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A PROBLEM WITH SECOND KIND'S NONLOCAL CONDITION FOR PSEUDOHYPERBOLIC EQUATION

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ABSTRACT. In this article, the Faeod-Galerkin method is proposed for solving a pseudohyperbolic type equation with an integral condition. We construct a discrete numerical solution of the approximate problem.

1. INTRODUCTION

Boundary value problems with integral conditions constitute a very interesting and important class of problems. These nonlocal conditions arise mostly when the data on the boundary cannot be measured directly. Recall that the presence of an integral term in boundary conditions can complicate the application of classical methods of functional analysis in the theoretical study of nonlocal problems, therefore, several methods have been proposed for overcoming the difficulties arising from nonlocal conditions; see Beilin [1] Cannon [5] and Bouziani [2]-[4].

Numerical solutions are introduced to obtain approximations for the solution of partial differential equations when the analytical solutions are difficult or impossible to obtain due to complicated geometry or boundary conditions. In the area of numerical analysis, the Faeod-Galerkin method is a class of methods for converting a continuous operator problem to a discrete problem. In principle, it is the equivalent of applying the method of a variation of parameters to a function space, by converting the equation to a weak formulation, hence in this approach we choose a system of linearly independent functions such that they satisfy the given homogeneous boundary condition, and they are dense in a function space containing the exact solution of the above boundary value problem. Now there are a number of results on the nonlocal problems with integral conditions solvability for parabolic and hyperbolic equations with Faeod-Galerkin method [8, 9, 10]

The present paper is devoted to study the uniqueness and existence of a generalized solution for pseudohyperbolic equation of the form

$$Lu = \frac{\partial^2}{\partial t^2}(u - (b(x,t)u_x)_x) - (a(x,t)u_x)_x + c(x,t)u = f(x,t), \tag{1}$$

in a cylinder $Q_T = (0,l) \times (0,T)$ with $l, T < \infty$.
Subject to the initial conditions

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \tag{2}$$

and the integral condition of second kind

$$\frac{\partial^2}{\partial t^2}\left(b(0,t)u_x(0,t)\right) + a(0,t)u_x(0,t) + \int_0^l K_1(x,t)u(x,t)dx = 0, \tag{3}$$

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$$\frac{\partial^2}{\partial t^2}((b(l,t)u_x(l,t)) + a(l,t)u_x(l,t) + \int_0^l K_2(x,t)u(x,t)dx = 0. \tag{4}$$

let us define the space

$$W(Q_T) = \{u \in H^1(Q_T), \quad u_{xt} \in L^2(Q_T)\},$$

with the norme

$$\|u\|^2_{W(Q_T)} = \|u\|^2_{H^1(Q_T)} + \|u_{xt}\|^2_{L^2(Q_T)}.$$

and denote the space

$$\hat{W}(Q_T) = \{v \in W(Q_T), \quad v(x,T) = 0\}.$$

2. Solvability of problem

Now we introduce the concept of the generalized solution. Suppose $u(x,t) \in W(Q_T)$ satisfy (1)-(4). Multiply the equation (1) by a function $v(x,t) \in \hat{W}(Q_T)$ and integrate it over the cylinder Q_T. After integrating by parts, we obtain

$$\begin{aligned} &\int_0^T \int_0^l (-u_t v_t - (bu_x)_t v_{xt} + a u_x v_x + cuv) dxdt \\ &- \int_0^T v(l,t) \int_0^l K_1(x,t)u(x,t)dx + \int_0^T v(0,t) \int_0^l K_2(x,t)u(x,t)dx \\ = \; &\int_0^T \int_0^l fv dxdt + \int_0^l (b_t(x,0) v_x(x,0) \varphi'(x) + b(x,0) v_x(x,0) \psi'(x)) dx \\ &+ \int_0^l \psi(x) v(x,0) dx. \end{aligned} \tag{5}$$

Definition 1. *A function $u(x,t) \in W(Q_T)$ is called a generalized solution of (1)-(4), if it satisfies (5) for every $v(x,t) \in \hat{W}(Q_T)$ and $u(x,0) = \varphi(x)$.*

Theorem 1. *if*

$$f \in L^2(Q_T), \; c, a, a_t \in C(\bar{Q}_T), \; b, b_t, b_{tt}, b_{ttt} \in C(\bar{Q}_T), \quad a(x,t) \geq a_0 > 0, \; b(x,t) \geq b_0$$

$$\varphi, \psi \in H^1(0,l), \; K_i \in C(\bar{Q}_T),$$

then there exists a unique generalized solution to problem (1)-(4).

We need the folowing lemma for the theorem's proof

Lemma 1. (*see* [8]) *Let V be a vectoriel normed space separable with infinite dimension. There existe denombrable independent familly $\{v_i\}_{i\in N}$ $v_i \in V$, such that the vectoriel subespace $\cup_{i=1}^{\infty} v_i$ is dense in V. We can choose $v_i \in V$ such that the sequence of vectoriels subspaces $V_m = spam\{v_i, 0 \leq i \leq m\}$ is incresing.*

Proof. **Uniqueness of the solutions.**

Suppose that there exist two different generalized solutions u_1 and u_2 for the problem (1)-(4). Then the difference $u = u_1 - u_2$ is a generalized solution of the problem (1)-(4) with homogeneous equation and homogeneous conditions, that is

$$f(x,t) = \varphi(x) = \psi(x) = 0, \qquad \forall x \in [0,l].$$

The identity (5) becomes

$$\int_0^T \int_0^l (-u_t v_t - (bu_x)_t\, v_{xt} + au_x v_x + cuv) dxdt \tag{6}$$

$$-\int_0^T v\,(l,t) \int_0^l K_2(x,t)u(x,t)dxdt + \int_0^T v\,(0,t) \int_0^l K_1(x,t)u(x,t)dxdt = 0. \tag{7}$$

Now, we shall introduce a new function :

$$v\,(x,t) = \begin{cases} \int_\tau^t u\,(x,\eta)\,d\eta, & 0 \le t \le \tau, \\ 0 \qquad\qquad , & \tau \le t \le T. \end{cases}$$

Substituting the function v into (7), integrating by parts, then using the fact that $v_t(x,t) = -u(x,t)$, it follows

$$\int_0^l \left((v_t\,(x,\tau))^2 + b\,(x,\tau)\,(v_{xt}\,(x,\tau))^2 + a\,(x,0)\,(v_x\,(x,0))^2 \right) dx$$

$$+ \int_0^\tau \int_0^l \left(2cvv_t - a_t\,(v_x)^2 - b_t\,(v_{xt})^2 \right) dxdt + 2\int_0^\tau v\,(l,t) \int_0^l K_2(x,t)v_t(x,t)dxdt$$

$$-2\int_0^\tau v\,(0,t) \int_0^l K_1(x,t)v_t(x,t)dxdt = 0. \tag{8}$$

Let us now estimate the left-hand side of equality (8) applying the Cauchy and Cauchy–Bunyakovsky inequalities together with assumptions of the theorem, it comes:

$$2 \mid \int_0^\tau \int_0^l cvv_t dxdt \mid \le c_0 \left(\int_0^\tau \int_0^l \left(v^2 + v_t^2\right) dxdt \right),$$

$$\mid \int_0^\tau \int_0^l a_t\,(v_x)^2\,dxdt \mid \le a_1 \int_0^\tau \int_0^l (v_x)^2\,dxdt,$$

$$\mid \int_0^\tau \int_0^l b_t\,(v_{xt})^2\,dxdt \mid \le b_1 \int_0^\tau \int_0^l (v_{xt})^2\,dxdt,$$

$$2 \mid \int_0^\tau v\,(l,t) \int_0^l K_2(x,t)v_t(x,t)dx \mid \le \int_0^\tau v\,(l,t)^2\,dt + k_1 \int_0^\tau \int_0^l v_t^2 dxdt,$$

$$2 \mid \int_0^\tau v\,(0,t) \int_0^l K_2(x,t)v_t(x,t)dx \mid \le \int_0^\tau v\,(0,t)^2\,dt + k_2 \int_0^\tau \int_0^l v_t^2 dxdt,$$

We assume next the notation

$$k_i = \max_{[0,T]} \int_0^l K_i^2 dx, \qquad \forall i = 1,2$$

where

$$k = \max_i k_i, \qquad \forall i = 1,2.$$

Later on, we shall need the following inequalities

$$\begin{aligned} (v\,(0,t))^2 &\le 2l \int_0^l v_x^2\,(x,t)\,dx + \frac{2}{l}\int_0^l v^2\,(x,t)\,dx, \\ (v\,(l,t))^2 &\le 2l \int_0^l v_x^2\,(x,t)\,dx + \frac{2}{l}\int_0^l v^2\,(x,t)\,dx. \end{aligned} \tag{9}$$

Both of them arise from equalities [7]

$$v\,(0,t) = \int_x^0 v_\xi\,(\xi,t)\,d\xi + v\,(x,t) \quad \text{and} \quad v\,(l,t) = \int_x^l v_\xi\,(\xi,t)\,d\xi + v\,(x,t)\,.$$

Having applied (9), we get

$$\frac{1}{2}\int_0^\tau \left[v\left(l,t\right)^2+v\left(0,t\right)^2\right]dt\leq 2l\int_0^\tau\int_0^l v_x^2\left(x,t\right)dxdt+\frac{2}{l}\int_0^\tau\int_0^l v^2\left(x,t\right)dxdt,$$

Note that from representation of v it follows

$$v^2\left(x,t\right)=\left(\int_\tau^t u(x,\eta)d\eta\right)^2\leq\tau\int_0^\tau u^2(x,t)dt=\tau\int_0^\tau v_t^2(x,t)dt,\quad \tau\in\left[0,T\right].$$

we arrive at the inequality

$$\begin{aligned}&\int_0^l\left[\left(v_t\left(x,\tau\right)\right)^2+b\left(x,\tau\right)\left(v_{xt}\left(x,\tau\right)\right)^2+a\left(x,0\right)\left(v_x\left(x,0\right)\right)^2\right]dx\\ \leq\;& C_1\int_0^\tau\int_0^l v_t^2dxdt+\left(a_1+4l\right)\int_0^\tau\int_0^l\left(v_x\right)^2dxdt+b_1\int_0^\tau\int_0^l\left(v_{xt}\right)^2dxdt.\end{aligned}\tag{10}$$

Put here $\omega\left(x,t\right)=\int_0^t u_x(x,\eta)d\eta$, and arrive at the relation

$$v_x\left(x,t\right)=\omega\left(x,t\right)-\omega\left(x,\tau\right),\qquad v_x\left(x,0\right)=-\omega\left(x,\tau\right).$$

Then (10) turns into

$$\begin{aligned}&\int_0^l\left[\left(v_t\left(x,\tau\right)\right)^2+a\left(x,0\right)\omega^2\left(x,\tau\right)+b\left(x,\tau\right)\left(v_{xt}\left(x,\tau\right)\right)^2\right]dx\\ \leq\;& C_2\int_0^\tau\int_0^l\left(v_t^2+\omega^2\left(x,t\right)+\omega_t^2\left(x,t\right)\right)dxdt+2\tau\left(a_1+4l\right)\int_0^l\omega^2\left(x,\tau\right)dx,\end{aligned}\tag{11}$$

As τ is arbirary we choose it in such a way that an inequality $a_0-2\tau\left(a_1+4l\right)\geq 0$ holds. Let $\tau\in\left[0,\frac{a_0}{4(a_1+4l)}\right]$. Then $a_0-2\tau\left(a_1+4l\right)\geq\frac{a_0}{2}$. And we can carry out $2\tau\left(a_1+4l\right)\int_0^l\omega^2\left(x,\tau\right)dx$ from the right side to the left side of inequality. We now have

$$\int_0^l\left[\left(v_t\left(x,\tau\right)\right)^2+\omega^2\left(x,\tau\right)+\omega_t^2\left(x,\tau\right)\right]dx\leq C_3\int_0^\tau\int_0^l\left(v_t^2+\omega^2\left(x,t\right)+\omega_t^2\left(x,t\right)\right)dxdt$$

and by virtue of Gronwall's inequality $v_t\left(x,\tau\right)=0$, hence, $u\left(x,\tau\right)=0\ \forall\tau\in\left[0,\frac{a_0}{4(a_1+4l)}\right]$. Following we repeat these arguments for $\tau\in\left[\frac{a_0}{4(a_1+4l)},\frac{a_0}{2(a_1+4l)}\right]$ and then continue this procedure. It follows that $u\left(x,\tau\right)=0\ \forall\tau\in[0,T]$. It means that problem (1)-(4) has at most one solution.

Existence. Let $w_k\left(x\right)$ be a fundamental system in $H^1\left(0,l\right)$, such that $\left(w_k,w_j\right)_{L^2(0,l)}=\delta_{kj}=\begin{cases}1, & k=j\\ 0, & k\neq j\end{cases}$.

Now we will try to find an approximate solution $u^m(x,t)$ of the problem (1)-(4) by using the Faedo-Galerkin method in the form

$$u^m(x,t)=\sum_{k=1}^m c_k\left(t\right)w_k\left(x\right),\tag{12}$$

where $c_k(t)$ are solutions to the Cauchy problem

$$\begin{aligned} & \int_0^l u_{tt}^m w_j + (b(x,t)u_x^m)_{tt} w_j' + a u_x^m w_j' + c u^m w_j' dx \\ & + w_j(l) \int_0^l K_2 u^m dx - w_j(0) \int_0^l K_1 u^m dx \\ = & \int_0^l f w_j dx, \quad j = 1, ..m, \end{aligned} \tag{13}$$

$$c_k(0) = \alpha_k, \quad c_k'(0) = \beta_k, \tag{14}$$

where α_k, β_k coefficients of the finite sums

$$\psi^m(x) = \sum_{k=1}^m \beta_k w_k(x), \quad \varphi^m(x) = \sum_{k=1}^m \alpha_k w_k(x)$$

approximating of the functions ψ and φ as $m \to \infty$ in $H^1(0,l)$:

$$\sum_{k=1}^m \beta_k w_k(x) \to \psi, \quad \sum_{k=1}^m \alpha_k w_k(x) \to \varphi, \quad m \to \infty \tag{15}$$

We write the Cauchy problem $(13) - (14)$ as

$$\sum_{k=1}^m c_k''(t) A_{kj}(t) + c_k'(t) D_{kj}(t) + c_k(t) B_{kj}(t) = f_j(t) \tag{16}$$

where

$$\begin{aligned} & A_{kj}(t) = \int_0^l w_j(x) w_k(x) + b(x,t) w_j'(x) w_k(x) dx, \\ & B_{kj}(t) = \int_0^l \left(b_{tt}(x,t) w_j(x) w_k'(x) + a w_k'(x) w_j'(x) \right) dx \\ & + w_j(l) \int_0^l K_2 w_j(x) dx - w_j(0) \int_0^l K_1 w_k(x) dx, \\ & D_{kj}(t) = \int_0^l 2 b_t w_k'(x) w_j(x) dx, \\ & f_j(t) = \int_0^l f w_j dx \end{aligned}$$

We obtain a system of differential equations of second order with respect to the variable t with smooth coefficients and the initial conditions $c_k(0) = \alpha_k$, $c_k'(0) = \beta_k$ consequently we get a Cauchy problem of linear differential equations with smooth coefficients that is uniquely solvable. So it has a unique solution u^m satisfying (13).

Next we need a priori estimates to pass to limit as $m \to \infty$.

Multipling (13) by $c_j'(t)$, summing from $j = 1$ to $j = m$ and integrating with respect to t from 0 to τ, we obtain

$$\begin{aligned} & \int_0^\tau \int_0^l (u_{tt}^m u_t^m + (b(x,t)u_x^m)_{tt} u_{xt}^m + a u_x^m u_{xt}^m + c u^m u_t^m) dx dt \\ & + \int_0^\tau u_t^m(l,t) \int_0^l K_1 u^m dx dt - \int_0^\tau u_t^m(0,t) \int_0^l K_2 u^m dx dt \\ = & \int_0^\tau \int_0^l f u_t^m dx dt. \end{aligned} \tag{17}$$

Integrating by parts the left side of (17), we get

$$\begin{aligned}
&\int_0^l \Big(\left(u_t^m(x,\tau)\right)^2 + \Big(a(x,\tau)\left(u_t^m(x,\tau)\right)^2\Big) + b_{tt}(x,\tau)\left(u_x^m(x,\tau)\right)^2 + b(x,\tau)\left(u_{xt}^m(x,\tau)\right)^2 \Big) dx \\
= \; & 2\int_0^\tau \int_0^l f u_t^m dxdt - 2\int_0^\tau u_t^m(l,t)\int_0^l K_2 u^m dxdt + 2\int_0^\tau u_t^m(0,t)\int_0^l K_1 u^m dxdt \\
&+ \int_0^l \Big(\left(u_t^m(x,0)\right)^2 + \left(u_x^m(x,0)\right)^2 \left(a(x,0) + b_{tt}(x,0)\right) + b(x,0)\left(u_{xt}^m(x,0)\right)^2 \Big) dx \\
&+ \int_0^\tau \int_0^l \left(b_{ttt} + a_t\right)\left(u_x^m\right)^2 dxdt - 3\int_0^\tau \int_0^l b_t \left(u_{xt}^m\right)^2 dxdt - 2\int_0^\tau \int_0^l c u^m u_t^m dxdt. \qquad (18)
\end{aligned}$$

Consider the right side of (18) and focus our attention on terms generated by nonlocal conditions. By applying Cauchy-Bunyakovskii inequality, we obtain

$$2 \mid \int_0^\tau u_t^m(0,t)\int_0^l K_1 u^m dxdt \mid \leq \int_0^\tau \left(u_t^m(0,t)\right)^2 dt + k_1 \int_0^\tau \int_0^l \left(u^m\right)^2 dxdt,$$

$$2 \mid \int_0^\tau u_t^m(l,t)\int_0^l K_2 u^m dxdt \mid \leq \int_0^\tau \left(u_t^m(l,t)\right)^2 dt + k_2 \int_0^\tau \int_0^l \left(u^m\right)^2 dxdt,$$

where

$$k_i = \max_{t\in[0,T]} \int_0^l K_i^2 dx, \quad i = 1,2.$$

In order to estimate the other terms of the right side we use the following inequalities

$$\left(u_t^m(0,t)\right)^2 \leq 2l\int_0^l \left(u_{xt}^m(x,t)\right)^2 dx + \frac{2}{l}\int_0^l \left(u_t^m(x,t)\right)^2,$$

$$\left(u_t^m(l,t)\right)^2 \leq 2l\int_0^l \left(u_{xt}^m(x,t)\right)^2 dx + \frac{2}{l}\int_0^l \left(u_t^m(x,t)\right)^2.$$

These inequalities follow from relations

$$u_t^m(0,t) = \int_x^0 u_{xt}^m(\xi,t)\, d\xi + u_t^m(x,t), \qquad u_t^m(l,t) = \int_x^l u_{xt}^m(\xi,t)\, d\xi + u_t^m(x,t).$$

Then

$$\mid \int_0^\tau \left(u_t^m(0,t)\right)^2 dt \mid + \mid \int_0^\tau \left(u_t^m(l,t)\right)^2 dt \mid \leq 4l\int_0^\tau \int_0^l \left(u_{xt}^m\right)^2 dxdt + \frac{4}{l}\int_0^\tau \int_0^l \left(u_t^m\right)^2 dxdt. \qquad (19)$$

Now we apply Cauchy inequality to estimate the other terms in the right side of (18), it becomes

$$2 \mid \int_0^\tau \int_0^l c u^m u_t^m dxdt \mid \leq c_0 \int_0^\tau \int_0^l \left(u^m\right)^2 + \left(u_t^m\right)^2 dxdt,$$

$$2 \mid \int_0^\tau \int_0^l f u_t^m dxdt \mid \leq \int_0^\tau \int_0^l \Big(f^2 + \left(u_t^m\right)^2\Big) dxdt,$$

$$2 \mid \int_0^\tau \int_0^l \left(b_{ttt} + a_t\right)\left(u_x^m\right)^2 dxdt \mid \leq \left(b_1 + a_1\right)\int_0^\tau \int_0^l \left(u_x^m\right)^2 dxdt,$$

With this result we can now obtain from (18) and (19)

$$\begin{aligned}
&\int_0^l (u^m(x,\tau))^2 + (u_t^m(x,\tau))^2 + (a(x,\tau) + b_{tt}(x,\tau))(u_x^m(x,\tau))^2 + b(x,\tau)(u_{xt}^m(x,\tau))^2\,dx \\
\leq\ &\int_0^l (u_t^m(x,0))^2 + (u_x^m(x,0))^2 (a(x,0) + b_{tt}(x,0) + 4) + b(x,0)(u_{xt}^m(x,0))^2\,dx \\
&+\int_0^\tau\int_0^l f^2 dxdt + (b_1 + a_1)\int_0^\tau\int_0^l (u_x^m)^2\,dxdt + c_0\int_0^\tau\int_0^l (u^m)^2\,dxdt \\
&+\left(c_0 + \frac{4}{l} + 1 + 2T\right)\int_0^\tau\int_0^l (u_t^m)^2\,dxdt + (3b_1 + 4l)\int_0^\tau\int_0^l (u_{xt}^m)^2\,dxdt. \qquad (20)
\end{aligned}$$

It easy to see that a relation

$$u^m(x,\tau) = \int_0^\tau u_t^m(x,t)\,dx + u^m(x,0)$$

implies the following inequality

$$\frac{1}{2} \mid \int_0^\tau (u^m(x,\tau))^2\,dx \mid \leq \tau \int_0^\tau\int_0^l (u_t^m)^2\,dxdt + \int_0^l (u^m(x,0))^2\,dx.$$

Adding it to (20), we get

$$\begin{aligned}
&C_4\int_0^l \left[(u^m(x,\tau))^2 + u_t^m(x,\tau)^2 + (u_x^m(x,\tau))^2 + (u_{xt}^m(x,\tau))^2\right] dx \\
\leq\ &C_5\left(\int_0^l (u_t^m(x,0))^2 + (u_x^m(x,0))^2 + (u_{xt}^m(x,0))^2\,dx\right) + C_6\left(\int_0^\tau\int_0^l f^2 dxdt\right. \\
&\left.+\int_0^\tau\int_0^l (u_x^m)^2\,dxdt + \int_0^\tau\int_0^l (u^m)^2\,dxdt + \int_0^\tau\int_0^l (u_t^m)^2\,dxdt + \int_0^\tau\int_0^l (u_{xt}^m)^2\,dxdt\right),
\end{aligned}$$

where $C_5 > 0$, $C_6 > 0$ depend only on c_0, a_1, b_1, l, T By Gronwall's lemma, we conclude that, for all $m \geq 1$,

$$\|u^m\|^2_{W(Q_T)} \leq C\left(\|u^m(x,0)\|^2_{H^1(0,l)} + \|u_t^m(x,0)\|^2_{H^1(0,l)} + \|f\|^2_{L(Q_T)}\right),$$

Taking into hypotheses of the Theorem 1 we can state that $\|u^m(x,0)\|^2_{H^1(0,l)}$, $\|u_t^m(x,0)\|^2_{H^1(0,l)}$ are uniformely bounded with respect to m. Consequently,

$$\|u^m\|^2_{W(Q_T)} \leq L, \qquad (21)$$

where $L > 0$ and does not depend on m.

Note that $W(Q_T)$ is Hilbert space. Therefore because of (21) we can extarct from $\{u^m\}$ a sunsequence convergent weakly in $W(Q_T)$ to $u \in W(Q_T)$.

We need to show that this limit function is a required generalized solution.

For this, we prove that the limit of the subsequence $\{u^m\}$ satisfies the identity (5) for any function $\eta^m(x,t) = \sum_{k=1}^m d_k(t)\,w_k(x)$.Taking into account lemma 1 we can see that the set $S_n = \left\{\eta(x,t) = \sum_{j=1}^m d_j(t)\,w_j(x), \quad d_j(t) \in H^1[0,T], d_j(T) = 0\right\}$ is such that $\overline{\cup_{n=1}^\infty S_n} = \hat{W}(Q_T)$, it suffices to prove (5) for $\eta^m(x,t) \in S_n$.

Multiplying (13) by $d_j(t) \in H^1[0,T]$, $d_j(T) = 0$, then taking the sum from $i = 0$ to n, then integrate with respect to t from 0 to T. After integrating by parts we get

$$\int_0^T \int_0^l \left(-u_t^m \eta_t(x,t) - (b(x,t)u_x^m)_t \eta_{xt}(x,t) + a u_x^m \eta_{xt}(x,t) + c u^m \eta(x,t)\right) dxdt$$
$$+ \int_0^T \eta(l,t) \int_0^l K_2 u^m dx - \int_0^T \eta(0,t) \int_0^l K_1 u^m dx = \int_0^T \int_0^l f(x,t)\eta(x,t)\, dx$$
$$+ \int_0^l \left(u_x^m(x,0) b_t(x,0) + u_t^m(x,0) + b(x,0) u_{xt}^m(x,0)\right) \eta(x,0)\, dx, \tag{22}$$

$$\begin{cases} u^m \rightharpoonup u \text{ in } W(Q_T), \\ \varphi^m \to \varphi \text{ in } H^1(0,l), \\ \psi^m \to \psi \text{ in } H^1(0,l). \end{cases} \tag{23}$$

Taking account convergences (23) we can pass to limit in (22) as $m \to \infty$. We get that u satisfies (5). □

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