TJMM
**12** (2020), No. 1, 45-58

# THREE - DIMENSIONAL INFLUENCE FUNCTIONS AND INTEGRATION FORMULAS FOR MANY BOUNDARY VALUE PROBLEMS WITHIN A THERMOELASTIC HALF-LAYER

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Abstract. The aim of this paper is the constructing of the main thermoelastic displacements Green's functions (MTDGFs) for a generalized 3D BVP of uncoupled thermoelasticity for a half-layer. To reach this aim are derived structural formulas for MTDGFs expressed via respective Green's functions for Poisson's equation (GFPE) by using harmonic integral representations method (HIRM). These structural formulas are validated by the checking the equations of thermoelasticity with respect to point of application the heat source and the nonhomogeneous Poisson's equation with respect to point of response in which the thermoelastic displacements appeared. In addition, they satisfy boundary conditions for temperature Green's function with respect to point application the displacements and to mechanical boundary conditions with respect to point of application the heat source. The thermoelastic volume dilatation (TVD) derived separately from respective integral representations have to be equal to the TVD derived by using structural formulas for MTDGFs. The final analytical expressions for MTDGFs obtained on the base of mentioned above structural formulas for eight new 3D BVPs of thermoelasticity within half-layer contain Bessel functions of the zero-order and of the second type. Finally, the integration formulas for thermal displacements and stresses, created by inner heat source and by the thermal data given of the surface of the half-layer are presented also.

## Abbreviations

MTDGFs – main thermoelastic displacements Green's functions;
3D – three dimensional;
BVP – boundary value problem;
GFPE – Green's function for Poisson equation;
MTSGFs – main thermal stresses Green's functions;
HIRM – harmonic integral representations method;
GFs – Green's functions;
TVD – thermoelastic volume dilatation;
IF – integration formula.

## 1. Introduction

The obtained in this paper results are considered for uncoupled thermoelasticity, in special for theory of thermal stresses, theories of which are presented in the classical [1]-[7] and modern [8] scientific literature. The MTDGFs and the integration formulas (IFs) were derived by using HIRM in the works [9]-[16]. In this paper for the first time is proposed the development of the HIRM to derivation of thermoelastic structural formulas

2010 *Mathematics Subject Classification.* code, code.
*Key words and phrases.* harmonic integral representation method, structural formulas, integration formulas, main thermoelastic Green's functions, thermoelastic volume dilatation, thermoelastic half-layer.

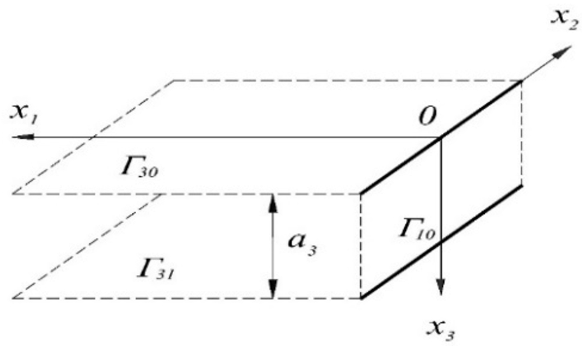


FIGURE 1. The scheme of the half-layer $V \equiv (0 \le x_1 < \infty, -\infty < x_2 < \infty, 0 \le x_3 \le a_3)$ with boundary half-planes $\Gamma_{30}(0 \le x_1 < \infty, -\infty < x_2 < \infty, x_3 = 0)$, $\Gamma_{31}(0 \le x_1 < \infty, -\infty < x_2 < \infty, x_3 = a_3)$ and boundary strip $\Gamma_{10}(x_1 = 0, -\infty < x_2 < \infty, 0 \le x_3 \le a_3)$.

for a generalized BVP, which permitted us to obtain analytical expressions for MTDGFs and IFs for eight BVPs for the half-layer (Figure 1).

**Objectives**

The main objective of this paper is to develop HIRM for constructing the MTDGFs and IFs for locally-mixed 3D BVPs of thermoelastic half-layer $V$ for eight new locally-mixed 3D BVPs of thermoelasticity. This paper is organized in such a way that the sections of investigations are given in the following consequence: Section 2. Formulation of the generalized BVP for thermoelastic half-layer; Section 3. General and particular integral representations for TVD and MTDGFs; Section 4. Structural formulas for MTDGF; Section 5. The checking of the derived final structural formulas for MTDGFs and TVD; Section 6. Analytical expressions for MTDGFs and TVD; Section 7. Integration formulas for thermoelastic displacements and stresses within half-layer.

## 2. Formulation of the generalized BVP for thermo-elastic half-layer

The generalized BVP to uncoupled thermoelasticity for determining structural formulas for MTDGFs for displacements $U_i(x,\xi)$ within the half-layer consist from Lame's and Poisson's equations:

$$\begin{aligned} &\mu \bigtriangledown^2_\xi U_i(x,\xi) + (\lambda+\mu)\Theta_{,\xi_i}(x,\xi) - \gamma G_{T,\xi_i}(x,\xi) = 0; i = 1,2,3; \\ &\nabla^2_\xi G_T(x,\xi) = -\delta(x-\xi); x \equiv (x_1,x_2,x_3), \xi \equiv (\xi_1,\xi_2,\xi_3), \end{aligned} \tag{1}$$

where $\lambda$ and $\mu$ are Lame constants of elasticity, $\gamma = \alpha_T(3\lambda + 2\mu)$ is the thermoelastic constant, $\alpha_T$ is the coefficient of linear thermal dilatation, $\Theta(x,\xi)$ is thermoelastic volume dilatation (TVD), $\delta(x-\xi)$ is delta Dirac function, $x \equiv (x_1,x_2,x_3)$ is the point of application of the unit interior heat source, $\xi \equiv (\xi_1,\xi_2,\xi_3)$ is the point in which MTDGFs $U_i(x,\xi)$ appeared. In addition on the surface of the half-layer are given the following eight possible combinations of the boundary conditions for $U_i(x,y); i = 1,2,3$, thermal stresses $\sigma^*_{ij}(y,\xi); j = 1,2,3$, and GF for temperature $G_T(y,\xi)$ or its derivative on external normal $\partial G_T(y,\xi)/\partial n_\Gamma$ :

$$U_3(y,\xi) = \sigma^*_{31}(y,\xi) = \sigma^*_{32}(y,\xi) = 0, \partial G_T(y,\xi)/\partial n_{\Gamma_{30}} = 0; \\ \xi \in V; y \equiv (y_1, y_2, 0) \in \Gamma_{30}; \tag{2}$$

or

$$\sigma^*_{33}(y,\xi) = U_1(y,\xi) = U_2(y,\xi) = 0, G_T(y,\xi) = 0; \xi \in V; y \equiv (y_1, y_2, 0) \in \Gamma_{30}; \tag{3}$$

- on the boundary half-plane $\Gamma_{30}(0 \leq y_1 < \infty, -\infty < y_2 < \infty, y_3 = 0)$, and

$$U_3(y,\xi) = \sigma^*_{31}(y,\xi) = \sigma^*_{32}(y,\xi) = 0, \partial G_T(y,\xi)/\partial n_{\Gamma_{31}} = 0; \\ \xi \in V; y \equiv (y_1, y_2, a_3) \in \Gamma_{31}; \tag{4}$$

or

$$\sigma^*_{33}(y,\xi) = U_1(y,\xi) = U_2(y,\xi) = 0, G_T(y,\xi) = 0; \xi \in V; y \equiv (y_1, y_2, a_3) \in \Gamma_{31}; \tag{5}$$

- on the boundary half-plane $\Gamma_{31}(0 \leq y_1 < \infty, -\infty < y_2 < \infty, y_3 = a_3)$, and

$$\sigma^*_{11}(y,\xi) = U_2(y,\xi) = U_3(y,\xi) = 0, G_T(y,\xi) = 0; \xi \in V; y \equiv (0, y_2, y_3) \in \Gamma_{10}; \tag{6}$$

or

$$U_1(y,\xi) = \sigma^*_{12}(y,\xi) = \sigma^*_{13}(y,\xi) = 0, G_T(y,\xi) = 0; \xi \in V; y \equiv (0, y_2, y_3) \in \Gamma_{10}; \tag{7}$$

- on the boundary strip $\Gamma_{10}(y_1 = 0, -\infty < y_2 < \infty, 0 \leq y_3 \leq a_3)$.

## 3. General and particular integral representations for TVD and MTDGFs

To derive the structural formulas for TVD and MTDGFs, we use the following general integral representations of the Eqs. (1) [10]-[13]:

$$\Theta(x,\xi) = \frac{\gamma}{\lambda + 2\mu} G_\Theta(x,\xi) + \int\limits_\Gamma \left[ \frac{\partial \Theta(x,y)}{\partial n_\Gamma} - \Theta(x,y) \frac{\partial}{\partial n_\Gamma} \right] G_\Theta(y,\xi) d\Gamma(y); \tag{8}$$

- for TVD $\Theta$, and

$$U_i(x,\xi) = \frac{\gamma \xi_i}{2\mu} G_T(x,\xi) - \frac{\lambda + \mu}{2\mu} \xi_i \Theta(x,\xi) - \frac{\gamma}{2(\lambda + 2\mu)} x_i G_i(x,\xi)$$

$$+ \int\limits_\Gamma \left[ G_i(y,\xi) \frac{\partial}{\partial n_\Gamma} - \frac{\partial G_i(y,\xi)}{\partial n_\Gamma} \right] \left[ U_i(x,y) + \frac{y_i}{2\mu} \left[ (\lambda + \mu) \Theta(x,y) - \gamma G_T(x,y) \right] \right] d\Gamma(y); \tag{9}$$

$$i = 1, 2, 3;$$

- for MTDGFs, where $G_i(x,\xi)$ are the GFPE.

The general representations (8) and (9) can be rewritten for half-layer $V$ as following:

$$\Theta(x,\xi) = \frac{\gamma}{\lambda + 2\mu} G_\Theta(x,\xi) - \int\limits_{\Gamma_{10}} \left[ \frac{\partial \Theta(x,y)}{\partial n_{\Gamma_{10}}} - \Theta(x,y) \frac{\partial}{\partial n_{\Gamma_{10}}} \right] G_\Theta(y,\xi) d\Gamma_{10}(y)$$

$$- \sum_{j=0}^{1} \int\limits_{\Gamma_{3j}} \left[ \frac{\partial \Theta(x,y)}{\partial n_{\Gamma_{3j}}} - \Theta(x,y) \frac{\partial}{\partial n_{\Gamma_{3j}}} \right] G_\Theta(y,\xi) d\Gamma_{3j}(y); \tag{10}$$

- for TVD $\Theta$, and

$$U_i(x,\xi) = \frac{\gamma\xi_i}{2\mu}G_T(x,\xi) - \frac{\lambda+\mu}{2\mu}\xi_i\Theta(x,\xi) - \frac{\gamma}{2(\lambda+2\mu)}x_iG_i(x,\xi)$$

$$-\int\limits_{\Gamma_{10}}\left(\frac{\partial G_i(y,\xi)}{\partial n_{\Gamma_{10}}} - G_i(y,\xi)\frac{\partial}{\partial n_{\Gamma_{10}}}\right)\left[U_i(x,y) + \frac{y_i}{2\mu}\left((\lambda+\mu)\Theta(x,y) - \gamma G_T(x,y)\right)\right]d\Gamma_{10}(y)$$

$$-\sum_{j=0}^{1}\int\limits_{\Gamma_{3j}}\left(\frac{\partial G_i(y,\xi)}{\partial n_{\Gamma_{3j}}} - G_i(y,\xi)\frac{\partial}{\partial n_{\Gamma_{3j}}}\right) \tag{11}$$

$$\times\left[U_i(x,y) + \frac{y_i}{2\mu}\left((\lambda+\mu)\Theta(x,y) - \gamma G_T(x,y)\right)\right]d\Gamma_{3j}(y); i = 1,2,3;$$

- for MTDGFs.

In addition, the boundary conditions (2) - (7) can be transformed to equivalent conditions on the base of the following links between displacements $U_i$, stresses $\sigma^*_{ij}$, TVD $\Theta$ and respective GFPE $G_i$, $G_\Theta$ on boundary half-planes $\Gamma_{3j}$; $j = 0,1$ and on boundary strip $\Gamma_{10} = 0$ of the half-layer $V$. So, if $U_i = 0$, then $G_i = 0$; if $U_{i,n} = 0$, then $G_{i,n} = 0$. If zero normal displacements, zero tangential stresses and $G_{T,n} = 0$ are given, then $\Theta_{,n} = 0$ and $G_{\Theta,n} = 0$. Finally, if zero normal stresses, zero tangential displacements and $G_T = 0$, then $\Theta = 0$ and $G_\Theta = 0$ [9]-[12], [14], [15]. So, in these cases the equivalent boundary conditions look as following:

$$\begin{gathered} U_3(y,\xi) = \sigma^*_{31}(y,\xi) = \sigma^*_{32}(y,\xi) = 0, G_{T,y_3}(y,\xi) = 0 \Rightarrow \\ U_3(y,\xi) = U_{1,y_3}(y,\xi) = U_{3,y_1}(y,\xi) = U_{3,y_2}(y,\xi) = U_{2,y_3}(y,\xi) = 0 \Rightarrow \\ \Theta_{,y_3}(y,\xi) = G_{1,y_3}(y,\xi) = G_{2,y_3}(y,\xi) = G_3(y,\xi) = G_{\Theta,y_3}(y,\xi) = \\ G_{T,y_3}(y,\xi) = 0; \xi \in V; y \equiv (y_1,y_2,0) \in \Gamma_{30}; \end{gathered} \tag{12}$$

or

$$\begin{gathered} \sigma^*_{33}(y,\xi) = U_1(y,\xi) = U_2(y,\xi) = 0, G_T(y,\xi) = 0 \Rightarrow \\ U_1(y,\xi) = U_{1,y_1}(y,\xi) = U_{1,y_2}(y,\xi) = U_{3,y_3}(y,\xi) = U_{2,y_1}(y,\xi) = U_{2,y_2}(y,\xi) = 0 \Rightarrow \\ \Theta(y,\xi) = G_1(y,\xi) = G_{3,y_3}(y,\xi) = G_2(y,\xi) = G_\Theta(y,\xi) = \\ G_T(y,\xi) = 0; \xi \in V; y \equiv (y_1,y_2,0) \in \Gamma_{30}; \end{gathered} \tag{13}$$

- on the boundary half-plane $\Gamma_{30}(0 \le y_1 < \infty, -\infty < y_2 < \infty, y_3 = 0)$,

$$\begin{gathered} U_3(y,\xi) = \sigma^*_{31}(y,\xi) = \sigma^*_{32}(y,\xi) = 0, G_{T,y_3}(y,\xi) = 0 \Rightarrow \\ U_3(y,\xi) = U_{1,y_3}(y,\xi) = U_{3,y_1}(y,\xi) = U_{3,y_2}(y,\xi) = U_{2,y_3}(y,\xi) = 0 \Rightarrow \\ \Theta_{,y_3}(y,\xi) = G_{1,y_3}(y,\xi) = G_{2,y_3}(y,\xi) = G_3(y,\xi) = G_{\Theta,y_3}(y,\xi) = \\ G_{T,y_3}(y,\xi) = 0; \xi \in V; y \equiv (y_1,y_2,a_3) \in \Gamma_{31}; \end{gathered} \tag{14}$$

or

$$\begin{gathered} \sigma^*_{33}(y,\xi) = U_1(y,\xi) = U_2(y,\xi) = 0, G_T(y,\xi) = 0 \Rightarrow \\ U_1(y,\xi) = U_{1,y_1}(y,\xi) = U_{1,y_2}(y,\xi) = U_{3,y_3}(y,\xi) = U_{2,y_1}(y,\xi) = U_{2,y_2}(y,\xi) = 0 \Rightarrow \\ \Theta(y,\xi) = G_1(y,\xi) = G_{3,y_3}(y,\xi) = G_2(y,\xi) = G_\Theta(y,\xi) = \\ G_T(y,\xi) = 0; \xi \in V; y \equiv (y_1,y_2,a_3) \in \Gamma_{31}; \end{gathered} \tag{15}$$

- on the boundary half-plane $\Gamma_{31}(0 \le y_1 < \infty, -\infty < y_2 < \infty, y_3 = a_3)$, and

$$\begin{gathered}\sigma^*_{11}(y,\xi)=U_2(y,\xi)=U_3(y,\xi)=0,G_T(y,\xi)=0\Rightarrow\\ U_3(y,\xi)=U_{3,y_3}(y,\xi)=U_{3,y_2}(y,\xi)=U_{2,y_2}(y,\xi)=U_{2,y_3}(y,\xi)=0\Rightarrow\\ \Theta(y,\xi)=G_{1,y_1}(y,\xi)=G_2(y,\xi)=G_3(y,\xi)=G_\Theta(y,\xi)=\\ G_T(y,\xi)=0;\xi\in V;y\equiv(0,y_2,y_3)\in\Gamma_{10};\end{gathered}\tag{16}$$

or

$$\begin{gathered}U_1(y,\xi)=\sigma^*_{12}(y,\xi)=\sigma^*_{13}(y,\xi)=0,G_{T,y_1}(y,\xi)=0\Rightarrow\\ U_1(y,\xi)=U_{1,y_3}(y,\xi)=U_{3,y_1}(y,\xi)=U_{1,y_2}(y,\xi)=U_{2,y_1}(y,\xi)=0\Rightarrow\\ \Theta_{,y_1}(y,\xi)=G_{1,y_3}(y,\xi)=G_{2,y_1}(y,\xi)=G_{3,y_1}(y,\xi)=G_1(y,\xi)=G_{\Theta,y_1}(y,\xi)=\\ G_{T,y_1}(y,\xi)=0;\xi\in V;y\equiv(0,y_2,y_3)\in\Gamma_{10};\end{gathered}\tag{17}$$

- on the boundary strip $\Gamma_{10}(y_1=0,-\infty<y_2<\infty,0\le y_3\le a_3)$.

Taken into account the boundary conditions (12) - (17) for $\Theta$ and $G_\Theta$ in the representation (10), the TVD $\Theta(x,\xi)$ can be rewritten in the form:

$$\Theta(x,\xi)=\frac{\gamma}{\lambda+2\mu}G_\Theta(x,\xi),\tag{18}$$

because for each boundary half-planes $\Gamma_{3j}$; $j=0,1$ and for the strip $\Gamma_{10}$ of the half-layer $V$, due the mentioned above four links for any eight combinations of the boundary conditions (12) - (17), the integrals in Eq. (10) vanish. Therefore, the final formula for TVD looks as in Eq. (18). Also, from boundary conditions (12) - (17) for $G_\Theta$ and $G_T$ follows $G_\Theta(x,\xi)=G_T(x,\xi)$. So, we obtain the following structural formula for TVD:

$$\Theta(x,\xi)=\frac{\gamma}{\lambda+2\mu}G_T(x,\xi).\tag{19}$$

## 4. Structural formulas for MTDGFs

Substituting, the proved formula (19) and boundary conditions (12) - (17) into (11), the integral representations will be simplified and look as follows:

$$U_1(x,\xi)=\frac{\gamma}{2(\lambda+2\mu)}\left[\xi_1G_T(x,\xi)-x_1G_1(x,\xi)\right];\tag{20}$$

$$U_2(x,\xi)=\frac{\gamma}{2(\lambda+2\mu)}\left[\xi_2G_T(x,\xi)-x_2G_2(x,\xi)\right];\tag{21}$$

$$\begin{aligned}U_3(x,\xi)=\frac{\gamma}{2(\lambda+2\mu)}&\left\{\xi_3G_T(x,\xi)-x_3G_3(x,\xi)\right.\\ &\left.+a_3\int\limits_{\Gamma_{31}}\left(\frac{\partial G_3(y,\xi)}{\partial n_{\Gamma_{31}}}-G_3(y,\xi)\frac{\partial}{\partial n_{\Gamma_{31}}}\right)G_T(x,y)d\Gamma_{31}(y)\right\};\end{aligned}\tag{22}$$

- for structural formulas of MTDGFs.

The integral in Eq. (22) can be calculated without the knowing the analytical expressions for GFPE $G_T(x,y)$ and $G_3(y,\xi)$ as follows:

$$\begin{aligned}I_3(x,\xi)=a_3\int\limits_{\Gamma_{31}}&\left(\frac{\partial G_3(y,\xi)}{\partial n_{\Gamma_{31}}}-G_3(y,\xi)\frac{\partial}{\partial n_{\Gamma_{31}}}\right)G_T(x,y)d\Gamma_{31}(y)\\ &=x_3G_3(x,\xi)-\xi_3G_T(x,\xi)\end{aligned}\tag{23}$$

$$-\int\left[\xi_1 G_{T,\xi_1}(x,\xi)-x_1 G_{1,\xi_1}(x,\xi)+\xi_2 G_{T,\xi_2}(x,\xi)-x_2 G_{2,\xi_2}(x,\xi)\right]d\xi_3.$$

Here was taking into account that $I_3(x,\xi)$ is a harmonic function with respect the coordinates of both points $x\equiv(x_1,x_2,x_3)$ and $\xi\equiv(\xi_1,\xi_2,\xi_3)$:

$$\nabla_x^2 I_3(x,\xi)=\nabla_\xi^2 I_3(x,\xi)=0, \tag{24}$$

which follow from the convolution [11], [15], [16]:

$$U_i(x,\xi)=\int\limits_V G_T(x,z)\Theta^{(i)}(z,\xi)dV(z), \tag{25}$$

where $\Theta^{(i)}(z,\xi)$ is the influence function for elastic volume dilatation, caused by the unit concentrated force, applied in the point $z\equiv(z_1,z_2,z_3)$ in the direction of the axis $oi; i=1,2,3$.

Also, the integral $I_3(x,\xi)$ have to satisfy with respect coordinates of the points $x\equiv(x_1,x_2,x_3)$ and $\xi\equiv(\xi_1,\xi_2,\xi_3)$ the following boundary conditions:

$$\begin{gathered}I_3(x_1,x_2,x_3=0;\xi)=0; I_3(x_1,x_2,x_3=a_3;\xi)=a_3 G_3(x_1,x_2,x_3=a_3;\xi);\\ I_3(x;\xi_1,\xi_2,\xi_3,=0)=0; I_3(x;\xi_1,\xi_2,\xi_3=a_3)=-a_3 G_T(x;\xi_1,\xi_2,\xi_3=a_3),\end{gathered} \tag{26}$$

which follow from Eq. (23).

Thus, substituting (23) into (22) we obtain the following structural formula for MTGFs $U_3(x,\xi)$:

$$\begin{gathered}U_3(x,\xi)=-\frac{\gamma}{2(\lambda+2\mu)}\int\left[\xi_1 G_{T,\xi_1}(x,\xi)\right.\\ \left.-x_1 G_{1,\xi_1}(x,\xi)+\xi_2 G_{T,\xi_2}(x,\xi)-x_2 G_{2,\xi_2}(x,\xi)\right]d\xi_3.\end{gathered} \tag{27}$$

## 5. The checking of the derived final structural formulas for MTDGFs and TVD

It is necessary to note that the checking procedure and validation of the obtained structural formulas for MTDGFs and TVD can be proved for generalized BVP (1), (12) - (17) at any combinations of the boundary conditions without knowing the analytical expressions for GFPE. However, for more understanding in this section the checking procedure and validation of MTDGFs and TVD is demonstrated for concrete BVP described by Eqs. (1), (2), (4) and (7) showed in the Figure 2.

The displacements $U_i(x,\xi)$ in the Eqs (20), (21) and (28) have to satisfy with respect to variables the following proved before [9]-[11] equation:

$$\nabla_x^2 U_i(x,\xi)=-\gamma\Theta^{(i)}(x,\xi), \tag{28}$$

which follows from the convolution (25). According to the boundary conditions (2), (4) and (7) and handbook [17] the influence functions for mechanical volume dilatation $\Theta^{(i)}(x,\xi)$, created by a unit concentrated force acting in the direction of axis $oi; i=1,2,3$ is determining as follows:

$$\Theta^{(i)}(x,\xi)=-\frac{1}{\lambda+2\mu}\frac{\partial}{\partial\xi_i}G_\Theta(x,\xi). \tag{29}$$

Thus, according to the given above mechanical boundary conditions (2), (4) and (7) and to the Problem 19.L.6 of the handbook [17] at boundary conditions:

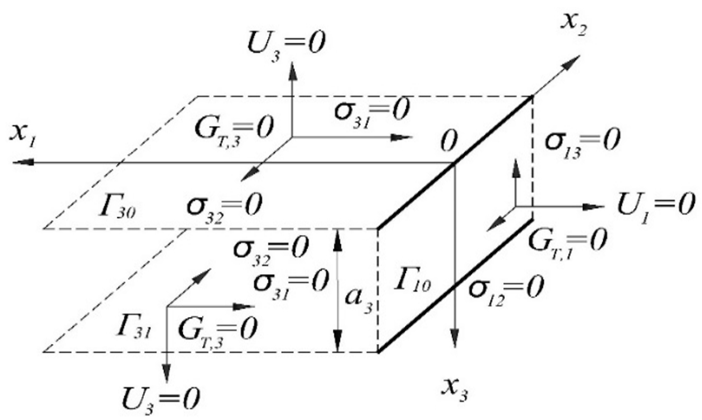


FIGURE 2. The scheme of the half-layer $V \equiv (0 \le x_1 < \infty, -\infty < x_2 < \infty, 0 \le x_3 \le a_3)$ with boundary half-planes $\Gamma_{30}(0 \le x_1 < \infty, -\infty < x_2 < \infty, x_3 = 0)$, $\Gamma_{31}(0 \le x_1 < \infty, -\infty < x_2 < \infty, x_3 = a_3)$ and boundary strip $\Gamma_{10}(x_1 = 0, -\infty < x_2 < \infty, 0 \le x_3 \le a_3)$ on which are given the homogeneous mechanical locally-mixed boundary conditions as normal displacements, tangential stresses and . Also, on the boundaries are given homogeneous normal derivatives from Green's function for temperature $G_T$.

$$\begin{gathered} U_3^{(k)} = \sigma_{31}^{(k)} = \sigma_{32}^{(k)} = 0; 0 \le x_1 < \infty, -\infty < x_2 < \infty, x_3 = 0,a_3; \\ U_1^{(k)} = \sigma_{12}^{(k)} = \sigma_{13}^{(k)} = 0; x_1 = 0, -\infty < x_2, x_3 < \infty; k = 1,2,3. \end{gathered} \tag{30}$$

We have the following GFPE given in the handbook [17] (see the Problems 19.P.2, 19.P.3, 19.P.6 and answers to them):

$$\begin{gathered} G_1(x,\xi) = G^{(3)}(x,\xi) = \\ \frac{1}{\pi a_3}\ln\frac{r_1}{r} + \frac{1}{\pi a_3}\sum_{n=1}^{\infty}[K_0(\mu_1 r) - K_0(\mu_1 r_1)]\cos\mu_1 x_3\cos\mu_1\xi_3; \mu_1 = \frac{n\pi}{a_3}; \end{gathered} \tag{31}$$

$$\begin{gathered} G_T(x,\xi) = G_2(x,\xi) = G_\Theta(x,\xi) = G^{(2)}(x,\xi) = \\ b - \frac{1}{\pi a_3}\ln r r_1 + \frac{1}{\pi a_3}\sum_{n=1}^{\infty}[K_0(\mu_1 r) + K_0(\mu_1 r_1)]\cos\mu_1 x_3\cos\mu_1\xi_3; \mu_1 = \frac{n\pi}{a_3}; \end{gathered} \tag{32}$$

$$G_3(x,\xi) = G^{(6)}(x,\xi) = \frac{1}{\pi a_3}\sum_{n=1}^{\infty}[K_0(\mu_1 r) + K_0(\mu_1 r_1)]\sin\mu_1 x_3\sin\mu_1\xi_3; \mu_1 = \frac{n\pi}{a_3}. \tag{33}$$

Here, and hereafter in this section $K_0(\mu_1 r)$ and $K_0(\mu_1 r_1)$ are modified Bessel functions (or cylindrical functions) of the zero-order of the second type, respectively:

$$r = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}; r_1 = \sqrt{(x_1+\xi_1)^2 + (x_2-\xi_2)^2}.$$

So, according to the Eqs. (29) and (32) the analytical expression for elastic volume dilatation has the form:

$$\Theta^{(i)}(x,\xi) = -\frac{1}{\lambda+2\mu}\frac{\partial}{\partial\xi_i}G^{(2)}(x,\xi) = -\frac{1}{\lambda+2\mu}\frac{\partial}{\partial\xi_i}$$
$$\times\left[b-\frac{1}{\pi a_3}\ln rr_1+\frac{1}{\pi a_3}\sum_{n=1}^{\infty}[K_0(\mu_1 r)+K_0(\mu_1 r_1)]\cos\mu_1 x_3\cos\mu_1\xi_3\right];\mu_1=\frac{n\pi}{a_3}. \quad (34)$$

5.1. **The checking the equations** (28) **and** (29)**.**

$$\nabla_x^2 U_1(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)}\nabla_x^2[\xi_1 G_T(x,\xi)-x_1G_1(x,\xi)] =$$
$$\frac{\gamma}{2(\lambda+2\mu)}\left[\xi_1\nabla_x^2 G_T(x,\xi)-\nabla_x^2(x_1G_1(x,\xi))\right] = -\frac{\gamma}{2(\lambda+2\mu)}\left(2\frac{\partial}{\partial x_1}G_1(x,\xi)\right) = \quad (35)$$
$$-\frac{\gamma}{(\lambda+2\mu)}\left(\frac{\partial}{\partial x_1}G^{(3)}(x,\xi)\right) = \frac{\gamma}{(\lambda+2\mu)}\frac{\partial}{\partial\xi_1}G^{(2)}(x,\xi) = -\gamma\Theta^{(1)}(x,\xi);$$

$$\nabla_x^2 U_2(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)}\nabla_x^2[\xi_2 G_T(x,\xi)-x_2G_2(x,\xi)] =$$
$$\frac{\gamma}{2(\lambda+2\mu)}\left[\xi_2\nabla_x^2 G_T(x,\xi)-\nabla_x^2(x_2G_2(x,\xi))\right] = -\frac{\gamma}{2(\lambda+2\mu)}\left(2\frac{\partial}{\partial x_2}G_2(x,\xi)\right) = \quad (36)$$
$$-\frac{\gamma}{(\lambda+2\mu)}\left(\frac{\partial}{\partial x_2}G^{(2)}(x,\xi)\right) = \frac{\gamma}{(\lambda+2\mu)}\frac{\partial}{\partial\xi_2}G^{(2)}(x,\xi) = -\gamma\Theta^{(2)}(x,\xi);$$

$$\nabla_x^2 U_3(x,\xi) = -\frac{\gamma}{2(\lambda+2\mu)}\nabla_x^2\left\{\int[\xi_1 G_{T,\xi_1}(x,\xi)-x_1G_{1,\xi_1}(x,\xi)]\,d\xi_3+\right.$$
$$\int[\xi_2 G_{T,\xi_2}(x,\xi)-x_2G_{2,\xi_2}(x,\xi)]\,d\xi_3 = \frac{\gamma}{2(\lambda+2\mu)}\int 2\frac{\partial}{\partial x_1}G^{(2)}_{,\xi_1}(x,\xi)d\xi_3 \quad (37)$$
$$+\frac{\gamma}{2(\lambda+2\mu)}\int 2\frac{\partial}{\partial x_2}G^{(2)}_{,\xi_2}(x,\xi)d\xi_3 = -\frac{\gamma}{(\lambda+2\mu)}\int\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)G^{(2)}(x,\xi)d\xi_3 =$$
$$\frac{\gamma}{(\lambda+2\mu)}\int\frac{\partial^2}{\partial\xi_3^2}G^{(2)}(x,\xi)d\xi_3 = \frac{\gamma}{(\lambda+2\mu)}\frac{\partial}{\partial\xi_3}G^{(2)}(x,\xi)d\xi_3 = -\gamma\Theta^{(3)}(x,\xi),$$

where the expressions for GFPE (31) - (33) were used.

In addition, according to works [9]-[11] and convolution (25), the displacements $U_i(x,\xi)$ have satisfy with respect to variable $\xi\equiv(\xi_1,\xi_2,\xi_3)$ the equations:

$$\mu\nabla_\xi^2 U_i(x,\xi)+(\lambda+\mu)\Theta_{,\xi_i}(x,\xi)-\gamma G_{T,\xi_i}(x,\xi)=0; i=1,2,3. \quad (38)$$

5.2. **The checking the equation** (38)**.**

$$\mu\nabla_\xi^2 U_1(x,\xi) = \frac{\gamma\mu}{2(\lambda+2\mu)}\nabla_\xi^2[\xi_1 G_T(x,\xi)-x_1G_1(x,\xi)] = \frac{\gamma\mu}{(\lambda+2\mu)}\frac{\partial}{\partial\xi_1}G_T(x,\xi);$$
$$(\lambda+\mu)\Theta_{,\xi_1}(x,\xi)-\gamma G_{T,\xi_1}(x,\xi) = \frac{\gamma(\lambda+\mu)}{(\lambda+2\mu)}\frac{\partial}{\partial\xi_1}G_T(x,\xi)-\frac{\gamma\partial}{\partial\xi_1}G_T(x,\xi) =$$
$$-\frac{\gamma\mu}{(\lambda+2\mu)}\frac{\partial}{\partial\xi_1}G_T(x,\xi);$$
$$\mu\nabla_\xi^2 U_1(x,\xi)+(\lambda+\mu)\Theta_{,\xi_1}(x,\xi)-\gamma G_{T,\xi_1}(x,\xi) =$$
$$\frac{\gamma\mu}{(\lambda+2\mu)}\frac{\partial}{\partial\xi_1}G_T(x,\xi)-\frac{\gamma\mu}{(\lambda+2\mu)}\frac{\partial}{\partial\xi_1}G_T(x,\xi)=0; \quad (39)$$

$$\mu \bigtriangledown_\xi^2 U_2(x,\xi) = \frac{\gamma\mu}{2(\lambda+2\mu)} \bigtriangledown_\xi^2 [\xi_2 G_T(x,\xi) - x_2 G_2(x,\xi)] = \frac{\gamma\mu}{(\lambda+2\mu)} \frac{\partial}{\partial\xi_2} G_T(x,\xi);$$

$$(\lambda+\mu)\Theta_{,\xi_2}(x,\xi) - \gamma G_{T,\xi_2}(x,\xi) = \frac{\gamma(\lambda+\mu)}{(\lambda+2\mu)} \frac{\partial}{\partial\xi_2} G_T(x,\xi) - \frac{\gamma\partial}{\partial\xi_2} G_T(x,\xi) =$$

$$-\frac{\gamma\mu}{(\lambda+2\mu)} \frac{\partial}{\partial\xi_2} G_T(x,\xi);$$

$$\begin{gathered} \mu \bigtriangledown_\xi^2 U_2(x,\xi) + (\lambda+\mu)\Theta_{,\xi_1}(x,\xi) - \gamma G_{T,\xi_1}(x,\xi) = \\ \frac{\gamma\mu}{(\lambda+2\mu)} \frac{\partial}{\partial\xi_2} G_T(x,\xi) - \frac{\gamma\mu}{(\lambda+2\mu)} \frac{\partial}{\partial\xi_2} G_T(x,\xi) = 0; \end{gathered} \tag{40}$$

$$\mu \bigtriangledown_\xi^2 U_3(x,\xi) = -\frac{\gamma\mu}{2(\lambda+2\mu)} \bigtriangledown_\xi^2 \left\{ \int [\xi_1 G_{T,\xi_1}(x,\xi) - x_1 G_{1,\xi_1}(x,\xi)]\, d\xi_3 \right.$$

$$\left. + \int [\xi_2 G_{T,\xi_2}(x,\xi) - x_2 G_{2,\xi_2}(x,\xi)]\, d\xi_3 \right\} = -\frac{\gamma\mu}{(\lambda+2\mu)} \int \left[ \frac{\partial^2}{\partial\xi_1^2} G_T(x,\xi) + \frac{\partial^2}{\partial\xi_2^2} G_T(x,\xi) \right] d\xi_3$$

$$= \frac{\gamma\mu}{(\lambda+2\mu)} \int \frac{\partial^2}{\partial\xi_3^2} G_T(x,\xi) d\xi_3 = \frac{\gamma\mu}{(\lambda+2\mu)} \frac{\partial}{\partial\xi_3} G_T(x,\xi);$$

$$(\lambda+\mu)\Theta_{,\xi_3}(x,\xi) - \gamma G_{T,\xi_3}(x,\xi) = \frac{\gamma(\lambda+\mu)}{(\lambda+2\mu)} \frac{\partial}{\partial\xi_3} G_T(x,\xi) - \frac{\gamma\partial}{\partial\xi_3} G_T(x,\xi) =$$

$$-\frac{\gamma\mu}{(\lambda+2\mu)} \frac{\partial}{\partial\xi_3} G_T(x,\xi);$$

$$\begin{gathered} \mu \bigtriangledown_\xi^2 U_3(x,\xi) + (\lambda+\mu)\Theta_{,\xi_3}(x,\xi) - \gamma G_{T,\xi_3}(x,\xi) = \\ \frac{\gamma\mu}{(\lambda+2\mu)} \frac{\partial}{\partial\xi_3} G_T(x,\xi) - \frac{\gamma\mu}{(\lambda+2\mu)} \frac{\partial}{\partial\xi_3} G_T(x,\xi) = 0. \end{gathered} \tag{41}$$

Finally, the TVD calculated on the basis of structural formulas (20), (21) and (27) have to be equal to the respective TVD given by equation (19):

$$U_{i,i}(x,\xi) = \Theta(x,\xi) = \frac{\gamma}{\lambda+2\mu} G_T(x,\xi). \tag{42}$$

5.3. **The checking the equation** (42)**.**

$$U_{1,1}(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)} \left[ G_T(x,\xi) + \xi_1 G_{T,\xi_1}(x,\xi) - x_1 G_{1,\xi_1}(x,\xi) \right]; \tag{43}$$

$$U_{2,2}(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)} \left[ G_T(x,\xi) + \xi_2 G_{T,\xi_2}(x,\xi) - x_2 G_{2,\xi_2}(x,\xi) \right]; \tag{44}$$

$$\begin{gathered} U_{3,3}(x,\xi) = -\frac{\gamma}{2(\lambda+2\mu)} [\xi_1 G_{T,\xi_1}(x,\xi) \\ -x_1 G_{1,\xi_1}(x,\xi) + \xi_2 G_{T,\xi_2}(x,\xi) - x_2 G_{2,\xi_2}(x,\xi)]. \end{gathered} \tag{45}$$

Thus, from Eqs (43) - (45) follows that Eqs. (42) or (19) is satisfied.

## 6. Analytical expressions for MTDGFs and TVD

Finally, to obtain analytical expressions for MTDGFs $U_i(x,\xi)$ we have to substitute the respective analytical expressions for GFPE $G_1$, $G_2$, $G_3$ and $G_T$ from Eqs. (31) - (33) (which are in accordance to respective boundary (12), (14) and (17)) into Eqs. (20), (21), (27) and (19). Thus in such a way we obtain the respective analytical expressions:

$$U_1(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)}\left\{\xi_1 b + \frac{1}{\pi a_3}\left[(x_1-\xi_1)\left(\ln r - \sum_{n=1}^{\infty} K_0(\mu_1 r)\cos\mu_1 x_3 \cos\mu_1\xi_3\right)\right.\right.$$
$$\left.\left. -\ (x_1+\xi_1)\left(\ln r_1 - \sum_{n=1}^{\infty} K_0(\mu_1 r_1)\cos\mu_1 x_3\cos\mu_1\xi_3\right)\right]\right\}; \tag{46}$$

$$U_2(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)}(\xi_2 - x_2)$$
$$\times\left\{b - \frac{1}{\pi a_3}\ln rr_1 + \frac{1}{\pi a_3}\sum_{n=1}^{\infty}[K_0(\mu_1 r) + K_0(\mu_1 r_1)]\cos\mu_1 x_3\cos\mu_1\xi_3\right\}; \tag{47}$$

$$U_3(x,\xi) = \frac{\gamma}{2\pi a_3(\lambda+2\mu)}$$
$$\times\left\{\xi_3(r+r_1) + \sum_{n=1}^{\infty}[rK_1(\mu_1 r) + r_1 K_1(\mu_1 r_1)]\cos\mu_1 x_3\cos\mu_1\xi_3\right\}; \tag{48}$$

- for MTDGF $U_i(x,\xi)$

were $K_1(\mu_1 r) = -\partial K_0(\mu_1 r)/\partial(\mu_1 r)$ and $K_1(\mu_1 r_1) = -\partial K_0(\mu_1 r_1)/\partial(\mu_1 r_1)$; and

$$\Theta(x,\xi) = \frac{\gamma}{\lambda+2\mu}(\xi_2 - x_2)$$
$$\times\left\{b - \frac{1}{\pi a_3}\ln rr_1 + \frac{1}{\pi a_3}\sum_{n=1}^{\infty}[K_0(\mu_1 r) + K_0(\mu_1 r_1)]\cos\mu_1 x_3\cos\mu_1\xi_3\right\}; \tag{49}$$

- for TVD.

Note that in Eqs. (46), (47) and (49) the arbitrary constant of integration $b = 0$, because the displacements $U_1(x,\xi), U_2(x,\xi)$ and TVD $\Theta$ have to vanish at infinity.

As could be seen the MTDGFs (46) - (48) satisfy boundary conditions (2), (5) and (7) for temperature Green's function $G_T(x,\xi)$ with respect to point of application $x \equiv (x_1, x_2, x_3)$ of the heat source and to mechanical boundary conditions (2), (4) and (7) with respect to point $\xi \equiv (\xi_1,\xi_2,\xi_3)$, in which MTDGF are aperies.

Next, we can obtain the analytical expressions for TSGFs by using the Duhamel-Newman law:

$$\sigma^*_{ij} = \mu(U_{i,j} + U_{j,i}) + \delta_{ij}(\lambda\Theta - \gamma G_T) \tag{50}$$

and expressions (46) - (48).

In addition, we could note that on the base of the structural formulas (20), (21), (27) and respective GFPE it is possible to derive concrete analytical expressions for MTDGFs for eight BVPs of thermoelasticity. So, to do this we have to take in structural formulas (20), (21) and (27) the respective GFPE linked with the following eight combinations of the boundary conditions:

$$G_1 = G^{(2)}; G_3 = G^{(1)}; G_2 = G_T = G^{(3)}; \tag{51}$$

- for the combination (12), (14), (17) of boundary conditions;

$$G_1 = G^{(3)}; G_2 = G_T = G^{(2)}; G_3 = G^{(6)}; \tag{52}$$

- for the combination (12), (14), (16) of boundary conditions;

$$G_1 = G^{(4)}; G_2 = G_T = G^{(7)}; G_3 = G^{(2)}; \tag{53}$$

- for the combination (12), (15), (16) of boundary conditions;

$$G_2 = G_T = G^{(4)}; G_1 = G^{(7)}; G_3 = G^{(5)}; \tag{54}$$

- for the combination (12), (15), (17) of boundary conditions;

$$G_2 = G_T = G^{(1)}; G_1 = G^{(6)}; G_3 = G^{(3)}; \tag{55}$$

- for the combination (13), (15), (17) of boundary conditions;

$$G_2 = G_T = G^{(6)}; G_1 = G^{(1)}; G_3 = G^{(2)}; \tag{56}$$

- for the combination (13), (15), (16) of boundary conditions;

$$G_2 = G_T = G^{(5)}; G_1 = G^{(8)}; G_3 = G^{(4)}; \tag{57}$$

- for the combination (13), (14), (17) of boundary conditions;

$$G_2 = G_T = G^{(8)}; G_1 = G^{(5)}; G_3 = G^{(7)}; \tag{58}$$

- for the combination (13), (14), (16) of boundary conditions.

According to the Eqs. (48) - (58) and the handbook [17] (see the Problems 19.P.1-19.P.8 and answers to them) the expressions of the GFPE are written in the following form:

$$G^{(1)}(x,\xi) = \frac{1}{\pi a_3}\sum_{n=1}^{\infty}[K_0(\mu_1 r) - K_0(\mu_1 r_1)]\sin\mu_1 x_3 \sin\mu_1\xi_3; \mu_1 = \frac{n\pi}{a_3}; \tag{59}$$

$$G^{(2)}(x,\xi) =$$
$$b - \frac{1}{\pi a_3}\ln r r_1 + \frac{1}{\pi a_3}\sum_{n=1}^{\infty}[K_0(\mu_1 r) + K_0(\mu_1 r_1)]\cos\mu_1 x_3 \cos\mu_1\xi_3; \mu_1 = \frac{n\pi}{a_3}; \tag{60}$$

$$G^{(3)}(x,\xi) =$$
$$\frac{1}{\pi a_3}\ln\frac{r_1}{r} + \frac{1}{\pi a_3}\sum_{n=1}^{\infty}[K_0(\mu_1 r) - K_0(\mu_1 r_1)]\cos\mu_1 x_3 \cos\mu_1\xi_3; \mu_1 = \frac{n\pi}{a_3}; \tag{61}$$

$$G^{(4)}(x,\xi) = \frac{1}{\pi a_3}\sum_{n=1}^{\infty}[K_0(\mu_1 r) - K_0(\mu_1 r_1)]\sin\mu_1 x_3 \sin\mu_1\xi_3; \mu_1 = (2n-1)\frac{\pi}{2a_3}; \tag{62}$$

$$G^{(5)}(x,\xi) = \frac{1}{\pi a_3}\sum_{n=1}^{\infty}[K_0(\mu_1 r) - K_0(\mu_1 r_1)]\cos\mu_1 x_3 \cos\mu_1\xi_3; \mu_1 = (2n-1)\frac{\pi}{2a_3}; \tag{63}$$

$$G^{(6)}(x,\xi) = \frac{1}{\pi a_3}\sum_{n=1}^{\infty}[K_0(\mu_1 r) + K_0(\mu_1 r_1)]\sin\mu_1 x_3 \sin\mu_1\xi_3; \mu_1 = \frac{n\pi}{a_3}; \tag{64}$$

$$G^{(7)}(x,\xi) = \frac{1}{\pi a_3}\sum_{n=1}^{\infty}[K_0(\mu_1 r) + K_0(\mu_1 r_1)]\sin\mu_1 x_3 \sin\mu_1\xi_3; \mu_1 = (2n-1)\frac{\pi}{2a_3}; \tag{65}$$

$$G^{(8)}(x,\xi)=\frac{1}{\pi a_3}\sum_{n=1}^{\infty}\left[K_0(\mu_1 r)+K_0(\mu_1 r_1)\right]\cos\mu_1 x_3\cos\mu_1\xi_3;\mu_1=(2n-1)\frac{\pi}{2a_3};\quad (66)$$

## 7. Integration formulas for thermoelastic displacements and stresses within half-layer

To obtain IFs for thermoelastic displacements and stresses within half-layer it is necessary to use the general integration formulas for displacements $u_i(x,\xi)$ and stresses $\sigma_{ij}(x); i,j=1,2,3$ [9]-[14]:

$$u_i(\xi)=a^{-1}\int\limits_V F(x)U_i(x,\xi)dV(x)-\int\limits_{\Gamma_D}T(y)\frac{\partial U_i(y,\xi)}{\partial n_y}d\Gamma_D(y)$$

$$+\int\limits_{\Gamma_N}\frac{\partial T(y)}{\partial n_y}U_i(y,\xi)d\Gamma_N(y)+a^{-1}\int\limits_{\Gamma_M}\left[\alpha T(y)+a\frac{\partial T(y)}{\partial n_y}\right]U_i(y,\xi)d\Gamma_M(y); i=1,2,3;\quad (67)$$

$$\sigma_{ij}(\xi)=a^{-1}\int\limits_V F(x)\sigma^*_{ij}(x,\xi)dV(x)$$

$$-\int\limits_{\Gamma_D}T(y)\frac{\partial\sigma^*_{ij}(y,\xi)}{\partial n_y}d\Gamma_D(y)+\int\limits_{\Gamma_N}\frac{\partial T(y)}{\partial n_y}\sigma^*_{ij}(y,\xi)d\Gamma_N(y)\quad (68)$$

$$+a^{-1}\int\limits_{\Gamma_M}\left[\alpha T(y)+a\frac{\partial T(y)}{\partial n_y}\right]\sigma^*_{ij}(y,\xi)d\Gamma_M(y); i,j=1,2,3,$$

where $a$ is the coefficient of thermal conductivity; $\alpha$ is a coefficient of heat conductivity.

So, the respective IFs for displacements $u_i(x)$ and for thermal stresses $\sigma_{ij}(x)$ within half-layer, created by the inner heat source $F(x)$, temperature $T(y)$ and thermal flux $\partial T(y)/\partial n_y$ given on the boundaries half-planes $\Gamma_{30}$ and $\Gamma_{31}$ on the boundary strip $\Gamma_{10}$ look as follow:

$$u_i(\xi)=a^{-1}\int\limits_V F(x)U_i(x,\xi)dV(x)-\int\limits_{\Gamma_{10}}\left[\frac{\partial U_i(y,\xi)}{\partial n_{\Gamma_{10}}}-U_i(y,\xi)\frac{\partial}{\partial n_{\Gamma_{10}}}\right]T_{10}(y)d\Gamma_{10}(y)$$

$$-\sum_{j=0}^{1}\int\limits_{\Gamma_{3j}}\left[\frac{\partial U_i(y,\xi)}{\partial n_{\Gamma_{3j}}}-U_i(y,\xi)\frac{\partial}{\partial n_{\Gamma_{3j}}}\right]T_{3j}(y)d\Gamma_{3j}(y);\quad (69)$$

$$\sigma_{ij}(\xi)=a^{-1}\int\limits_V F(x)\sigma^*_{ij}(x,\xi)dV(x)-\int\limits_{\Gamma_{10}}\left[\frac{\partial\sigma^*_{ij}(y,\xi)}{\partial n_{\Gamma_{10}}}-\sigma^*_{ij}(y,\xi)\frac{\partial}{\partial n_{\Gamma_{10}}}\right]T_{10}(y)d\Gamma_{10}(y)$$

$$-\sum_{j=0}^{1}\int\limits_{\Gamma_{3j}}\left[\frac{\partial\sigma^*_{ij}(y,\xi)}{\partial n_{\Gamma_{3j}}}-\sigma^*_{ij}(y,\xi)\frac{\partial}{\partial n_{\Gamma_{3j}}}\right]T_{3j}(y)d\Gamma_{3j}(y).\quad (70)$$

As example, on the base of the IFs (69) and (70) and analytical expressions for MTDGF given in the Eqs. (46)-(48) for BVP in Eqs (1), (2), (4) and (7) we can to write the following integration formulas:

$$u_i(\xi) = a^{-1} \int_V F(x) U_i(x,\xi) dV(x) + \int_{\Gamma_{10}} U_i(y,\xi) \frac{\partial}{\partial n_{\Gamma_{10}}} T_{10}(y) d\Gamma_{10}(y)$$
$$+ \sum_{j=0}^{1} \int_{\Gamma_{3j}} U_i(y,\xi) \frac{\partial}{\partial n_{\Gamma_{3j}}} T_{3j}(y) d\Gamma_{3j}(y); \tag{71}$$

- for thermoelastic displacements, and

$$\sigma_{ij}(\xi) = a^{-1} \int_V F(x) \sigma^*_{ij}(x,\xi) dV(x) + \int_{\Gamma_{10}} \sigma^*_{ij}(y,\xi) \frac{\partial}{\partial n_{\Gamma_{10}}} T_{10}(y) d\Gamma_{10}(y)$$
$$+ \sum_{j=0}^{1} \int_{\Gamma_{3j}} \sigma^*_{ij}(y,\xi) \frac{\partial}{\partial n_{\Gamma_{3j}}} T_{3j}(y) d\Gamma_{3j}(y); \tag{72}$$

- for thermoelastic stresses.

So, in analogical way the readers can obtain the integration formulas for seven additional combinations of the boundary conditions (52) - (57) which could be obtained by using GFPE (59) - (66) and structural formulas (20), (21) and (27).

## CONCLUSION

The structural formulas and analytical expressions for MTDGFs and TVD for eight new BVPs for a thermoelastic half-layer are obtained for the first time by using HIRM. Obtained structural formulas are validated by the checking presented in the subsections 5.1 - 5.3. The integration formulas for displacements and stresses within half-layer, created by internal and by surface thermal actions are derived also. The readers can obtain the integration formulas for seven additional BVPs within half-layer which contain seven combinations of the boundary conditions (52) - (57) by using GFPE (59) - (66) and structural formulas half-layer. In addition, if in these 3D MTDGFs for eight BVPs for the thermoelastic half-layer we remove the boundary strip $\Gamma_{10}$ at infinity (equivalently we have to omit the functions containing $r_1$) then we obtain the respective 3D MTDGFs to four BVPs for the thermoelastic layer [18].

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