_α or, if $\alpha<\frac{1}{2}$, then assign the value $-b$ to z_α.

Step 7. Stop.

The subprogram **FUNCTION quantile(alfa:EXTENDED):EXTENDED;** implements this algorithm.

[3]The *kod* variable will have the value 1 as long as the approximate value of the $\alpha-quantile\ z_\alpha$ has not been determined and when this value is determined, the variable will receive the value 0.

4. Algorithmic efficiency and result precision

In this section, by referring to the concepts of absolute error and relative error, we demonstrate that our study is rigorously substantiated both from a mathematical point of view and from the point of view of implementing the proposed algorithms in a high level programming language.

Therefore, we analyzed the following two cases:

1. When solving the direct problem, after starting from the initial value z_α, the probability $\alpha = \Phi(z_\alpha)$ is determined and then, for α, the inverse problem is solved, thus finally determining a value $z_\alpha^{'} = \Phi^{-1}(\alpha)$. If all calculations are made without any error, according to the following scheme:

$$z_\alpha \xrightarrow{\Phi} \alpha \xrightarrow{\Phi^{-1}} z_\alpha^{'}$$

the value $z_\alpha^{'}$ should be equal to z_α, but this is generally not the case because two types of errors appear in the data processing process: *method errors* and *calculation errors*. The effect of method errors can be diminished by adopting the best mathematical model and computational errors (that are due to cropping or rounding out results) can be diminished by working with data represented in a double-precision floating-point format. The difference between the calculated value $z_\alpha^{'}$ and the ground truth value z_α is measured by absolute error and relative error. The absolute error of $z_\alpha^{'}$ is determined by the formula $E_a(z_\alpha^{'}) = |z_\alpha - z_\alpha^{'}|$, and the relative error is determined by the formula $E_r(z_\alpha^{'}) = |\frac{z_\alpha - z_\alpha}{z_\alpha}|$. The lower the values of these errors, the more efficient are the mathematical models that solve the two problems (direct and inverse) and this directly translated into a higher precision of the obtained results. From the multiple numerical experiments we performed on the computer, we came to the conclusion that both the absolute error and the relative error are, in most cases, smaller than 10^{-16} and this finding proves that the pursued objective has been achieved. An example in this sense is the following:

```
The solution to the direct problem:
If zalfa = 1.400000000000000000 then alfa= 0.919243340766228954
The solution to the inverse problem:
If alfa = 0.919243340766228954 then zalfap = 1.400000000000000010
Errors for zalfap:
Absolute error is Ea= 7.04731412115578E-0018
Relative error is Er= 5.03379580082556E-0018
```

2. When solving the inverse problem, after starting from the initial value α, the quantile $z_\alpha^{'} = \Phi^{-1}(\alpha)$ is determined, and then, for $z_\alpha^{'}$, the direct problem is solved, thus finally determining a probability value $\alpha^{'} = \Phi(z_\alpha^{'})$. If all calculations are performed accurately, according to the following scheme:

$$\alpha \xrightarrow{\Phi^{-1}} z_\alpha^{'} \xrightarrow{\Phi} \alpha^{'}$$

the value $\alpha^{'}$ should be equal to the ground truth value of α, but in this case also the value $\alpha^{'}$ is affected by the same types of errors as in the previous case. This time the absolute error is $E_a(\alpha^{'}) = |\alpha - \alpha^{'}|$ and the relative error is $E_r(\alpha^{'}) = |\frac{\alpha - \alpha^{'}}{\alpha}|$. Following the computer experiments we came to the same conclusion as in the previous case, namely that both the absolute error and the relative error are lower than 10^{-16}. An example in this sense is the following:

```
The solution to the direct problem:
If alfa = 0.919243340766228954 then zalfap = 1.400000000000000010
The solution to the inverse problem:
If zalfap = 1.400000000000000010 then alfap= 0.919243340766228955
Errors for alfa:
The absolute error is Ea= 9.21571846612679E-0019
The relative error is Er= 1.00253306795185E-0018
```

5. The Borland Pascal program which solves the direct problem and the inverse problem of the standard normal distribution

The scientific approach presented in this article reaches its finality if based on it a software product can be realized that implements all the encountered cases. The program presented in this section is a rather expressive solution and provides interested specialists with a useful tool to solve the practical problems they face in day-to-day work. The program has a modular structure and is quite easy to read. The underlying algorithm of this program structurally consists of the following steps:

Step 1. Interrogate what kind of problem to solve (D / I), i.e., direct or inverse.

Step 2. If the direct problem has to be solved, then the value of z_α (coded with $zalfa$) is read, the function **probability(zalfa)** is called assigning to α (coded with $alfa$) the return value of this function and the value of $alfa$ is finally printed. If the inverse problem has to be solved, the value of $alfa$ is read, the function **quantile(alfa)** is called assigning to $zalfa$ the return value of this function and the value of $zalfa$ is finally printed.

The obtained software product is the following:

```
PROGRAM PDI_Normal_Distribution;{$F+} {$N+}

{ The Direct problem and the Inverse problem of the standard normal
distribution.}

TYPE func=FUNCTION(x:EXTENDED):EXTENDED;
VAR alfa,zalfa:EXTENDED;
    tp:CHAR;
CONST eps=1.0E-18;

FUNCTION g(x:EXTENDED):EXTENDED;
BEGIN
  g:=EXP(-x*x/2);
END;
PROCEDURE Simpson(g:func;n:INTEGER;a,b:EXTENDED;VAR vi:EXTENDED);
VAR h,s1,s2:EXTENDED;
    i:INTEGER;
BEGIN
  h:=(b-a)/n;
  vi:=(g(a)+g(b))/2;
  s1:=0;
  FOR i:=1 TO n-1 DO s1:=s1+g(a+i*h);
  s2:=0;
  FOR i:=0 TO n-1 DO s2:=s2+g(a+i*h+h/2);
  vi:=h*(vi+s1+2*s2)/3;
END;
```

```
FUNCTION probability(zalfa:EXTENDED):EXTENDED;
VAR vi:EXTENDED;
    n:INTEGER;
BEGIN
   n:=5000;
   IF ABS(zalfa)<=eps THEN probability:=0.5
   ELSE
     BEGIN
       Simpson(g,n,0,ABS(zalfa),vi);
       vi:=vi/SQRT(2*PI);
       IF zalfa> eps THEN probability:=0.5+vi
       IF zalfa<-eps THEN probability:=0.5-vi;
     END;
END;
FUNCTION quantile(alfa:EXTENDED):EXTENDED;
VAR a,b,p,vi,s,za,gama:EXTENDED;
    n,kod:INTEGER;
BEGIN
  IF alfa=0.5 THEN quantile:=0
  ELSE
    BEGIN
      IF alfa>0.5 THEN gama:=SQRT(2*PI)*(alfa-0.5);
      IF alfa<0.5 THEN gama:=SQRT(2*PI)*(0.5-alfa);
      a:=0; p:=0.1; s:=0;
      n:=5000;kod:=1;
      WHILE kod<>0 DO
        BEGIN
          REPEAT
            b:=a+p;
            Simpson(g,n,a,b,vi);
            s:=s+vi; a:=b;
          UNTIL s>gama;
          IF ABS(s-gama)<eps THEN kod:=0
          ELSE
            BEGIN
              kod:=1;  a:=b-p;
              s:=s-vi; p:=p/10;
            END;
        END;
      IF alfa>0.5 THEN quantile:=b;
      IF alfa<0.5 THEN quantile:=-b;
    END;
END;
    { M A I N   P R O G R A M }
BEGIN
  WRITE('Problem type (D/I): ');
  READLN(tp);
  IF UPCASE(tp)='D' THEN
    BEGIN
```

```
      WRITE('Input the value of the alfa quantile: zalfa= ');
      READLN(zalfa);
      alfa:=probability(zalfa);
      WRITELN('The solution of the direct problem:');
      WRITELN('If zalfa = ',zalfa:20:18,'  then alfa = ',alfa:20:18);
    END
  ELSE
    BEGIN
      WRITE('Input the probability value: alfa= ');
      READLN(alfa);
      zalfa:=quantile(alfa);
      WRITELN('The solution of the inverse problem:');
      WRITELN('If alfa =  ',alfa:20:18,'  then zalfa = ',zalfa:20:18);
    END;
END.
```

6. Final conclusions

1. In this article, we synthesized the main theoretical results regarding the normal distribution and we also emphasized that the field of investigations on this issue is not closed. We justify this last statement by the fact that the solutions for the direct and the inverse problem of the standard normal probability distribution returned by the statistical functions NORMSDIST and NORMSINV included in the Microsoft Office Excel spreadsheet program have a much lower precision than those obtained in this work.

2. The mathematical models and the algorithms by which we have solved the two problems (i.e., direct and inverse) are quite expressive entities and the program we have designed is a useful and efficient tool for both statistics and computer science practitioners.

3. The study is also important for researchers working in the field of numerical simulation because they have the possibility to generate random variables with a normal distribution by the inverse method although it is not possible to determine the analytical expression of the inverse of the cumulative distribution function $F(x)$ corresponding to the random variable $X \sim N(m, \sigma)$. The inverse method of generating random variables is grounded on the Smirnov-Hincin lemma that states the following: *if X is a random variable that has the cumulative distribution function $F(x)$, and U is uniformly distributed random variable over $(0, 1)$ then the cumulative distribution function $Y = F^{-1}(U)$ is $F(x)$*. Basically, in order to generate a value x_α of a random variable $X \sim N(m, \sigma)$, one can proceed as follows:

Step 1. Generate a random number α uniformly distributed over $(0, 1)$.

Step 2. Determine $z_\alpha = \Phi^{-1}(\alpha)$ and then the value $x_\alpha = m + \sigma \cdot z_\alpha$.

Obviously, if the requirement is to generate an array of random numbers $x_1, x_2, \ldots, x_n,\ n \in \mathbb{N}$, steps 1 and 2 are to be repeated, in this order, n times.

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