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CHEBYSHEV TYPE INEQUALITIES INVOLVING GENERALIZED PROPORTIONAL FRACTIONAL INTEGRAL OPERATORS

İLKER MUMCU, ERHAN SET, AND ARTION KASHURI

ABSTRACT. A number of definition of fractional integral operators have, recently, been presented. In [11], Jarad et al. introduced the proportional generalized fractional integrals. In this paper, we motivated essentially by the earlier works and established some Chebyshev type inequalities for synchronous functions involving generalized proportional fractional integral operators. Also we have results containing confluent hypergeometric functions and incomplete gamma functions.

1. INTRODUCTION

Many integral inequalities of various types have been presented in the literature. Among them, we choose to recall the following Chebyshev inequality (see [4]):

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \geq \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{1}$$

where f and g are two integrable and synchronous functions on $[a,b]$. Here, two functions f and g are called *synchronous* on $[a,b]$ if

$$(f(x)-f(y))\,(g(x)-g(y)) \geq 0 \quad (x,\, y \in [a,b]).$$

In recent years, many papers related to (1) inequality for written, for some of them please see [5, 7, 15, 20, 24, 27, 28] For $\rho > 0$, the Gamma function is

$$\Gamma(\nu) = \int_0^\infty e^{-\rho}\rho^{\nu-1}d\rho.$$

The confluent hypergeometric function is defined by the absolutely convergent infinite power series

$$M(a,b,z) = {}_1F_1(a,b,z) = \sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\frac{z^k}{k!} \tag{2}$$

where $(a)_k$ and $(b)_k$ are Pochhammer's symbols [14] defined by

$$(a)_k = \prod_{i=1}^{k-1}(a+i) = \frac{\Gamma(a+k)}{\Gamma(a)}$$

with $(a)_0 = 1$. If a a negative integer and $k \geq -a$, then $(a)_k = 0$. Also $(1)_n = n!$.
In [1], Abramowitz and Stegun give a useful integral form for $b > a > 0$

$${}_1F_1(a,b,z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}dt \tag{3}$$

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In [1], the lower incomplete gamma function is defined:

$$\gamma(u,\nu) = \int_0^{\nu} e^{-\rho}\rho^{u-1}d\rho,$$

and the relations with the confluent Hypergeometric function are as follows

$$\gamma(v,z) = v^{-1}z^{v}\,{}_1F_1(v,v+1,-z) \tag{4}$$

Many new extensions, generalizations and variations of classically integral inequalities by used fractional calculus technique have been established in the literature. Riemann-Liouville fractional integrals that pioneers in the fractional calculus approach used to obtain new inequalities are defined as:

Definition 1. *Let $f \in L_1[a,b]$. The Riemann–Liouville integrals $J^{\mu}_{a+}f$ and $J^{\mu}_{b-}f$ of order $\alpha > 0$ are defined by*

$$J^{\mu}_{a+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}f(t)dt, \qquad x > a$$

and

$$J^{\mu}_{b-}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b (t-x)^{\mu-1}f(t)dt, \qquad x < b.$$

Here $J^{0}_{a+}f(x) = J^{0}_{b-}f(x) = f(x)$

In the case of $\mu = 1$, the fractional integral reduces to classical integral. For detail about this integrals and some recent results, please refer to [2, 8, 9, 10, 12, 13, 16, 22, 23, 25, 26]. In [6], Belarbi and Dahmani established following theorems for the Chebyshev inequalities.

Theorem 1. *Let f and g be two synchronous functions on $[0,+\infty)$. Then for all $t > 0,\ \alpha > 0$, we have:*

$$J^{\alpha}(fg) \geq \frac{\Gamma(\alpha+1)}{t^{\alpha}}J^{\alpha}f(t)J^{\alpha}g(t). \tag{5}$$

Theorem 2. *Let f and g be two synchronous functions on $[0,+\infty)$. Then for all $t > 0,\ \alpha > 0,\ \beta > 0$, we have:*

$$\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}(fg)(t) + \frac{t^{\beta}}{\Gamma(\beta+1)}J^{\alpha}(fg)(t) \geq J^{\alpha}f(t)J^{\beta}g(t) + J^{\beta}f(t)J^{\alpha}g(t) \tag{6}$$

Theorem 3. *Let $(f_i)_{i=1,\ldots,n}$ be n positive increasing functions on $[0,+\infty)$. Then for any $t > 0,\ \alpha > 0$, we have*

$$J^{\alpha}\left(\prod_{i=1}^{n} f_i\right)(t) \geq (J^{\alpha}(1))^{1-n}\prod_{i=1}^{n} J^{\alpha}f_i(t). \tag{7}$$

Theorem 4. *Let f and g be two functions defined on $[0,+\infty)$, such that f is increasing, g is differentiable and there exist a real number $m := \inf_{t\geq 0} g'(t)$. Then the inequality*

$$J^{\alpha}(fg)(t) \geq (J^{\alpha}(1))^{-1}J^{\alpha}f(t)J^{\alpha}g(t) - \frac{mt}{\alpha+1}J^{\alpha}f(t) + mJ^{\alpha}(tf(t)) \tag{8}$$

is valid for all $t > 0,\ \alpha > 0$.

Jarad et al. in [11], introduced the generalized proportional fractional integrals as follows:

Definition 2. *The left and right generalized proportional fractional integral operators are respectively defined by*

$$({}_a\mathfrak{I}^{\lambda,\eta}g)(\vartheta)=\frac{1}{\eta^{\lambda}\Gamma(\lambda)}\int_a^{\vartheta}e^{[\frac{\eta-1}{\eta}(\vartheta-\rho)]}(\vartheta-\rho)^{\lambda-1}g(\rho)d\rho \tag{9}$$

and

$$(\mathfrak{I}_b^{\lambda,\eta}g)(\vartheta)=\frac{1}{\eta^{\lambda}\Gamma(\lambda)}\int_{\vartheta}^{b}e^{[\frac{\eta-1}{\eta}(\rho-\vartheta)]}(\rho-\vartheta)^{\lambda-1}g(\rho)d\rho, \tag{10}$$

where $\eta\in(0,1]$, $\lambda\in\mathbb{C}$ *and* $\Re(\lambda)>0$.

We also let

$$({}_0\mathfrak{I}^{\lambda,\eta}g)(\vartheta)=(\mathfrak{I}^{\lambda,\eta}g)(\vartheta).$$

Recently, various types of inequalities by employing generalized proportional fractional integral operators have been introduced, see [3, 17, 18, 19, 21].

The main purpose of this work is to establish Chebyshev type inequalities with synchronous functions for the generalized proportional fractional integral operators.

2. Chebyshev type inequalities involving generalized proportional fractional integral operators

Theorem 5. *Let f and g be two integrable functions which are synchronous on* $[0,+\infty)$. *Then*

$$(\mathfrak{I}^{\lambda,\eta}fg)(\vartheta)\geq\frac{\eta^{\lambda}\Gamma(\lambda+1)}{\vartheta^{\lambda}\,{}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)}(\mathfrak{I}^{\lambda,\eta}f)(\vartheta)(\mathfrak{I}^{\lambda,\eta}g)(\vartheta), \tag{11}$$

with $\eta\in(0,1]$, $\lambda\in\mathbb{C}$ *and* $\Re(\lambda)>0$.

Proof. Since f and g are synchronous on $[0,+\infty)$, we have

$$(f(\rho)-f(\tau))(g(\rho)-g(\tau))\geq 0,$$

or, equivalently,

$$f(\rho)g(\rho)+f(\tau)g(\tau)\geq f(\rho)g(\tau)+f(\tau)g(\rho). \tag{12}$$

Multiplying both sides of (12) by

$$\frac{1}{\eta^{\lambda}\Gamma(\lambda)}e^{\frac{\eta-1}{\eta}(\vartheta-\rho)}(\vartheta-\rho)^{\lambda-1}$$

and integrating both sides of the resulting inequality with respect to the variable ρ from 0 and ϑ, we get

$$\begin{aligned}&(\mathfrak{I}^{\lambda,\eta}fg)(\vartheta)+f(\tau)g(\tau)\frac{1}{\eta^{\lambda}\Gamma(\lambda)}\int_0^{\vartheta}e^{\frac{\eta-1}{\eta}(\vartheta-\rho)}(\vartheta-\rho)^{\lambda-1}d\rho\\ \geq\;& g(\tau)\frac{1}{\eta^{\lambda}\Gamma(\lambda)}\int_0^{\vartheta}e^{\frac{\eta-1}{\eta}(\vartheta-\rho)}(\vartheta-\rho)^{\lambda-1}f(\rho)d\rho\\ &+f(\tau)\frac{1}{\eta^{\lambda}\Gamma(\lambda)}\int_0^{\vartheta}e^{\frac{\eta-1}{\eta}(\vartheta-\rho)}(\vartheta-\rho)^{\lambda-1}g(\rho)d\rho.\end{aligned}$$

From (3) and (9), we obtain

$$\begin{aligned}&(\mathfrak{I}^{\lambda,\eta}fg)(\vartheta)+f(\tau)g(\tau)\frac{\vartheta^{\lambda}}{\eta^{\lambda}\Gamma(\lambda+1)}\,{}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)\\ \geq\;& g(\tau)(\mathfrak{I}^{\lambda,\eta}f)(\vartheta)+f(\tau)(\mathfrak{I}^{\lambda,\eta}g)(\vartheta)\end{aligned} \tag{13}$$

Multiplying both sides of (13) by

$$\frac{1}{\eta^{\lambda}\Gamma(\lambda)}e^{\frac{\eta-1}{\eta}(\vartheta-\tau)}(\vartheta-\tau)^{\lambda-1}$$

and integrating both sides of the resulting inequality with respect to the variable τ from 0 and ϑ, similarly as above, we have

$$2(\mathfrak{I}^{\lambda,\eta}fg)\frac{\vartheta^{\lambda}}{\eta^{\lambda}\Gamma(\lambda+1)}\,{}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)\geq 2(\mathfrak{I}^{\lambda,\eta}f)(\vartheta)(\mathfrak{I}^{\lambda,\eta}g)(\vartheta)$$

which, upon simplifying, leads to inequality of (11). The proof of Theorem 5 is completed. □

Remark 1. *In Theorem 5, if we take $\eta=1$, then the inequality of (11) becomes inequality of (5).*

Corollary 1. *In Theorem 5, if we using equality of (4), then the inequality of (11) becomes as follows*

$$(\mathfrak{I}^{\lambda,\eta}fg)(\vartheta)\geq\frac{(\eta-1)^{\lambda}\Gamma(\lambda)}{\gamma\left(\lambda,\frac{(\eta-1)\vartheta}{\eta}\right)}(\mathfrak{I}^{\lambda,\eta}f)(\vartheta)(\mathfrak{I}^{\lambda,\eta}g)(\vartheta).$$

Theorem 6. *Let f and g be two integrable functions which are synchronous on $[0,+\infty)$. Then*

$$\begin{aligned}&\frac{\vartheta^{\kappa}}{\eta^{\kappa}\Gamma(\kappa+1)}\,{}_1F_1\left(\kappa,\kappa+1,\frac{(1-\eta)\vartheta}{\eta}\right)(\mathfrak{I}^{\lambda,\eta}fg)(\vartheta)\\&+\frac{\vartheta^{\lambda}}{\eta^{\lambda}\Gamma(\lambda+1)}\,{}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)(\mathfrak{I}^{\kappa,\eta}fg)(\vartheta)\\\geq\;&(\mathfrak{I}^{\lambda,\eta}f)(\vartheta)(\mathfrak{I}^{\kappa,\eta}g)(\vartheta)+(\mathfrak{I}^{\kappa,\eta}g)(\vartheta)(\mathfrak{I}^{\lambda,\eta}f)(\vartheta)\end{aligned}\tag{14}$$

with $\eta\in(0,1]$, $\lambda,\ \kappa\in\mathbb{C}$ and $\Re(\lambda)>0$.

Proof. Multiplying both sides of (13) by

$$\frac{1}{\eta^{\kappa}\Gamma(\kappa)}e^{\frac{\eta-1}{\eta}(\vartheta-\tau)}(\vartheta-\tau)^{\kappa-1}$$

and integrating both sides of the resulting inequality with respect to the variable τ from 0 and ϑ, we have

$$\begin{aligned}&\frac{\vartheta^{\kappa}}{\eta^{\kappa}\Gamma(\kappa+1)}\,{}_1F_1\left(\kappa,\kappa+1,\frac{(1-\eta)\vartheta}{\eta}\right)(\mathfrak{I}^{\lambda,\eta}fg)(\vartheta)\\&+\frac{\vartheta^{\lambda}}{\eta^{\lambda}\Gamma(\lambda+1)}\,{}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)(\mathfrak{I}^{\kappa,\eta}fg)(\vartheta)\\\geq\;&(\mathfrak{I}^{\lambda,\eta}f)(\vartheta)(\mathfrak{I}^{\kappa,\eta}g)(\vartheta)+(\mathfrak{I}^{\kappa,\eta}g)(\vartheta)(\mathfrak{I}^{\lambda,\eta}f)(\vartheta)\end{aligned}$$

and this ends proof. □

Remark 2. *In Theorem 6, if we take $\kappa=\lambda$, then the inequality of (14) becomes inequality of (11).*

Remark 3. *In Theorem 6, if we take $\eta=1$, then the inequality of (14) becomes inequality of (6).*

Corollary 2. *In Theorem 6, if we using equality of (4), then the inequality (14) becomes as follows*

$$\begin{aligned} &\frac{1}{(\eta-1)^{\kappa}\Gamma(\kappa)}\gamma\left(\kappa, \frac{(\eta-1)\vartheta}{\eta}\right)(\mathfrak{I}^{\lambda,\eta}fg)(\vartheta) \\ &+\frac{1}{(\eta-1)^{\lambda}\Gamma(\lambda)}\gamma\left(\lambda, \frac{(\eta-1)\vartheta}{\eta}\right)(\mathfrak{I}^{\kappa,\eta}fg)(\vartheta) \\ \geq\ & (\mathfrak{I}^{\lambda,\eta}f)(\vartheta)(\mathfrak{I}^{\kappa,\eta}g)(\vartheta)+(\mathfrak{I}^{\kappa,\eta}g)(\vartheta)(\mathfrak{I}^{\lambda,\eta}f)(\vartheta) \end{aligned}$$

Theorem 7. *Let $(f_i)_{i=1,\dots,n}$ be n positive increasing functions on $[0,+\infty)$. Then*

$$\left(\mathfrak{I}^{\lambda,\eta}\prod_{i=1}^{n} f_i\right)(\vartheta) \geq \left(\frac{\eta^{\lambda}\Gamma(\lambda+1)}{\vartheta^{\lambda}\ {}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)}\right)^{n-1}\prod_{i=1}^{n}\left(\mathfrak{I}^{\lambda,\eta}f_i\right)(\vartheta), \tag{15}$$

with $\eta\in(0,1]$, $\lambda\in\mathbb{C}$ and $\Re(\lambda)>0$.

Proof. We prove this theorem by induction.
For $n=1$, inequality of (15) is evident. For $n=2$, inequality of (15) coincide with inequality of (11) of Theorem 5. Now, suppose that (induction hypothesis)

$$\left(\mathfrak{I}^{\lambda,\eta}\prod_{i=1}^{n-1} f_i\right)(\vartheta) \geq \left(\frac{\eta^{\lambda}\Gamma(\lambda+1)}{\vartheta^{\lambda}\ {}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)}\right)^{n-2}\prod_{i=1}^{n-1}\left(\mathfrak{I}^{\lambda,\eta}f_i\right)(\vartheta). \tag{16}$$

Since $(f_i)_{i=1,\dots,n}$ are n positive increasing functions on $[0,+\infty)$, then $\prod_{i=1}^{n-1} f_i = g$ is an increasing function. Hence applying Theorem 5 to the functions $f_n=f$ and g, we get

$$\begin{aligned} \left(\mathfrak{I}^{\lambda,\eta}\prod_{i=1}^{n} f_i\right)(\vartheta) = \left(\mathfrak{I}^{\lambda,\eta}fg\right)(\vartheta) &\geq \left(\frac{\eta^{\lambda}\Gamma(\lambda+1)}{\vartheta^{\lambda}\ {}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)}\right) \\ &\times\left(\mathfrak{I}^{\lambda,\eta}\prod_{i=1}^{n-1} f_i\right)(\vartheta)\left(\mathfrak{I}^{\lambda,\eta}f_n\right)(\vartheta). \end{aligned}$$

Taking into account the hypothesis of (16), we obtain

$$\begin{aligned} \left(\mathfrak{I}^{\lambda,\eta}\prod_{i=1}^{n} f_i\right)(\vartheta) &\geq \left(\frac{\eta^{\lambda}\Gamma(\lambda+1)}{\vartheta^{\lambda}\ {}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)}\right)\left(\frac{\eta^{\lambda}\Gamma(\lambda+1)}{\vartheta^{\lambda}\ {}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)}\right)^{n-2} \\ &\times\prod_{i=1}^{n-1}\left(\mathfrak{I}^{\lambda,\eta}f_i\right)(\vartheta)\left(\mathfrak{I}^{\lambda,\eta}f_n\right)(\vartheta). \end{aligned}$$

The proof of Theorem 7 is completed. □

Remark 4. *In Theorem 7, if we take $\eta=1$, then the inequality of (15) becomes inequality of (7).*

Corollary 3. *In Theorem 7, if we using equality of (4), then the inequality of (15) becomes as follows:*

$$\left(\mathfrak{I}^{\lambda,\eta}\prod_{i=1}^{n} f_i\right)(\vartheta) \geq \left(\frac{(\eta-1)^{\lambda}\Gamma(\lambda)}{\gamma\left(\lambda, \frac{(\eta-1)\vartheta}{\eta}\right)}\right)^{n-1}\prod_{i=1}^{n}\left(\mathfrak{I}^{\lambda,\eta} f_i\right)(\vartheta),$$

Let derive the following new result from Theorem 5.

Theorem 8. *Let f and g be two functions defined on $[0,+\infty)$, such that f is increasing, g is differentiable and there exist a real number $m := \inf_{t\geq 0} g'(t)$. Then the inequality*

$$(\mathfrak{I}^{\lambda,\eta} fg)(\vartheta) \geq \frac{\eta^{\lambda}\Gamma(\lambda+1)}{\vartheta^{\lambda}\,{}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)} \tag{17}$$

$$\times\Big[(\mathfrak{I}^{\lambda,\eta} f)(\vartheta)(\mathfrak{I}^{\lambda,\eta} g)(\vartheta) - m(\mathfrak{I}^{\lambda,\eta} t)(\vartheta)(\mathfrak{I}^{\lambda,\eta} f)(\vartheta)\Big] + m(\mathfrak{I}^{\lambda,\eta} tf)(\vartheta),$$

with $\eta \in (0,1]$, $\lambda \in \mathbb{C}$ and $\Re(\lambda) > 0$.

Proof. We consider the function $h(t) = g(t) - mt$. It is clear that h is differentiable and it is increasing on $[0,+\infty)$. Then using Theorem 5, we have

$$\begin{aligned}
&(\mathfrak{I}^{\lambda,\eta} hf)(\vartheta) = (\mathfrak{I}^{\lambda,\eta}(g-mt)f)(\vartheta)\\
\geq\ & \frac{\eta^{\lambda}\Gamma(\lambda+1)}{\vartheta^{\lambda}\,{}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)}(\mathfrak{I}^{\lambda,\eta}(g-mt))(\vartheta)(\mathfrak{I}^{\lambda,\eta} f)(\vartheta) \qquad (18)\\
=\ & \frac{\eta^{\lambda}\Gamma(\lambda+1)}{\vartheta^{\lambda}\,{}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)}\Big[(\mathfrak{I}^{\lambda,\eta} f)(\vartheta)(\mathfrak{I}^{\lambda,\eta} g)(\vartheta) - m(\mathfrak{I}^{\lambda,\eta} t)(\vartheta)(\mathfrak{I}^{\lambda,\eta} f)(\vartheta)\Big].
\end{aligned}$$

On the other hand,

$$(\mathfrak{I}^{\lambda,\eta} hf)(\vartheta) = (\mathfrak{I}^{\lambda,\eta}(g-mt)f)(\vartheta) = (\mathfrak{I}^{\lambda,\eta} fg)(\vartheta) - m(\mathfrak{I}^{\lambda,\eta} tf)(\vartheta). \tag{19}$$

From (18) and (19), we obtain our result of (17). The proof of Theorem 8 is completed. □

Remark 5. *In Theorem 8, if we take $\eta = 1$, then the inequality of (17) becomes inequality of (8).*

Finally, from Theorem 8 we can derive the following corollaries.

Corollary 4. *Let f and g be two functions defined on $[0,+\infty)$, such that f is decreasing, g is differentiable and there exist a real number $M := \sup_{t\geq 0} g'(t)$. Then the inequality*

$$(\mathfrak{I}^{\lambda,\eta} fg)(\vartheta) \geq \frac{\eta^{\lambda}\Gamma(\lambda+1)}{\vartheta^{\lambda}\,{}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)}$$

$$\times\Big[(\mathfrak{I}^{\lambda,\eta} f)(\vartheta)(\mathfrak{I}^{\lambda,\eta} g)(\vartheta) - M(\mathfrak{I}^{\lambda,\eta} t)(\vartheta)(\mathfrak{I}^{\lambda,\eta} f)(\vartheta)\Big] + M(\mathfrak{I}^{\lambda,\eta} tf)(\vartheta),$$

with $\eta \in (0,1]$, $\lambda \in \mathbb{C}$ and $\Re(\lambda) > 0$.

Proof. Taking the function $H(t) = g(t) - Mt$ and using the same idea of the proof as in Theorem 8. □

Corollary 5. *Let f and g be two increasing functions defined on $[0,+\infty)$ and differentiable and there exists two real numbers $m_1 := \inf_{t\geq 0} f'(t)$, $m_2 := \inf_{t\geq 0} g'(t)$. Then the inequality*

$$(\mathfrak{I}^{\lambda,\eta} fg)(\vartheta) \geq \frac{\eta^{\lambda}\Gamma(\lambda+1)}{\vartheta^{\lambda}\,{}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)}$$

$$\times\Big[(\mathfrak{I}^{\lambda,\eta} f)(\vartheta)(\mathfrak{I}^{\lambda,\eta} g)(\vartheta) - m_1(\mathfrak{I}^{\lambda,\eta} t)(\vartheta)(\mathfrak{I}^{\lambda,\eta} g)(\vartheta)$$

$$-m_2(\mathfrak{I}^{\lambda,\eta} t)(\vartheta)(\mathfrak{I}^{\lambda,\eta} f)(\vartheta) + m_1m_2(\mathfrak{I}^{\lambda,\eta} t)^2(\vartheta)\Big]$$

$$+m_1(\mathfrak{I}^{\lambda,\eta} tg)(\vartheta) + m_2(\mathfrak{I}^{\lambda,\eta} tf)(\vartheta) - m_1m_2(\mathfrak{I}^{\lambda,\eta} t^2)(\vartheta),$$

with $\eta \in (0,1]$, $\lambda \in \mathbb{C}$ and $\Re(\lambda) > 0$.

Proof. Choosing the functions $p(t) = f(t) - m_1t$, $q(t) = g(t) - m_2t$ and using the same idea of the proof as in Theorem 8. □

Corollary 6. *Let f and g be two decreasing functions defined on $[0,+\infty)$ and differentiable and there exists two real numbers $M_1 := \sup_{t\geq 0} f'(t)$, $M_2 := \sup_{t\geq 0} g'(t)$. Then the inequality*

$$(\mathfrak{I}^{\lambda,\eta} fg)(\vartheta) \geq \frac{\eta^{\lambda}\Gamma(\lambda+1)}{\vartheta^{\lambda}\,{}_1F_1\left(\lambda,\lambda+1,\frac{(1-\eta)\vartheta}{\eta}\right)}$$

$$\times\Big[(\mathfrak{I}^{\lambda,\eta} f)(\vartheta)(\mathfrak{I}^{\lambda,\eta} g)(\vartheta) - M_1(\mathfrak{I}^{\lambda,\eta} t)(\vartheta)(\mathfrak{I}^{\lambda,\eta} g)(\vartheta)$$

$$-M_2(\mathfrak{I}^{\lambda,\eta} t)(\vartheta)(\mathfrak{I}^{\lambda,\eta} f)(\vartheta) + M_1M_2(\mathfrak{I}^{\lambda,\eta} t)^2(\vartheta)\Big]$$

$$+M_1(\mathfrak{I}^{\lambda,\eta} tg)(\vartheta) + M_2(\mathfrak{I}^{\lambda,\eta} tf)(\vartheta) - M_1M_2(\mathfrak{I}^{\lambda,\eta} t^2)(\vartheta),$$

with $\eta \in (0,1]$, $\lambda \in \mathbb{C}$ and $\Re(\lambda) > 0$.

Proof. Choosing the functions $F(t) = f(t) - M_1t$, $G(t) = g(t) - M_2t$ and using the same idea of the proof as in Theorem 8. □

Corollary 7. *In Theorem 8, if we using equality of (4), then the inequality (17) becomes as follows*

$$(\mathfrak{I}^{\lambda,\eta} fg)(\vartheta) \geq \frac{(\eta-1)^{\lambda}\Gamma(\lambda)}{\gamma\left(\lambda,\frac{(\eta-1)\vartheta}{\eta}\right)}$$

$$\times\Big[(\mathfrak{I}^{\lambda,\eta} f)(\vartheta)(\mathfrak{I}^{\lambda,\eta} g)(\vartheta) - m(\mathfrak{I}^{\lambda,\eta} t)(\vartheta)(\mathfrak{I}^{\lambda,\eta} f)(\vartheta)\Big] + m(\mathfrak{I}^{\lambda,\eta} tf)(\vartheta).$$

References

[1] M. Abramovitz, I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standarts Applied Mathematics Series, 55, Washington D. C., U. S. Printing Office.

[2] A. O. Akdemir, *Inequalities of Ostrowski's type for m- and (alpha; m)-logarithmically convex functions via Riemann-Liouville fractional integrals* . J. Comput. Anal. Appl., 16(2) (2014), 375-383.

[3] J. Alzabut , T. Abdeljawad, F. Jarad, W. Sudsutad, *Gronwall inequality via the generalized proportional fractional derivative with applications* , Journal of Inequalities and Applications, 2019:101 (2019), 1-12.

[4] P. L. Chebyshev, Sur les expressions approximatives des integrales definies par les autres prises entre les mmes limites, *Proc. Math. Soc. Charkov* **2** (1882), 93-98.

[5] Z. Dahmani, *New inequalities in fractional integrals*, Int. J. of Nonlinear Sci., 9(4) (2010), 493-497.

[6] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities, *J. Inequal. Pure Appl. Math.*, **10**(3) (2009), Article 86.

[7] Z.Dahmani, O.Mechoua, S.Brahami, *Certain inequalities related to the Chebyshevs functional involving a type Riemann-Liouville operator* , Bull. of Math. Anal. andAppl., 3(4) (2011), 38-44.

[8] R. Goreno, F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, Springer Verlag, Wien (1997), 223-276.

[9] İ. İşcan, *Hermite-hadamard-fejer type inequalities for convex functions via fractional integrals*, Stud.Univ. Babes-Bolyai Math., 60(3) (2015), 355–366.

[10] İ. İşcan, S. Wu, *Hermite-hadamard type inequalities for harmonically convex functions via fractional integrals,* Applied Mathematics and Computation, 238 (2014), 237–244.

[11] F. Jarad, T. Abdeljawad, J. Alzabut, Generalized fractional derivatives generated by a class of local proportional derivatives, *Eur. Phys. J. Spec. Top.*, **226** (2017), 3457–3471.

[12] A. A. Kilbas, H. M. Srivastavaand J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, vol. 204. Elsevier (North-Holland) Science Publishers, Amsterdam, 2006.

[13] S. Mille, B. Ross, *An introduction to the Fractional Calculus and Fractional Differential Equations* , John Wiley&Sons, USA, 1993, p.2.

[14] K. E. Muller, Computing the Confluent Hypergeometric function, $M(a, b, x)$, *Numer. Math.*, **90**(1) (2001), 179–196.

[15] M.E. Ozdemir, E. Set, A.O. Akdemir, M.Z. Sarıkaya, *Some new Chebyshev type inequalities for functions whose derivatives belongs to Lp-spaces* , Afr. Mat., 26 (2015), 1609-1619.

[16] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of their Applications*, Academic Press,San Diego, 1999.

[17] G. Rahman, T. Abdeljawad, A. Khan, K. S. Nisar, *Some fractional proportional integral inequalities*, Journal of Inequalities and Applications, 2019:244(2019), 1-13.

[18] G. Rahman, A. Khan, T. Abdeljawad, K. S. Nisar, *The Minkowski inequalities via generalized proportional fractional integral operators* , Advances in Difference Equations 2019:287(2019), 1-14.

[19] G. Rahman, K. S. Nisar , T. Abdeljawad, S. Ullah, *Certain fractional proportional integral inequalities via convex functions*, Mathematics8(2020), 1-11; doi:10.3390/math8020222.

[20] G. Rahman, Z. Ullah, A. Khan, E. Set, K. S. Nisar, *Certain Chebyshev-Type inequalities involving fractional conformable integral operators* , Mathematics, 7 (364) (2019), 1-8, 2019.

[21] S. Rashid, F. Jarad, M. A. Noor, *Grüss-type integrals inequalities via generalized proportional fractional operators* , RACSAM, 114:93(2020), 1-16.

[22] M.Z. Sarıkaya, E. Set, H. Yaldiz, N., Basak, *Hermite-Hadamard's inequalities forfractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, DOI:10.1016/j.mcm.2011.12.048, 57 (2013) 2403-2407.

[23] E. Set, B. Çelik, A.O. Akdemir, *Some new Hermite-Hadamard type inequalities for quasi-convex functions via fractional integral operator* , AIP Conference Proceedings 1833 (2017) 020021.

[24] E. Set, Z. Dahmani, İ. Mumcu, *New extensions of Chebyshev type inequalities using generalized Katugampola integrals via P´olya-Szego inequality* , IJOCTA, 8(2)(2018), 137-144.

[25] E. Set, İ. İşcan, F. Zehir, *On some new inequalities of Hermite-Hadamard type involving harmonically convex functions via fractional integrals* , Konuralp J. Math. 3 (2015) 42–55.

[26] E. Set, S.S.Karataş, M.A.Khan, *Hermite–Hadamard type inequalities obtained via fractional integrals for differentiable m-convex and (α,m)-convex functions.* Int. J. Anal. 2016, Article ID 4765691 (2016).

[27] E. Set, İ.Mumcu, S. Demirbaş, *Conformable fractional integral inequalities of Chebyshev type* , RACSAM, 113(3), 2253–2259 (2019).
[28] E. Set, M. Z. Sarıkaya, F. Ahmad, *A generalization of Cebysev type inequalities for first differentiable mappings* , Miskolc Math. Notes 12, 2 (2011), 245–253.

Ordu University
Department of Mathematics, Faculty of Science and Arts
Ordu, Turkey
Email address: mumcuilker@msn.com

Ordu University
Department of Mathematics, Faculty of Science and Arts
Ordu, Turkey
Email address: erhanset@yahoo.com

University Ismail Qemali
Department of Mathematics, Faculty of Technical Science
Vlora, Albania
Email address: artionkashuri@gmail.com