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MODIFIED S-METRIC SPACES AND SOME FIXED POINT RESULTS

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Abstract. In this paper, a structure of modified S-metric spaces is introduced which can be viewed a generalization of both S-metric and S_b-metric spaces. Also, the notions of $\widetilde{S}$-contractive mappings in the modified S-metric spaces is given. We also investigate the existence of fixed point for such mappings under various contractive conditions. We provide example and an application to illustrate the results presented herein.

1. Introduction and preliminaries

There is a large number of generalizations of Banach contraction principle via using different forms of contractive conditions in various generalized metric spaces. Some of such generalizations are given in [1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14].

Let Ψ be a family of all mappings $\Omega : [0,\infty) \to [0,\infty)$ such that

(1) Ω is strictly increasing and continuous,
(2) $\Omega^{-1}(t) \le t \le \Omega(t)$,
(3) $\Omega(0) = 0$.

Define $\Omega_i : [0,\infty) \to [0,\infty), i = 1,2,3$, by
$\Omega_1(t) = e^t - 1$, $\Omega_2(t) = te^t$, $\Omega_3(t) = t^2 + 2t$,
then $\Omega_i \in \Psi, i = 1,2,3$.

Parvaneh introduced in [8] the concept of extended b-metric spaces as follows.

Definition 1. [8] *Let X be a (nonempty) set. A function $\widetilde{d} : X \times X \to \mathbb{R}^+$ is called a p-metric if there exists $\Omega \in \Psi$ such that for all $x, y, z \in X$, the following conditions hold:*

($\widetilde{d}_1$) $\widetilde{d}(x,y) = 0$ *if and only if* $x = y$,
($\widetilde{d}_2$) $\widetilde{d}(x,y) = \widetilde{d}(y,x)$,
($\widetilde{d}_3$) $\widetilde{d}(x,z) \le \Omega(\widetilde{d}(x,y) + \widetilde{d}(y,z))$.

The pair $(X,\widetilde{d})$ is called a p-metric space, or an extended b-metric space.

A b-metric (see [1]) is a p-metric, with $\Omega(t) = bt$, for some fixed $b \ge 1$. If $\Omega \in \Psi$ then every metric is a p-metric, while a p-metric space becomes an ordinary metric space with $\Omega(t) = t$.

In [10], Sedghi et al. introduced the notion of an S-metric space as a generalization of a G-metric space (see [7]) as follows.

Definition 2. [10] *Let X be a nonempty set and $S : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:*

(S1) $S(x,y,z) = 0$ if and only if $x = y = z$;

(S2) $S(x,y,z) \le S(x,x,a) + S(y,y,a) + S(z,z,a)$, for all $x,y,z,a \in X$ (rectangle inequality).

The pair (X,S) is called an S-metric space.

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Example 1. [10] *Let $\mathbb{R}$ be the real line. Then $S(x,y,z) = |x-z| + |y-z|$, for all $x,y,z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric on $\mathbb{R}$ is called the usual S-metric on $\mathbb{R}$.*

Sedghi et al. in the paper [11] initiated the concept of S_b-metric spaces and studied some basic properties of such spaces.

The following is the definition of modified S-metric spaces which is a proper generalization of the notions of S-metric spaces and S_b-metric spaces.

Definition 3. *Let X be a nonempty set. A function $\widetilde{S} : X \times X \times X \to \mathbb{R}^+$ is called a modified S-metric ($\widetilde{S}$-metric) if there exists $\Omega \in \Psi$ such that for any $x,y,z,a \in X$, we have*

($\widetilde{S}1$) $\widetilde{S}(x,y,z) = 0$ *if and only if* $x = y = z$,

($\widetilde{S}2$) $\widetilde{S}(x,y,z) \leq \Omega[\widetilde{S}(x,x,a) + \widetilde{S}(y,y,a) + \widetilde{S}(z,z,a)]$ *(rectangle inequality).*

The pair $(X, \widetilde{S})$ is called a modified S-metric space or $\widetilde{S}$-metric space.

Definition 4. *A $\widetilde{S}$-metric is called symmetric if $\widetilde{S}(x,x,y) = \widetilde{S}(y,y,x)$, for all $x,y \in X$.*

An S-metric space is an $\widetilde{S}$-metric space for every $\Omega \in \Psi$ and every S_b-metric space with parameter $b \geq 1$ is an $\widetilde{S}$-metric space provided that $\Omega(t) = bt$.

Example 2. *Let $X = [0,\infty)$ and*

$$\widetilde{S}(x,y,z) = \begin{cases} 0, & x = y = z, \\ x+y+z, & \text{otherwise}. \end{cases}$$

Then $\widetilde{S}(x,y,z)$ is an $\widetilde{S}$-metric with $\Omega(t) = 2t$. But it is not symmetric.

Example 3. *Let (X,S) be an S-metric space. Then:*

1. $\widetilde{S}(x,y,z) = e^{S(x,y,z)} - 1$ *is an $\widetilde{S}$-metric with $\Omega(t) = e^t - 1$.*
2. $\widetilde{S}(x,y,z) = S(x,y,z)e^{S(x,y,z)}$ *is an $\widetilde{S}$-metric with $\Omega(t) = te^t$.*
3. $\widetilde{S}(x,y,z) = \Omega(S(x,y,z))$ *is an $\widetilde{S}$-metric and it is a symmetric for every $\Omega \in \Psi$.*

Definition 5. *Let X be an $\widetilde{S}$-metric space. A sequence $\{x_n\}$ in X is said to be:*

(1) $\widetilde{S}$-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m,n \geq n_0$, $\widetilde{S}(x_n,x_n,x_m) < \varepsilon$;

(2) $\widetilde{S}$-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n \geq n_0$, $\widetilde{S}(x,x,x_n) < \varepsilon$.

In these case we conclude for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\widetilde{S}(x_n,x_n,x) < \varepsilon$. Since,

$$\widetilde{S}(x_n,x_n,x) \leq \Omega(2\widetilde{S}(x_n,x_n,x_n) + \widetilde{S}(x,x,x_n) = \Omega(\widetilde{S}(x,x,x_n)),$$

we obtain

$$\lim_{n\longrightarrow\infty} \widetilde{S}(x_n,x_n,x) \leq \lim_{n\longrightarrow\infty} \Omega(\widetilde{S}(x,x,x_n)) = \Omega(0) = 0,$$

so, $\lim_{n\longrightarrow\infty} \widetilde{S}(x_n,x_n,x) = 0$. Also, similarly we conclude that

$$\lim_{n\longrightarrow\infty} \widetilde{S}(x_n,x,x_n) = \lim_{n\longrightarrow\infty} \widetilde{S}(x,x_n,x) = \lim_{n\longrightarrow\infty} \widetilde{S}(x,x_n,x_n) = 0.$$

(3) An $\widetilde{S}$-metric space X is called $\widetilde{S}$-complete, if every $\widetilde{S}$-Cauchy sequence is $\widetilde{S}$-convergent in X.

In general, an $\widetilde{S}$-metric mapping $\widetilde{S}(x,y,z)$ with nontrivial function Ω need not be jointly continuous in all its variables (see [6]). Thus, in some proofs we will need the following simple lemma about the $\widetilde{S}$-convergent sequences.

Lemma 1. *Let $(X,\widetilde{S})$ be an $\widetilde{S}$-metric space.*

1. Suppose that $\{x_n\}$ and $\{y_n\}$ are $\widetilde{S}$-convergent to x and y, respectively. Then we have

$$\Omega^{-2}[\widetilde{S}(x,x,y)] \leq \liminf_{n\longrightarrow\infty} \widetilde{S}(x_n,x_n,y_n) \leq \limsup_{n\longrightarrow\infty} \widetilde{S}(x_n,x_n,y_n) \leq \Omega^{2}[\widetilde{S}(x,x,y)].$$

In particular, if $x=y$, then we have $\lim_{n\to\infty} \widetilde{S}(x_n,x_n,y_n)=0$.

2. Suppose that $\widetilde{S}$-metric is symmetric and $\{x_n\}$ is $\widetilde{S}$-convergent to x and $z\in X$ is arbitrary. Then we have

$$\Omega^{-1}(\widetilde{S}(x,x,z)) \leq \liminf_{n\longrightarrow\infty} \widetilde{S}(x_n,x_n,z) \leq \limsup_{n\longrightarrow\infty} \widetilde{S}(x_n,x_n,z) \leq \Omega(\widetilde{S}(x,x,z)).$$

Proof. 1. Using the rectangle inequality in the $\widetilde{S}$-metric space it is easy to see that

$$\begin{aligned}\widetilde{S}(x,x,y) &\leq \Omega\big[2\widetilde{S}(x,x,x_n)+\widetilde{S}(y,y,x_n)\big]\\ &\leq \Omega\big[2\widetilde{S}(x,x,x_n)+\Omega\big[2\widetilde{S}(y,y,y_n)+\widetilde{S}(x_n,x_n,y_n)\big]\big]\end{aligned}$$

and

$$\begin{aligned}\widetilde{S}(x_n,x_n,y_n) &\leq \Omega\big[2\widetilde{S}(x_n,x_n,x)+\widetilde{S}(y_n,y_n,x)\big]\\ &\leq \Omega\big[2\widetilde{S}(x_n,x_n,x)+\Omega\big[2\widetilde{S}(y_n,y_n,y)+\widetilde{S}(x,x,y)\big]\big].\end{aligned}$$

Taking the lower limit as $n\to\infty$ in the first inequality and the upper limit as $n\to\infty$ in the second inequality we obtain the desired result.

2. Using the rectangle inequality we see that

$$\Omega^{-1}(\widetilde{S}(x,x,z)) \leq 2\widetilde{S}(x,x,x_n)+\widetilde{S}(x_n,x_n,z,)$$

and

$$\widetilde{S}(x_n,x_n,z) \leq \Omega(2\widetilde{S}(x_n,x_n,x)+\widetilde{S}(z,z,x)).$$

Taking the lower limit as $n\to\infty$ in the first inequality and the upper limit as $n\to\infty$ in the second inequality we obtain the desired result. □

2. THE MAIN RESULTS

Our main result is the following theorem.

Theorem 1. *Let $\widetilde{S}$-metric be symmetric and f,g,R,S be four self-mappings of a $\widetilde{S}$-complete metric space $(X,\widetilde{S})$ with $\Omega(a+b)\geq\Omega(a)+\Omega(b)$, for every $a,b\in[0,\infty)$ and:*
(i)*$f(X)\subseteq R(X)$, $g(X)\subseteq S(X)$ and $R(X)$ or $S(X)$ is a closed subset of X,*
(ii)

$$\Omega(\widetilde{S}(fx,fy,gz)) \leq q\max\left\{\begin{array}{l}\Omega^{-1}(\widetilde{S}(Sx,Sy,Rz)),\Omega^{-1}(\frac{\widetilde{S}(Sx,Sx,fx)+\widetilde{S}(Sy,Sy,fy)+\widetilde{S}(Rz,Rz,gz)}{3}),\\ \frac{1}{3}\Omega^{-1}(\frac{\widetilde{S}(Sx,Sx,gz)+\widetilde{S}(Sy,Sy,gz)+\widetilde{S}(Rz,Rz,fx)+\widetilde{S}(Rz,Rz,fy)}{2})\end{array}\right\} \tag{1}$$

for every $x,y,z\in X$, where $q\in(0,1)$,
(iii) *the pair (f,S) and (g,R) are weak compatible.*
Then f,g,R and S have a unique common fixed point in X.

Proof. Let $x_0\in X$ be an arbitrary point . By (i), there exist $x_1,x_2\in X$ such that

$$fx_0=Rx_1=y_0 \text{ and } gx_1=Sx_2=y_1.$$

Inductively, construct sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = Rx_{2n+1} \text{ and } y_{2n+1} = gx_{2n+1} = Sx_{2n+2},$$

for $n = 0, 1, 2, \dots$

Now, we prove that $\{y_n\}$ is a $\widetilde{S}$-Cauchy sequence. Let $\widetilde{S}_m = \widetilde{S}(y_m, y_m, y_{m+1})$. Then, we have

$$\begin{aligned}
&\Omega(\widetilde{S}_{2n}) = \Omega(\widetilde{S}(y_{2n}, y_{2n}, y_{2n+1})) = \Omega(\widetilde{S}(fx_{2n}, fx_{2n}, gx_{2n+1}))\\
\leq\ & q\max\begin{pmatrix} \Omega^{-1}(\widetilde{S}(Sx_{2n}, Sx_{2n}, Rx_{2n+1})), \\ \Omega^{-1}(\frac{\widetilde{S}(Sx_{2n},Sx_{2n},fx_{2n})+\widetilde{S}(Sx_{2n},Sx_{2n},fx_{2n})+\widetilde{S}(Rx_{2n+1},Rx_{2n+1},gx_{2n+1})}{3}), \\ \frac{1}{3}\Omega^{-1}(\frac{2\widetilde{S}(Sx_{2n},Sx_{2n},gx_{2n+1})+2\widetilde{S}(Rx_{2n+1},Rx_{2n+1},fx_{2n})}{2}) \end{pmatrix}\\
=\ & q\max\begin{pmatrix} \Omega^{-1}(\widetilde{S}(y_{2n-1}, y_{2n-1}, y_{2n})), \\ \Omega^{-1}(\frac{\widetilde{S}(y_{2n-1},y_{2n-1},y_{2n})+\widetilde{S}(y_{2n-1},y_{2n-1},y_{2n})+\widetilde{S}(y_{2n},y_{2n},y_{2n+1})}{3}), \\ \frac{1}{3}\Omega^{-1}(\widetilde{S}(y_{2n-1}, y_{2n-1}, y_{2n+1}) + \widetilde{S}(y_{2n}, y_{2n}, y_{2n})) \end{pmatrix}\\
=\ & q\max\begin{pmatrix} \Omega^{-1}(\widetilde{S}_{2n-1}), \Omega^{-1}(\frac{2\widetilde{S}_{2n-1}+\widetilde{S}_{2n}}{3}), \\ \frac{1}{3}\Omega^{-1}(\widetilde{S}(y_{2n-1}, y_{2n-1}, y_{2n+1})) \end{pmatrix}.
\end{aligned}$$

Also,

$$\widetilde{S}(y_{2n-1}, y_{2n-1}, y_{2n+1}) \leq \Omega(2\widetilde{S}(y_{2n-1}, y_{2n-1}, y_{2n}) + \widetilde{S}(y_{2n}, y_{2n}, y_{2n+1})),$$

that is

$$\frac{1}{3}\Omega^{-1}(\widetilde{S}(y_{2n-1}, y_{2n-1}, y_{2n+1})) \leq \frac{1}{3}(2\widetilde{S}(y_{2n-1}, y_{2n-1}, y_{2n}) + \widetilde{S}(y_{2n}, y_{2n}, y_{2n+1})).$$

We prove that $\widetilde{S}_{2n} \leq \widetilde{S}_{2n-1}$, for every $n \in \mathbb{N}$. If $\widetilde{S}_{2n} > \widetilde{S}_{2n-1}$ for some $n \in \mathbb{N}$, then we get

$$\frac{1}{3}\Omega^{-1}(\widetilde{S}(y_{2n-1}, y_{2n-1}, y_{2n+1})) \leq \widetilde{S}(y_{2n}, y_{2n}, y_{2n+1}).$$

Hence by above inequality we have $\widetilde{S}_{2n} \leq \Omega(\widetilde{S}_{2n}) < q\widetilde{S}_{2n}$, is a contradiction. Now, if $m = 2n + 1$, then

$$\begin{aligned}
&\Omega(\widetilde{S}_{2n+1}) = \Omega(\widetilde{S}(y_{2n+1}, y_{2n+1}, y_{2n+2})) = \Omega(\widetilde{S}(y_{2n+2}, y_{2n+2}, y_{2n+1}))\\
=\ & \Omega(\widetilde{S}(fx_{2n+2}, fx_{2n+2}, gx_{2n+1}))\\
\leq\ & q\max\begin{Bmatrix} \Omega^{-1}(\widetilde{S}(Sx_{2n+2}, Sx_{2n+2}, Rx_{2n+1})), \\ \Omega^{-1}(\frac{\widetilde{S}(Sx_{2n+2},Sx_{2n+2},fx_{2n+2})+\widetilde{S}(Sx_{2n+2},Sx_{2n+2},fx_{2n+2})+\widetilde{S}(Rx_{2n+1},Rx_{2n+1},gx_{2n+1})}{3}), \\ \frac{1}{3}\Omega^{-1}(\frac{2\widetilde{S}(Sx_{2n+2},Sx_{2n+2},gx_{2n+1})+2\widetilde{S}(Rx_{2n+1},Rx_{2n+1},fx_{2n+2})}{2}) \end{Bmatrix}\\
=\ & q\max\begin{Bmatrix} \Omega^{-1}(\widetilde{S}(y_{2n+1}, y_{2n+1}, y_{2n})), \\ \Omega^{-1}(\frac{\widetilde{S}(y_{2n+1},y_{2n+1},y_{2n+2})+\widetilde{S}(y_{2n+1},y_{2n+1},y_{2n+2})+\widetilde{S}(y_{2n},y_{2n},y_{2n+1})}{3}), \\ \frac{1}{3}\Omega^{-1}(\widetilde{S}(y_{2n+1}, y_{2n+1}, y_{2n+1}) + \widetilde{S}(y_{2n}, y_{2n}, y_{2n+2})) \end{Bmatrix}\\
=\ & q\max\begin{Bmatrix} \Omega^{-1}(\widetilde{S}_{2n}), \Omega^{-1}(\frac{\widetilde{S}_{2n}+2\widetilde{S}_{2n+1}}{3}), \\ \frac{1}{3}\Omega^{-1}(\widetilde{S}(y_{2n}, y_{2n}, y_{2n+2})) \end{Bmatrix}.
\end{aligned}$$

From

$$\widetilde{S}(y_{2n}, y_{2n}, y_{2n+2}) \leq \Omega(2\widetilde{S}(y_{2n}, y_{2n}, y_{2n+1}) + \widetilde{S}(y_{2n+1}, y_{2n+1}, y_{2n+2})),$$

we obtain

$$\frac{1}{3}\Omega^{-1}(\widetilde{S}(y_{2n}, y_{2n}, y_{2n+2})) \leq \frac{1}{3}(2\widetilde{S}(y_{2n}, y_{2n}, y_{2n+1}) + \widetilde{S}(y_{2n+1}, y_{2n+1}, y_{2n+2})).$$

Similarly, if $\widetilde{S}_{2n+1} > \widetilde{S}_{2n}$ for some $n \in \mathbb{N}$, then we get

$$\frac{1}{3}\Omega^{-1}(\widetilde{S}(y_{2n}, y_{2n}, y_{2n+2})) \leq \widetilde{S}(y_{2n+1}, y_{2n+1}, y_{2n+2}).$$

Hence by above inequality we have $\widetilde{S}_{2n+1} < q\widetilde{S}_{2n+1}$, is a contradiction.

Hence for every $n \in \mathbb{N}$ we have

$$\widetilde{S}_n \leq q\widetilde{S}_{n-1} < \widetilde{S}_{n-1}. \tag{2}$$

That is, $\{\widetilde{S}(y_n, y_n, y_{n+1})\}$ is a decreasing sequence. Then there exists $r \geq 0$ such that $\lim_{n\to\infty} \widetilde{S}(y_n, y_n, y_{n+1}) = r$. We will prove that $r = 0$. Suppose on contrary, that $r > 0$. Then, letting $n \to \infty$, from (2) we have

$$r \leq qr \leq r,$$

which implies that $q = 1$, a contradiction. Hence, the assumption that $r > 0$ is false. That is,

$$\lim_{n\to\infty} \widetilde{S}(y_n, y_n, y_{n+1}) = 0. \tag{3}$$

Now, we prove that the sequence $\{y_n\}$ is an $\widetilde{S}$-Cauchy sequence. Suppose the contrary, *i.e.*, there exists $\varepsilon > 0$ for which we can find two subsequences $\{y_{2m_i}\}$ and $\{y_{2n_i}\}$ of $\{y_n\}$ such that $2n_i$ is the smallest index for which

$$2n_i > 2m_i > i \text{ and } \widetilde{S}(y_{2m_i}, y_{2m_i}, y_{2n_i}) \geq \varepsilon. \tag{4}$$

This means that

$$\widetilde{S}(y_{2m_i}, y_{2m_i}, y_{2n_i-1}) < \varepsilon. \tag{5}$$

From (1), we get

$$\begin{aligned}
&\Omega(\widetilde{S}(y_{2n_i}, y_{2n_i}, y_{2m_i+1})) = \Omega(\widetilde{S}(fx_{2n_i}, fx_{2n_i}, gx_{2m_i+1}))\\
\leq\ & q\max\left\{\begin{array}{l}\Omega^{-1}(\widetilde{S}(Sx_{2n_i}, Sx_{2n_i}, Rx_{2m_i+1})),\\ \Omega^{-1}(\frac{\widetilde{S}(Sx_{2n_i},Sx_{2n_i},fx_{2n_i})+\widetilde{S}(Sx_{2n_i},Sx_{2n_i},fx_{2n_i})+\widetilde{S}(Rx_{2m_i+1},Rx_{2m_i+1},gx_{2m_i+1})}{3}),\\ \frac{1}{3}\Omega^{-1}(\frac{2\widetilde{S}(Sx_{2n_i},Sx_{2n_i},gx_{2m_i+1})+2\widetilde{S}(Rx_{2m_i+1},Rx_{2m_i+1},fx_{2n_i})}{2})\end{array}\right\}\\
=\ & q\max\left\{\begin{array}{l}\Omega^{-1}(\widetilde{S}(y_{2n_i-1}, y_{2n_i-1}, y_{2m_i})),\\ \Omega^{-1}(\frac{\widetilde{S}(y_{2n_i-1},y_{2n_i-1},y_{2n_i})+\widetilde{S}(y_{2n_i-1},y_{2n_i-1},y_{2n_i})+\widetilde{S}(y_{2m_i},y_{2m_i},y_{2m_i+1})}{3}),\\ \frac{1}{3}\Omega^{-1}(\widetilde{S}(y_{2n_i-1}, y_{2n_i-1}, y_{2m_i+1}) + \widetilde{S}(y_{2m_i}, y_{2m_i}, y_{2n_i})).\end{array}\right\}
\end{aligned}$$

Since

$$\begin{aligned}
\varepsilon \leq \widetilde{S}(y_{2n_i}, y_{2n_i}, y_{2m_i}) &\leq \Omega(2\widetilde{S}(y_{2n_i}, y_{2n_i}, y_{2n_i-1}) + \widetilde{S}(y_{2m_i}, y_{2m_i}, y_{2n_i-1}))\\
&\leq \Omega(2\widetilde{S}(y_{2n_i}, y_{2n_i}, y_{2n_i-1}) + \varepsilon),
\end{aligned}$$

taking the upper limit as $i \to \infty$, we get

$$\varepsilon \leq \limsup_{i\to\infty} \widetilde{S}(y_{2n_i}, y_{2n_i}, y_{2m_i}) \leq \Omega(2\limsup_{i\to\infty} \widetilde{S}(y_{2n_i}, y_{2n_i}, y_{2n_i-1}) + \varepsilon) \leq \Omega(\varepsilon).$$

Also, since

$$\begin{aligned}
&\widetilde{S}(y_{2n_i-1}, y_{2n_i-1}, y_{2m_i+1}) = \widetilde{S}(y_{2m_i+1}, y_{2m_i+1}, y_{2n_i-1})\\
\leq\ & \Omega(2\widetilde{S}(y_{2m_i+1}, y_{2m_i+1}, y_{2m_i}) + \widetilde{S}(y_{2n_i-1}, y_{2n_i-1}, y_{2m_i}))\\
\leq\ & \Omega(2\widetilde{S}(y_{2m_i+1}, y_{2m_i+1}, y_{2m_i}) + \varepsilon)
\end{aligned}$$

taking the upper limit as $i \to \infty$, we get

$$\limsup_{i\to\infty} \widetilde{S}(y_{2n_i-1}, y_{2n_i-1}, y_{2m_i+1}) \leq \Omega(2\limsup_{i\to\infty} \widetilde{S}(y_{2m_i+1}, y_{2m_i+1}, y_{2m_i}) + \varepsilon) \leq \Omega(\varepsilon).$$

Now, since

$\varepsilon \le \widetilde{S}(y_{2n_i}, y_{2n_i}, y_{2m_i}) = \widetilde{S}(y_{2m_i}, y_{2m_i}, y_{2n_i}) \le \Omega(2\widetilde{S}(y_{2m_i}, y_{2m_i}, y_{2m_i+1}) + \widetilde{S}(y_{2n_i}, y_{2n_i}, y_{2m_i+1}))$,

we get

$$\begin{aligned} \varepsilon &\le \limsup_{i\to\infty} \widetilde{S}(y_{2n_i}, y_{2n_i}, y_{2m_i}) \\ &\le \Omega(2\limsup_{i\to\infty} \widetilde{S}(y_{2m_i}, y_{2m_i}, y_{2m_i+1}) + \limsup_{i\to\infty} \widetilde{S}(y_{2n_i}, y_{2n_i}, y_{2m_i+1})) \\ &\le \Omega(0 + \limsup_{i\to\infty} \widetilde{S}(y_{2n_i}, y_{2n_i}, y_{2m_i+1})). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \varepsilon &\le \Omega(\limsup_{i\to\infty} \widetilde{S}(y_{2n_i}, y_{2n_i}, y_{2m_i})) \\ &\le q \max \left\{ \begin{array}{l} \Omega^{-1}(\limsup_{i\to\infty} \widetilde{S}(y_{2n_i-1}, y_{2n_i-1}, y_{2m_i})), \\ \Omega^{-1}(\limsup_{i\to\infty} \frac{\widetilde{S}(y_{2n_i-1}, y_{2n_i-1}, y_{2n_i}) + \widetilde{S}(y_{2n_i-1}, y_{2n_i-1}, y_{2n_i}) + \widetilde{S}(y_{2m_i}, y_{2m_i}, y_{2m_i+1})}{3}), \\ \frac{1}{3}\Omega^{-1}(\limsup_{i\to\infty} \widetilde{S}(y_{2n_i-1}, y_{2n_i-1}, y_{2m_i+1}) + \limsup_{i\to\infty} \widetilde{S}(y_{2m_i}, y_{2m_i}, y_{2n_i})). \end{array} \right\} \\ &\le q \max\{\Omega^{-1}(\varepsilon), \Omega^{-1}(0), \frac{1}{3}\Omega^{-1}(2\Omega(\varepsilon))\} \le q\frac{1}{3}\Omega^{-1}(2\Omega(\varepsilon)) < \frac{1}{3}\Omega^{-1}(2\Omega(\varepsilon)), \end{aligned}$$

this implies that

$$\Omega(3\varepsilon) < 2\Omega(\varepsilon).$$

Since, $\Omega(a+b) \ge \Omega(a)+\Omega(b)$, we have $3\Omega(\varepsilon) \le \Omega(3\varepsilon) < 2\Omega(\varepsilon)$, a contradiction. Therefore, $\{y_n\}$ is an $\widetilde{S}$-Cauchy sequence. Because of $\widetilde{S}$-completeness of X yields that $\{y_n\}$ $\widetilde{S}$-converges to a point $y \in X$. That is, $\lim_{n\to\infty} y_n = y$

$$\begin{aligned} \lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Rx_{2n+1} \\ &= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y. \end{aligned}$$

Let $R(X)$ be a closed subset of X, hence there exists $u \in X$ such that $Ru = y$. We prove that $gu = y$. Since

$$\begin{aligned} &\Omega(\widetilde{S}(fx_{2n}, fx_{2n}, gu)) \\ \le\ & q \max \left\{ \begin{array}{l} \Omega^{-1}(\widetilde{S}(Sx_{2n}, Sx_{2n}, Ru)), \Omega^{-1}(\frac{\widetilde{S}(Sx_{2n}, Sx_{2n}, fx_{2n}) + \widetilde{S}(Sx_{2n}, Sx_{2n}, fx_{2n}) + \widetilde{S}(Ru, Ru, gu)}{3}), \\ \frac{1}{3}\Omega^{-1}(\frac{\widetilde{S}(Sx_{2n}, Sx_{2n}, gu) + \widetilde{S}(Sx_{2n}, Sx_{2n}, gu) + \widetilde{S}(Ru, Ru, fx_{2n}) + \widetilde{S}(Ru, Ru, fx_{2n})}{2}) \end{array} \right\}, \end{aligned}$$

taking the upper limit as $n \to \infty$ by Lemma 1, we get

$$\begin{aligned} \widetilde{S}(y, y, gu) &= \Omega(\Omega^{-1}(\widetilde{S}(y, y, gu))) \\ &\le \Omega(\limsup_{n\to\infty} \widetilde{S}(y, y, gu)) \\ &\le q \max \left\{ \begin{array}{l} 0, \Omega^{-1}(\frac{0+0+\widetilde{S}(y,y,gu)}{3}), \\ \frac{1}{3}\Omega^{-1}(\frac{\limsup_{n\to\infty} \widetilde{S}(Sx_{2n}, Sx_{2n}, gu) + \limsup_{n\to\infty} \widetilde{S}(Sx_{2n}, Sx_{2n}, gu) + 0 + 0}{2}) \end{array} \right\} \\ &\le q \max\left\{ 0, \frac{\widetilde{S}(y,y,gu)}{3}, \frac{1}{3}\Omega^{-1}(\Omega(\widetilde{S}(y, y, gu))) \right\} \\ &\le q \max\left\{ 0, \frac{\widetilde{S}(y,y,gu)}{3}, \frac{\widetilde{S}(y,y,gu)}{3} \right\}. \end{aligned}$$

If $\widetilde{S}(y, y, gu) > 0$, then we have $\widetilde{S}(y, y, gu) < \frac{q}{3}\widetilde{S}(y, y, gu)$ is a contradiction. Thus $gu = y$. By the weak compatibility of the pair (R, g) we have $gRu = Rgu$. Hence $gy = Ry$. We prove that $gy = y$, if $gy \neq y$, then

$$\begin{aligned} & \Omega(\widetilde{S}(fx_{2n}, fx_{2n}, gy)) \\ \leq \quad & q \max \left\{ \begin{array}{l} \Omega^{-1}(\widetilde{S}(Sx_{2n}, Sx_{2n}, Ry)), \Omega^{-1}(\frac{\widetilde{S}(Sx_{2n},Sx_{2n},fx_{2n})+\widetilde{S}(Sx_{2n},Sx_{2n},fx_{2n})+\widetilde{S}(Ry,Ry,gy)}{3}), \\ \frac{1}{3}\Omega^{-1}(\frac{\widetilde{S}(Sx_{2n},Sx_{2n},gy)+\widetilde{S}(Sx_{2n},Sx_{2n},gy)+\widetilde{S}(Ry,Ry,fx_{2n})+\widetilde{S}(Ry,Ry,fx_{2n})}{2}) \end{array} \right\}. \end{aligned}$$

Similarly, taking the upper limit as $n \to \infty$ by Lemma 1, we get

$$\begin{aligned} \widetilde{S}(y, y, gy) \quad & \leq \quad q \max \left\{ \begin{array}{l} \Omega^{-1}(\Omega(\widetilde{S}(y, y, gy))), \Omega^{-1}(\frac{0+0+0}{3}), \\ \frac{1}{3}\Omega^{-1}(\frac{\Omega(\widetilde{S}(y,y,gy))+\Omega(\widetilde{S}(y,y,gy))+\Omega(\widetilde{S}(gy,gy,y))+\Omega(\widetilde{S}(gy,gy,y))}{2}) \end{array} \right\} \\ & \leq \quad q \max \left\{ \widetilde{S}(y, y, gy), \frac{1}{3}\Omega^{-1}(\Omega(2\widetilde{S}(y, y, gy))) \right\} \\ & \leq \quad \frac{2}{3}\widetilde{S}(y, y, gy) \end{aligned}$$

a contradiction. Therefore, $Ry = gy = y$, that is, y is a common fixed of R, g .
Since $y = gy \in g(X) \subseteq S(X)$, hence there exists $v \in X$ such that $Sv = y$. We prove that $fv = y$. For

$$\begin{aligned} & \Omega(\widetilde{S}(fv, fv, gx_{2n+1})) \\ \leq \quad & q \max \left\{ \begin{array}{l} \Omega^{-1}(\widetilde{S}(Sv, Sv, Rx_{2n+1})), \Omega^{-1}(\frac{\widetilde{S}(Sv,Sv,fv)+\widetilde{S}(Sv,Sv,fv)+\widetilde{S}(Rx_{2n+1},Rx_{2n+1},gx_{2n+1})}{3}), \\ \frac{1}{3}\Omega^{-1}(\frac{\widetilde{S}(Sv,Sv,gx_{2n+1})+\widetilde{S}(Sv,Sv,gx_{2n+1})+\widetilde{S}(Rx_{2n+1},Rx_{2n+1},fv)+\widetilde{S}(Rx_{2n+1},Rx_{2n+1},fv)}{2}) \end{array} \right\}, \end{aligned}$$

similarly, taking the upper limit as $n \to \infty$ by Lemma 1, we get

$$\widetilde{S}(fv, fv, y) \leq q \max \left\{ 0, \tfrac{2}{3}\widetilde{S}(y, y, fv), \tfrac{1}{3}\Omega^{-1}(\Omega(\widetilde{S}(y, y, fv))) \right\} \leq \frac{2}{3}\widetilde{S}(fv, fv, y)$$

Thus $fv = y$. By the weak compatibility of the pair (f, S) we have $Sfv = fSv$. Hence $fy = Sy$. We prove that $fy = y$, if $fy \neq y$, then

$$\begin{aligned} & \Omega(\widetilde{S}(fy, fy, gy)) \\ \leq \quad & q \max \left\{ \begin{array}{l} \Omega^{-1}(\widetilde{S}(Sy, Sy, Ry)), \Omega^{-1}(\frac{\widetilde{S}(Sy,Sy,fy)+\widetilde{S}(Sy,Sy,fy)+\widetilde{S}(Ry,Ry,gy)}{3}), \\ \frac{1}{3}\Omega^{-1}(\frac{\widetilde{S}(Sy,Sy,gy)+\widetilde{S}(Sy,Sy,gy)+\widetilde{S}(Ry,Ry,fy)+\widetilde{S}(Ry,Ry,fy)}{2}) \end{array} \right\}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \widetilde{S}(fy, fy, y) \quad & \leq \quad q \max \left\{ \Omega^{-1}(\widetilde{S}(fy, fy, y)), 0, \tfrac{1}{3}\Omega^{-1}(2\widetilde{S}(fy, fy, y)) \right\} \\ & \leq \quad q\widetilde{S}(fy, fy, y), \end{aligned}$$

a contradiction. Therefore,$fy = Sy = y$, that is, y is a common fixed of f, S. That is,

$$fy = Sy = gy = Ry = y.$$

To prove uniqueness, let v be another common fixed point of f, g, R, S.
If $\widetilde{S}(y, y, v) > 0$, hence

$$\begin{aligned} & \Omega(\widetilde{S}(y, y, v)) = \Omega(\widetilde{S}(fy, fy, gv)) \\ \leq \quad & q \max \left\{ \begin{array}{l} \Omega^{-1}(\widetilde{S}(Sy, Sy, Rv)), \Omega^{-1}(\frac{\widetilde{S}(Sy,Sy,fy)+\widetilde{S}(Sy,Sy,fy)+\widetilde{S}(Rv,Rv,gv)}{3}), \\ \frac{1}{3}\Omega^{-1}(\frac{\widetilde{S}(Sy,Sy,gv)+\widetilde{S}(Sy,Sy,gv)+\widetilde{S}(Rv,Rv,fy)+\widetilde{S}(Rv,Rv,fy)}{2}) \end{array} \right\} \\ \leq \quad & q\widetilde{S}(y, y, v), \end{aligned}$$

a contradiction. Therefore, $y = v$ is the unique common fixed point of self-maps f, g, R, S.

□

Corollary 1. *Let (X,S) be an S_b-metric for $b \geq 1$ and it be symmetric. Let f,g,R,T be four self-mappings of a S_b-complete metric space (X,S) with:*
(i)*$f(X) \subseteq R(X)$, $g(X) \subseteq T(X)$ and $R(X)$ or $T(X)$ is a closed subset of X,*
(ii)

$$S(fx,fy,gz)) \leq \frac{q}{b^2} \max \left\{ \begin{array}{c} S(Tx,Ty,Rz), \frac{S(Tx,Tx,fx)+S(Ty,Ty,fy)+S(Rz,Rz,gz)}{3}, \\ \frac{S(Tx,Tx,gz)+S(Ty,Ty,gz)+S(Rz,Rz,fx)+S(Rz,Rz,fy)}{6} \end{array} \right\},$$

for every $x,y,z \in X$, where $q \in (0,1)$,
(iii) *the pair (f,S) and (g,R) are weak compatible.*
Then f,g,R and S have a unique common fixed point in X.

Proof. It is enough set $\Omega(t) = bt$ in Theorem 1. □

Corollary 2. *Let $\widetilde{S}$-metric be symmetric and g,R be two self-mappings of a $\widetilde{S}$-complete metric space $(X,\widetilde{S})$ with $\Omega(a+b) \geq \Omega(a)+\Omega(b)$ for every $a,b \in [0,\infty)$ and*
(i)*$g(X) \subseteq R(X)$ and $R(X)$ is a closed subset of X,*
(ii)

$$\Omega(\widetilde{S}(gx,gy,gz)) \leq q \max \left\{ \begin{array}{c} \Omega^{-1}(\widetilde{S}(Rx,Ry,Rz)), \Omega^{-1}(\frac{\widetilde{S}(Rx,Rx,gx)+\widetilde{S}(Ry,Ry,gy)+\widetilde{S}(Rz,Rz,gz)}{3}), \\ \frac{1}{3}\Omega^{-1}(\frac{\widetilde{S}(Rx,Rx,gz)+\widetilde{S}(Ry,Ry,gz)+\widetilde{S}(Rz,Rz,gx)+\widetilde{S}(Rz,Rz,gy)}{2}) \end{array} \right\}$$

for every $x,y,z \in X$, where $q \in (0,1)$, (iii) *the pair (g,R) are weak compatible. Then g and R have a unique common fixed point in X.*

Corollary 3. *Let $\widetilde{S}$-metric be symmetric and f,g be two self-mappings of a $\widetilde{S}$-complete metric space $(X,\widetilde{S})$ with $\Omega(a+b) \geq \Omega(a)+\Omega(b)$ for every $a,b \in [0,\infty)$ and*

$$\Omega(\widetilde{S}(fx,fy,gz)) \leq q \max \left\{ \begin{array}{c} \Omega^{-1}(\widetilde{S}(x,y,z)), \Omega^{-1}(\frac{\widetilde{S}(x,x,gx)+\widetilde{S}(y,y,fy)+\widetilde{S}(z,z,gz)}{3}), \\ \frac{1}{3}\Omega^{-1}(\frac{\widetilde{S}(x,x,gz)+\widetilde{S}(y,y,gz)+\widetilde{S}(z,z,fx)+\widetilde{S}(z,z,fy)}{2}) \end{array} \right\}$$

for every $x,y,z \in X$, where $q \in (0,1)$. Then f and g have a unique common fixed point in X.

Corollary 4. *Let $\widetilde{S}$-metric be symmetric and T be a self-mapping of a $\widetilde{S}$-complete metric space $(X,\widetilde{S})$ with $\Omega(a+b) \geq \Omega(a)+\Omega(b)$ for every $a,b \in [0,\infty)$ and*

$$\Omega(\widetilde{S}(Tx,Ty,Tz)) \leq q \max \left\{ \begin{array}{c} \Omega^{-1}(\widetilde{S}(x,y,z)), \Omega^{-1}(\frac{\widetilde{S}(x,x,Tx)+\widetilde{S}(y,y,Ty)+\widetilde{S}(z,z,Tz)}{3}), \\ \frac{1}{3}\Omega^{-1}(\frac{\widetilde{S}(x,x,Tz)+\widetilde{S}(y,y,Tz)+\widetilde{S}(z,z,Tx)+\widetilde{S}(z,z,Ty)}{2}) \end{array} \right\}$$

for every $x,y,z \in X$, where $q \in (0,1)$. Then T has a unique fixed point in X.

Example 4. *Let $X = L^p(0,\infty), p > 1$ and the usual norm, that is,*

$$||f||_p = (\int_0^\infty |f(t)|^p dt)^{\frac{1}{p}}.$$

Consider $\widetilde{S} : X^3 \to [0,\infty)$ defined by

$$\widetilde{S}(f,g,h) = ||f-h||_p + ||g-h||_p,$$

for every $f,g,h \in X$ and $\Omega(t) = t$. Let $T : X \to X$ be a mapping such that

$$Tf(x) = \frac{q}{x}\int_0^x f(t)dt,$$

where $q \in \left(0, 1-\frac{1}{p}\right)$. Then T has a unique fixed point.

Proof. It is easy to see that, $(X, \widetilde{S})$ is a complete symmetric $\widetilde{S}$-metric space. Now, from

$$||Tf||_p \leq \frac{qp}{p-1}||f||_p,$$

we have

$$\begin{aligned}
\widetilde{S}(Tf, Tg, Th) &= ||Tf - Th||_p + ||Tg - Th||_p \\
&= ||T(f-h)||_p + ||T(g-h)||_p \\
&\leq \frac{qp}{p-1}(||f-h||_p + ||g-h||_p) = \frac{qp}{p-1}\widetilde{S}(f,g,h) \\
&\leq \frac{qp}{p-1}\max\left\{ \begin{array}{c} \widetilde{S}(f,g,h), \frac{\widetilde{S}(f,f,Tf)+\widetilde{S}(g,g,Tg)+\widetilde{S}(h,h,Th)}{3}, \\ \frac{1}{3}\frac{\widetilde{S}(f,f,Th)+\widetilde{S}(g,g,Th)+\widetilde{S}(h,h,Tf)+\widetilde{S}(h,h,Tg)}{2} \end{array} \right\},
\end{aligned}$$

for all $f, g, h \in X$. Therefore, all conditions of Corollary 4 are satisfied. Thus, T has a unique fixed point in $X = L^p(0, \infty)$, i.e there exists a unique $f \in X$ such that $Tf = f$. $\square$

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