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# AN APPLICATION OF THE ADMISSIBILITY TYPES IN $b$-METRIC SPACES

MARIA DOBRIȚOIU

Abstract. In this paper we present a fixed point theorem for the solution of the nonlinear Fredholm integral equation

$$x(t) = \int_a^b K(t, r, x(r), x(g(r)))dr + f(t),\ t \in [a, b],$$

as an application of a result of S. Radenović et al., in [14]. In order to obtain this existence result, we used also, the notions of admissibility types defined on a $b$-metric space, in [17].

## 1. Introduction

We consider the nonlinear Fredholm integral equation with modified argument:

$$x(t) = \int_a^b K(t, r, x(r), x(g(r)))dr + f(t),\ t \in [a, b] \tag{1}$$

where $K : [a, b] \times [a, b] \times \mathbb{R}^2 \to \mathbb{R}$, $g : [a, b] \to [a, b]$ and $f : [a, b] \to \mathbb{R}$.

Some properties of the solution of this integral equation can be found in the papers [2], [4]-[6]. Also, the properties of the solution of some similar equations were studied in [1], [3] [7]-[10], [12], [13] and [15].

In this paper we set out to present another property of the solution of this integral equation, established on the basis of some notions given in [14] and [17] and of a theorem in [17].

In [17], Sintunavarat defined the $\alpha$-admissible mapping and the $\alpha$-admissible mapping type S and attached to these two types of mappings the sets $\mathcal{A}(X, \alpha)$ and $\mathcal{A}_s(X, \alpha)$.

In their paper, [14], Radenović et al., defined the $\alpha_{s,\varepsilon}$-contraction mapping, with $\varepsilon > 1$, on a $b$-metric space with coefficient $s \geq 1$. They used this mapping to prove two fixed point theorems for an $\alpha_{s,\varepsilon}$-contraction mapping, with $\varepsilon > 1$, in a $b$-complete $b$-metric space with coefficient $s > 1$. Also, they applied the obtained results to study the existence of the solution of the nonlinear Fredholm integral equation:

$$x(t) = \int_a^b K(t, r, x(r))dr + f(t),$$

where $a, b \in \mathbb{R}$ such that $a < b$; $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ are given mappings and $x \in C[a, b]$ is the unknown mapping.

In what follows, we will use these notions and results to establish some new conditions of existence of the solution of the nonlinear Fredholm integral equation with modified argument (1).

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This paper contains three sections. In section 2, "Basic notions and results", we recall some definitions and results concerning the notions introduced in [14] and [17], that we apply in order to obtain our main result in section 3, i.e. a fixed point theorem for the solution of the nonlinear Fredholm integral equation with modified argument (1).

## 2. Basic notions and results

Let $f : X \to X$ a mapping defined on a nonempty set, $X$.

In this paper we shall use the following notations:

$C[a, b] := \{f : [a, b] \to \mathbb{R} | f \text{ is a continuous function}\}$

$F_f := \{x \in X | f(x) = x\}$ - the fixed points set of the mapping $f$.

The notations used in this paper will be completed during this section.

In what follows, we present the following notions: $b$-metric on a nonempty set $X$, $b$-metric space, altering distance function, $\alpha$-admissible mapping, $\alpha$-admissible mapping type S, $(\alpha, \psi, \varphi)_s$-contraction mapping.

Let $X$ be a nonempty set and let $s \geq 1$ be a given real number.

**Definition 1** ([14])**.** *A function $d : X \times X \to [0, +\infty)$ is a b-metric on X, if the following conditions hold:*

*(i)* $d(x, y) = 0$ *if and only if* $x = y$, *for all* $x, y \in X$*;*
*(ii)* $d(x, y) = d(y, x)$, *for all* $x, y \in X$*;*
*(iii)* $d(x, y) \leq s[d(x, z) + d(z, y)]$, *for all* $x, y, z \in X$.

The pair $(X, d)$ is called a $b$-metric space with coefficient $s \geq 1$.

**Remark 1.** *If $s = 1$, then the b-metric space is the usual metric space.*

**Remark 2.** *The notions of b-convergence, b-completeness and b-Cauchy sequence in a b-metric space, can be found by the reader in* [17].

**Definition 2** ([11])**.** *A function $\varphi : [0, +\infty) \to [0, +\infty)$ is an altering distance function, if it has the following two properties:*

*(i)* $\varphi$ *is continuous and nondecreasing;*
*(ii)* $\varphi(t) = 0$ *if and only if* $t = 0$.

Let $X$ be a nonempty set and let $\alpha : X \times X \to [0, +\infty)$ be a given mapping.

**Definition 3** ([16])**.** *A mapping $f : X \to X$ is an $\alpha$-admissible mapping if it satisfies the following condition:*

$$x, y \in X,\ \alpha(x, y) \geq 1 \implies \alpha(f(x), f(y)) \geq 1.$$

In addition, let $s$ be a given real number, such that $s \geq 1$.

**Definition 4** ([17])**.** *A mapping $f : X \to X$ is an $\alpha$-admissible mapping type S if it satisfies the following condition:*

$$x, y \in X,\ \alpha(x, y) \geq s \implies \alpha(f(x), f(y)) \geq s.$$

The sets of these two types of $\alpha$-admissible mappings were denoted

$$\mathcal{A}(X, \alpha) = \{f : X \to X / f \text{ is an } \alpha - \text{admissible mapping}\}$$

and with coefficient $s$

$$\mathcal{A}_s(X, \alpha) = \{f : X \to X / f \text{ is an } \alpha - \text{admissible mapping type } S\},$$

which is added to the notations used in this paper.

Also, in [17], were introduced the following two notions: weak $\alpha$-admissible mapping and respectively, weak $\alpha$-admissible mapping type S, that we present below.

**Definition 5** ([14]). *A mapping $f : X \to X$ is a weak $\alpha$-admissible mapping if the following condition holds:*

$$x \in X,\ \alpha(x, f(x)) \geq 1 \implies \alpha(f(x), f(f(x))) \geq 1.$$

**Definition 6** ([14]). *A mapping $f : X \to X$ is a weak $\alpha$-admissible mapping type $S$ if the following condition holds:*

$$x \in X,\ \alpha(x, f(x)) \geq s \implies \alpha(f(x), f(f(x))) \geq s.$$

The sets of these two types of weak $\alpha$-admissible mappings were denoted:

$$\mathcal{WA}(X, \alpha) = \{f : X \to X / f \textit{ is a weak } \alpha - \textit{admissible mapping}\}$$

and

$$\mathcal{WA}_s(X, \alpha) = \{f : X \to X / f \textit{ is a weak } \alpha - \textit{admissible mapping type } S\},$$

which concludes the presentation of the notations used in this paper.

**Remark 3.** *It is observed that $\mathcal{A}(X, \alpha) \subseteq \mathcal{WA}(X, \alpha)$ and $\mathcal{A}_s(X, \alpha) \subseteq \mathcal{WA}_s(X, \alpha)$.*

**Definition 7** ([14]). *Let $(X, d)$ be a $b$-metric space with coefficient $s \geq 1$, let $\alpha : X \times X \to [0, +\infty)$ be a given mapping and let $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ be two altering distance functions. Then a mapping $f : X \to X$ is an $(\alpha, \psi, \varphi)_s$-contraction mapping if the following conditions holds:*

$$x, y \in X, \alpha(x, y) \geq s \implies \psi(s^3 d(f(x), f(y))) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)),$$

*where*

$$M_s(x, y) := \max\left\{d(x, y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(y)) + d(y, f(x))}{2s}\right\}.$$

**Remark 4.** *The collection of all $(\alpha, \psi, \varphi)_s$-contraction mappings on a $b$-metric space $(X, d)$ with coefficient $s \geq 1$ was denoted with $\Omega_s(X, \alpha, \psi, \varphi)$ in* [17].

Let $(X, d)$ be a $b$-metric space with coefficient $s \geq 1$ and let $\alpha : X \times X \to [0, +\infty)$ be a given mapping.

**Definition 8** ([14]). *A mapping $f : X \to X$ is said to be an $\alpha_{s,\varepsilon}$-contraction mapping, where $\varepsilon > 1$, if the following condition holds:*

$$x, y \in X,\ \alpha(x, y) \geq s \implies s^{\varepsilon} d(f(x), f(x)) \leq M_s(x, y). \tag{2}$$

Using the notion of $\alpha_{s,\varepsilon}$-contraction mapping, where $\varepsilon > 1$, defined on a $b$-metric space with coefficient $s \geq 1$, Radenović et al. formulate the following fixed point theorem.

**Theorem 1** ([14]). *Let $(X, d)$ be a $b$-complete $b$-metric space with coefficient $s > 1$, let $\alpha : X \times X \to [0, +\infty)$ be a given mapping and $f : X \to X$ be an $\alpha_{s,\varepsilon}$-contraction mapping, where $\varepsilon > 1$. Suppose that the following conditions hold:*

- *($s_1$) $f \in \mathcal{WA}_s(X, \alpha)$;*
- *($s_2$) there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq s$;*
- *($s_3$) $f$ is $b$-continuous.*

*Then, $f$ has at least one fixed point in $X$, i.e. $F_f \neq \emptyset$.*

This theorem will be applied in the next section, in order to obtain our main result.

## 3. Main result

In this section we will establish a theorem of existence of the solution of the nonlinear Fredholm integral equation with modified argument (1)

$$x(t) = \int_a^b K(t, r, x(r), x(g(r)))dr + f(t),$$

where $a, b \in \mathbb{R}$ such that $a < b$, $K \in C([a, b] \times [a, b] \times \mathbb{R}^2)$, $g \in C([a, b], [a, b])$ and $f \in C[a, b]$ are given mappings and $x \in C[a, b]$ is the unknown mapping.

Now, we will use an $\alpha$-admissible mapping type S, [17], and we will apply the Theorem 1 above ([14]).

Let $X = C[a, b]$ and $d : X \times X \to [0, +\infty)$ a $b$-metric defined on $X$, by the relation:

$$d(x, y) := \sup_{t\in[a,b]} |x(t) - y(t)|^p, \ for\ all\ x, y \in X. \tag{3}$$

Then $(X, d)$ is a $b$-complete $b$-metric space with the coefficient $s = 2^{p-1}$ ([14]).

To this integral equation we attach the operator $A : X \to X$ defined by

$$A(x)(t) := \int_a^b K(t, r, x(r), x(g(r)), x(a), x(b))dr + f(t), \tag{4}$$

for all $x \in X$ and $t \in [a, b]$.

In what follows, we will establish the conditions so that the operator $A$ has at least one fixed point.

For this, we define the mapping $\alpha : X \times X \to [0, +\infty)$ by the relation:

$$\alpha(x, y) := \begin{cases} 2^{p-1}, & x(t) \le y(t) \\ \tau, & \text{otherwise} \end{cases} \tag{5}$$

where $\tau \in (0, 2^{p-1})$.

Suppose that the function K is nondecreasing in the last two arguments and therefore, we have:

$$A \in \mathcal{A}_s(X, \alpha) \subseteq \mathcal{WA}_s(X, \alpha).$$

Also, we suppose that, there exists $x_0 \in X$ such that

$$x_0(t) \le \int_a^b K(t, r, x(r), x(g(r)))dr + f(t)$$

for all $t \in [a, b]$.

Therefore, we have: $\alpha(x_0, A(x_0)) \ge 2^{p-1}$.

Let $p \ge 1$ and $q < \infty$, be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$.

Now, we suppose that

$$|K(t, r, x(r), x(g(r))) - K(t, r, y(r), y(g(r)))| \le \tag{6}$$

$$\le \gamma(t, r)\Big[|x(r) - y(r)|^p + |x(g(r)) - y(g(r))|^p\Big]$$

where $t, r \in [a, b]$, $x, y \in X$ with $x(s) \le y(s)$ for all $s \in [a, b]$, $p > 1$ and $\gamma \in C([a, b] \times [a, b], [0, \infty))$, a function that satisfies

$$\sup_{t\in[a,b]} \left(\int_a^b (\gamma(t, r))^p dr\right) < \frac{1}{2^{\varepsilon(p^2-p)+p}(b - a)^{p-1}}, \ for\ \varepsilon > 1. \tag{7}$$

Using (4), (6) and (3), we have:

$$\Big(2^{\varepsilon(p-1)}|A(x)(t)-A(y)(t)|\Big)^p \le 2^{\varepsilon(p^2-p)}\cdot$$
$$\cdot\Big(\int_a^b |K(t,r,x(r),x(g(r)))-K(t,r,y(r),y(g(r)))|dr\Big)^p$$
$$\le 2^{\varepsilon(p^2-p)}\Big[\Big(\int_a^b 1^q dr\Big)^{\frac{1}{q}}\cdot$$
$$\cdot\Big(\int_a^b \Big|K(t,r,x(r),x(g(r)))-K(t,r,y(r),y(g(r)))\Big|^p dr\Big)^{\frac{1}{p}}\Big]^p$$
$$\le 2^{\varepsilon(p^2-p)}(b-a)^{\frac{p}{q}}\Big(\int_a^b \big(\gamma(t,r)\big)^p\cdot$$
$$\cdot\big(|x(r)-y(r)|^p+|x(g(r))-y(g(r))|^p\big)^p dr\Big)$$
$$\le 2^{\varepsilon(p^2-p)}(b-a)^{\frac{p}{q}}\int_a^b \big(\gamma(t,r)\big)^p\big(2d(x,y)\big)^p dr$$
$$\le 2^{\varepsilon(p^2-p)+p}(b-a)^{p-1}\int_a^b \big(\gamma(t,r)\big)^p\big(M_s(x,y)\big)^p dr$$
$$\le 2^{\varepsilon(p^2-p)+p}(b-a)^{p-1}\big(M_s(x,y)\big)^p \sup_{t\in[a,b]}\Big(\int_a^b \big(\gamma(t,r)\big)^p dr\Big),$$

and from (7), it is obtain

$$\big(2^{\varepsilon(p-1)}|A(x)(t)-A(y)(t)|\big)^p \le \big(M_s(x,y)\big)^p.$$

Therefore, we have:

$$2^{\varepsilon(p-1)}|A(x)(t)-A(y)(t)| \le M_s(x,y),$$

i.e.

$$s^\varepsilon d(A(x),A(y)) \le M_s(x,y),$$

and thus, from (2), it results that the operator $A$ is an $\alpha_{s,\varepsilon}$-contraction, where $\varepsilon > 1$.

It is noted that the conditions of Theorem 1 are fulfilled.

Finally, applying the Theorem 1, we can formulate the following theorem of existence of the solution of the nonlinear Fredholm integral equation with modified argument (1), that we present below as the main result of this paper.

**Theorem 2** (Main Theorem). *Consider the nonlinear Fredholm integral equation with modified argument (1). Suppose that the following conditions hold:*

*(i)* $K \in C([a,b]\times[a,b]\times\mathbb{R}^2)$ *is nondecreasing in the last two arguments,* $g \in C([a,b],[a,b])$ *and* $f \in C[a,b]$*;*

*(ii) for each* $r,t\in[a,b]$ *and* $x,y\in X$ *with* $x(s)\le y(s)$ *for all* $s\in[a,b]$*, we have:*

$$|K(t,r,x(r),x(g(r)))-K(t,r,y(r),y(g(r)))|$$
$$\le \gamma(t,r)\Big[|x(r)-y(r)|^p+|x(g(r))-y(g(r))|^p\Big],\ p>1;$$

*(iii) there exists* $x_0\in X$ *such that*

$$x_0(t) \le \int_a^b K(t,r,x(r),x(g(r)))dr + f(t)$$

*for all* $t\in[a,b]$*.*

*Then, the nonlinear Fredholm integral equation with modified argument (1) has at least one solution in $C[a,b]$, i.e. $F_A \neq \emptyset$.*

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University of Petroşani
Department of Mathematics and Computer Science
Universităţii 20, 332006, Petroşani, România
*E-mail address*: mariadobritoiu@yahoo.com