TJMM
12 (2020), No. 1, 1-10

SOME NEW SIMPSON-LIKE TYPE INEQUALITIES VIA PREQAUSIINVEXITY

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ABSTRACT. In this paper, we first establish a new integral identity which represent a partielle result, by using this identity we derive some new Simpson like type integral inequalities for functions whose second derivatives are prequasiinvex functions. we also discuss some special cases where the second derivatives are monotonous functions. At the end we give some applications to special means.

1. INTRODUCTION

The following inequality is well known in the literature as Simpson's inequality

$$\left|\frac{1}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4,$$

where f be four times continuously differentiable function on (a,b), and $\|f\|_\infty=\sup\limits_{x\in(a,b)}\left|f^{(4)}(x)\right|$.

In recent years, many researchers have studied the error estimates of Simpson's inequality, in order to establish new refinements, generalizations as well as new Simpson-type inequalities for more details we refer readers [2, 3, 5, 7, 12, 13].

In [1] Alomari showed that for $f: I\subset\mathbb{R}\to\mathbb{R}$ an absolutely continuous function on I° and $a,b\in I$ with $a<b$ such that $f''\in L([a,b])$ with $|f''|$ a quasi-convex function

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(u)\,du-\frac{1}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right|\\&\leq\frac{(b-a)^2}{162}\left\{\max\left\{|f''(a)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{|f''(b)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right\}\right\}.\end{aligned}$$

In [9] Özdemir et al. gave the following Simpson's inequality for the function whose second derivative at certain power is quasi-convex

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(u)\,du-\frac{1}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)\right|\\&\leq\frac{(b-a)^2}{24}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{3}\right)^{\frac{2}{q}}\left(\frac{q+3+2^{q+2}}{(q+1)(q+2)}\right)\left(\max\left\{|f''(a)|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}.\end{aligned}$$

2010 *Mathematics Subject Classification.* 26D10, 26D15, 26A51.

Key words and phrases. Simpson inequality, Hölder inequality, preqausiinvex functions quasi-convex functions.

Recently, Hua et al.[4] established the following identity

$$\left|\tfrac{1}{6}\left(f\left(a\right)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f\left(b\right)\right)-\tfrac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right|$$
$$=\tfrac{(b-a)^2}{54}\int\limits_{0}^{1}t\left(1-t\right)\left(f''\left(\tfrac{2+t}{3}a+\tfrac{1-t}{3}b\right)+f''\left(\tfrac{1+t}{3}a+\tfrac{2-t}{3}b\right)\right.$$
$$\left.+\ f''\left(\tfrac{t}{3}a+\tfrac{3-t}{3}b\right)\right)dt,$$

and derived some Simpson type inequalities for strongly s-convex functions.

Motivated by the above results in the following study, we generalize the above identity, and then we establish some new Simpson-like type inequalities for twice differentiable prequasiinvex functions.

2. Preliminaries

In this section, we recall some definitions known in the literature

Definition 1. [8] *A set $I\subseteq\mathbb{R}^n$ is said to be convex if for any $x,y\in H$, and $\forall t\in[0,1]$, we have*

$$tx+\left(1-t\right)y\in I.$$

Definition 2. [10] *A function $f:I\to\mathbb{R}$ is said to be convex on I where I is an interval of $\mathbb{R}$, if*

$$f\left(tx+\left(1-t\right)y\right)\leq tf\left(x\right)+\left(1-t\right)f(y)$$

holds for all $x,y\in I$ and all $t\in[0,1]$.

Definition 3. [6] *A function $f:I\subset\mathbb{R}\to\mathbb{R}$ is said to be quasi-convex on I, if*

$$f\left(tx+\left(1-t\right)y\right)\leq\max\left\{f\left(x\right),f\left(y\right)\right\}$$

holds for all $x,y\in I$ and all $t\in[0,1]$.

Definition 4. [14] *A set $K\subset\mathbb{R}^n$ is said to be invex with respect to the map $\eta:K\times K\to\mathbb{R}^n$, if*

$$x+t\eta\left(y,x\right)\in K$$

holds for all $x,y\in K$ and $t\in[0,1]$.

Definition 5. [14] *A function $f:K\subset(0,+\infty)\to\mathbb{R}$ is said to be preinvex with respect to η, if*

$$f\left(x+t\eta\left(y,x\right)\right)\leq\left(1-t\right)f\left(x\right)+tf(y)$$

holds for all $x,y\in K$ and $t\in[0,1]$.

Definition 6. [11] *A function $f:K\subseteq\mathbb{R}\to\mathbb{R}$ is said to be prequasiinvex function with respect to the bifunction $\eta(.,.)$ if*

$$f\left(x+t\eta\left(y,x\right)\right)\leq\max\left\{f\left(y\right),f\left(x\right)\right\}$$

holds for all $x,y\in K$, and $t\in[0,1]$.

We also recall that the beta function for any complex numbers and non-positive integers ρ,τ such that $\mathrm{Re}\left(\rho\right)>0$ and $\mathrm{Re}\left(\tau\right)>0$ is defined by

$$B\left(\rho,\tau\right)=\int\limits_{0}^{1}\theta^{\rho-1}\left(1-\theta\right)^{\tau-1}d\theta=\tfrac{\Gamma(\rho)\Gamma(\tau)}{\Gamma(\rho+\tau)}.$$

3. Main results

Lemma 1. *Let $f : [a, a + \eta(b, a)] \to \mathbb{R}$ be a function be such that f' is absolutely continuous and f'' is integrable on $[a, a + \eta(b, a)]$, then the following equality holds*

$$F(a, b, f) = \frac{\eta^2(b,a)}{54} \int_0^1 t(1-t) \left\{ f''\left(a + \frac{1-t}{3}\eta(b,a)\right) + f''\left(a + \frac{2-t}{3}\eta(b,a)\right) + f''\left(a + \frac{3-t}{3}\eta(b,a)\right) \right\} dt, \tag{1}$$

where

$$F(a, b, f) = \frac{1}{6}\left(f(a) + 2f\left(a + \frac{\eta(b,a)}{3}\right) + 2f\left(a + \frac{2\eta(b,a)}{3}\right) + f(a + \eta(b,a)) \right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\, dx. \tag{2}$$

Proof. Let

$$I_1 = \int_0^1 t(1-t) f''\left(a + \frac{1-t}{3}\eta(b,a)\right) dt, \tag{3}$$

$$I_2 = \int_0^1 t(1-t) f''\left(a + \frac{2-t}{3}\eta(b,a)\right) dt, \tag{4}$$

and

$$I_3 = \int_0^1 t(1-t) f''\left(a + \frac{3-t}{3}\eta(b,a)\right) dt. \tag{5}$$

Integrating by parts the right side of (3), we get

$$\begin{aligned} I_1 &= -\frac{3}{\eta(b,a)} \left(t(1-t) f'\left(a + \frac{1-t}{3}\eta(b,a)\right)\Big|_0^1 - \int_0^1 (1-2t) f'\left(a + \frac{1-t}{3}\eta(b,a)\right) dt \right) \\ &= -\frac{9}{\eta^2(b,a)} (1-2t) f\left(a + \frac{1-t}{3}\eta(b,a)\right)\Big|_0^1 - \frac{18}{\eta^2(b,a)} \int_0^1 f\left(a + \frac{1-t}{3}\eta(b,a)\right) dt \\ &= \frac{9}{\eta^2(b,a)} \left(f(a) + f\left(a + \frac{1}{3}\eta(b,a)\right)\right) - \frac{18}{\eta^2(b,a)} \int_0^1 f\left(a + \frac{1-t}{3}\eta(b,a)\right) dt \\ &= \frac{9}{\eta^2(b,a)} \left(f(a) + f\left(a + \frac{1}{3}\eta(b,a)\right)\right) - \frac{54}{\eta^3(b,a)} \int_a^{a+\frac{1}{3}\eta(b,a)} f(u)\, du. \end{aligned} \tag{6}$$

Similarly, we can easily obtain

$$I_2 = \frac{9}{\eta^2(b,a)} \left(f\left(a + \frac{1}{3}\eta(b,a)\right) + f\left(a + \frac{2}{3}\eta(b,a)\right)\right) - \frac{54}{\eta^3(b,a)} \int_{a+\frac{1}{3}\eta(b,a)}^{a+\frac{2}{3}\eta(b,a)} f(u)\, du, \tag{7}$$

and

$$I_3 = \frac{9}{\eta^2(b,a)} \left(f\left(a + \frac{2}{3}\eta(b,a)\right) + f(a + \eta(b,a))\right) - \frac{54}{\eta^3(b,a)} \int_{a+\frac{2}{3}\eta(b,a)}^{a+\eta(b,a)} f(u)\, du. \tag{8}$$

Adding (6)-(8), and then multiplying the resulting equality by $\frac{\eta^2(b,a)}{54}$, we get the desired result. □

Remark 1. *Lemma 1 will be reduced to Lemma 2.1 from* [4] *if we take $\eta(b,a) = b - a$.*

Theorem 1. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a function such that f' is absolutely continuous and f'' is integrable on $[a,a+\eta(b,a)]$.If $|f''|$ is prequasiinvex, then the following inequality holds*

$$\begin{aligned}|F(a,b,f)|\leq&\tfrac{\eta^2(b,a)}{324}\left(\left(\max\left\{\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|,|f''(a)|\right\}\right)\right.\\&+\left(\max\left\{\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|,\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|\right\}\right)\\&\left.+\left(\max\left\{|f''(b)|,\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|\right\}\right)\right),\end{aligned}$$

where $F(a,b,f)$ is defined as in (2).

Proof. From Lemma 1, property of modulus, and prequasiinvexity of $|f''|$, we have

$$\begin{aligned}&|F(a,b,f)|\\\leq&\tfrac{\eta^2(b,a)}{54}\left(\int_0^1 t(1-t)\left|f''\left(a+\tfrac{1-t}{3}\eta(b,a)\right)\right|dt\right.\\&+\int_0^1 t(1-t)\left|f''\left(a+\tfrac{2-t}{3}\eta(b,a)\right)\right|dt\\&\left.+\int_0^1 t(1-t)\left|f''\left(a+\tfrac{3-t}{3}\eta(b,a)\right)\right|dt\right)\\\leq&\tfrac{\eta^2(b,a)}{54}\left(\int_0^1 t(1-t)\,dt\right)\left(\left(\max\left\{\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|,|f''(a)|\right\}\right)\right.\\&+\left(\max\left\{\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|,\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|\right\}\right)\\&\left.+\left(\max\left\{|f''(b)|,\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|\right\}\right)\right)\\=&\tfrac{\eta^2(b,a)}{324}\left(\left(\max\left\{\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|,|f''(a)|\right\}\right)\right.\\&+\left(\max\left\{\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|,\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|\right\}\right)\\&\left.+\left(\max\left\{|f''(b)|,\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|\right\}\right)\right).\end{aligned}$$

The proof is completed. □

Corollary 1. *In Theorem 1, Additionally, if*

1/ $|f''|$ is increasing, then we have

$$|F(a,b,f)|\leq\frac{\eta^2(b,a)}{324}\left(\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|+\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|+|f''(b)|\right).$$

2/ $|f''|$ is decreasing, then we have

$$|F(a,b,f)|\leq\frac{\eta^2(b,a)}{324}\left(|f''(a)|+\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|+\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|\right).$$

Corollary 2. *In Theorem 1 choosing $\eta(b,a)=b-a$, we obtain*

$$\begin{aligned}&\left|\tfrac{1}{6}\left(f(a)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(x)\,dx\right|\\\leq&\tfrac{(b-a)^2}{324}\left(\left(\max\left\{\left|f''\left(\tfrac{2a+b}{3}\right)\right|,|f''(a)|\right\}\right)\right.\\&+\left(\max\left\{\left|f''\left(\tfrac{a+2b}{3}\right)\right|,\left|f''\left(\tfrac{2a+b}{3}\right)\right|\right\}\right)\\&\left.+\left(\max\left\{|f''(b)|,\left|f''\left(\tfrac{a+2b}{3}\right)\right|\right\}\right)\right).\end{aligned}$$

Corollary 3. *In Corollary 2, Additionally, if*

1/ $|f''|$ is increasing, then we have

$$\left|\tfrac{1}{6}\left(f(a)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \tfrac{(b-a)^2}{324}\left(\left|f''\left(\tfrac{2a+b}{3}\right)\right|+\left|f''\left(\tfrac{a+2b}{3}\right)\right|+|f''(b)|\right).$$

2/ $|f''|$ is decreasing, then we have

$$\left|\tfrac{1}{6}\left(f(a)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \tfrac{(b-a)^2}{324}\left(|f''(a)|+\left|f''\left(\tfrac{2a+b}{3}\right)\right|+\left|f''\left(\tfrac{a+2b}{3}\right)\right|\right).$$

Theorem 2. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a function such that f' is absolutely continuous and f'' is integrable on $[a,a+\eta(b,a)]$. If $|f''|^q$ is prequasiinvex for $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds*

$$\begin{aligned}
&|F(a,b,f)|\\
\leq&\tfrac{\eta^2(b,a)}{54}\left(B(p+1,p+1)\right)^{\frac{1}{p}}\\
&\times\Bigg(\left(\max\left\{\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|^q,|f''(a)|^q\right\}\right)^{\frac{1}{q}}\\
&+\left(\max\left\{\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|^q,\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}}\\
&+\left(\max\left\{|f''(b)|^q,\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}}\Bigg),
\end{aligned}$$

where $F(a,b,f)$ is defined as in (2) and $B(.,.)$ is the beta function.

Proof. Using Lemma 1, properties of modulus, Hölder's inequality, and prequasiinvexity of $|f''|^q$, we have

$$\begin{aligned}
&|F(a,b,f)| \\
\leq &\tfrac{\eta^2(b,a)}{54}\left(\int_0^1 t(1-t)\left|f''\left(a+\tfrac{1-t}{3}\eta(b,a)\right)\right|dt\right. \\
&+\int_0^1 t(1-t)\left|f''\left(a+\tfrac{2-t}{3}\eta(b,a)\right)\right|dt \\
&\left.+\int_0^1 t(1-t)\left|f''\left(a+\tfrac{3-t}{3}\eta(b,a)\right)\right|dt\right) \\
\leq &\tfrac{\eta^2(b,a)}{54}\left(\int_0^1 (t(1-t))^p\,dt\right)^{\frac{1}{p}} \\
&\times\left(\left(\max\left\{\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|^q,|f''(a)|^q\right\}\int_0^1 dt\right)^{\frac{1}{q}}\right. \\
&+\left(\max\left\{\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|^q,\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|^q\right\}\int_0^1 dt\right)^{\frac{1}{q}} \\
&\left.+\left(\max\left\{|f''(b)|^q,\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|^q\right\}\int_0^1 dt\right)^{\frac{1}{q}}\right) \\
\leq &\tfrac{\eta^2(b,a)}{54}\left(B(p+1,p+1)\right)^{\frac{1}{p}} \\
&\times\left(\left(\max\left\{\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|^q,|f''(a)|^q\right\}\right)^{\frac{1}{q}}\right. \\
&+\left(\max\left\{\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|^q,\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
&\left.+\left(\max\left\{|f''(b)|^q,\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

The proof is completed. □

Corollary 4. *In Theorem 2, Additionally, if*

1/ $|f''|$ is increasing, then we have

$$\begin{aligned}
&|F(a,b,f)| \leq \tfrac{\eta^2(b,a)}{54}\left(B(p+1,p+1)\right)^{\frac{1}{p}} \\
&\times\left(\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|+\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|+|f''(b)|\right).
\end{aligned}$$

2/ $|f''|$ is decreasing, then we have

$$\begin{aligned}
&|F(a,b,f)| \leq \tfrac{\eta^2(b,a)}{54}\left(B(p+1,p+1)\right)^{\frac{1}{p}} \\
&\times\left(|f''(a)|+\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|+\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|\right).
\end{aligned}$$

Corollary 5. *In Theorem 2 choosing $\eta(b,a) = b-a$, we obtain*

$$\begin{aligned}&\left|\tfrac{1}{6}\left(f(a)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq&\tfrac{\eta^2(b,a)}{54}\left(B(p+1,p+1)\right)^{\frac{1}{p}}\left(\left(\max\left\{\left|f''\left(\tfrac{2a+b}{3}\right)\right|^q,\left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\ &+\left(\max\left\{\left|f''\left(\tfrac{a+2b}{3}\right)\right|^q,\left|f''\left(\tfrac{2a+b}{3}\right)\right|^q\right\}\right)^{\frac{1}{q}}\\ &\left.+\left(\max\left\{\left|f''(b)\right|^q,\left|f''\left(\tfrac{a+2b}{3}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Corollary 6. *In Corollary 5, Additionally, if*
1/ $|f''|$ is increasing, then we have

$$\begin{aligned}&\left|\tfrac{1}{6}\left(f(a)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq&\tfrac{(b-a)^2}{54}\left(B(p+1,p+1)\right)^{\frac{1}{p}}\left(\left|f''\left(\tfrac{2a+b}{3}\right)\right|+\left|f''\left(\tfrac{a+2b}{3}\right)\right|+\left|f''(b)\right|\right).\end{aligned}$$

2/ $|f''|$ is decreasing, then we have

$$\begin{aligned}&\left|\tfrac{1}{6}\left(f(a)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(x)\,dx\right|\\ \leq&\tfrac{(b-a)^2}{54}\left(B(p+1,p+1)\right)^{\frac{1}{p}}\left(\left|f''(a)\right|+\left|f''\left(\tfrac{2a+b}{3}\right)\right|+\left|f''\left(\tfrac{a+2b}{3}\right)\right|\right).\end{aligned}$$

Theorem 3. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a function such that f' is absolutely continuous and f'' is integrable on $[a,a+\eta(b,a)]$. If $|f''|^q$ is prequasiinvex for $q\geq 1$, then the following inequality holds*

$$\begin{aligned}|F(a,b,f)|\leq&\tfrac{\eta^2(b,a)}{324}\left(\left(\max\left\{\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|^q,\left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\ &+\left(\max\left\{\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|^q,\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}}\\ &\left.+\left(\max\left\{\left|f''(b)\right|^q,\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}}\right),\end{aligned}$$

where $F(a,b,f)$ is defined as in (2).

Proof. Using Lemma 1, properties of modulus, power mean inequality, and prequasiinvexity of $|f''|^q$ we get

$$
\begin{aligned}
&\left|F(a,b,f)\right| \\
\leq&\tfrac{\eta^2(b,a)}{54}\left(\int_0^1 t(1-t)\left|f''\left(a+\tfrac{1-t}{3}\eta(b,a)\right)\right|dt\right. \\
&+\int_0^1 t(1-t)\left|f''\left(a+\tfrac{2-t}{3}\eta(b,a)\right)\right|dt \\
&\left.+\int_0^1 t(1-t)\left|f''\left(a+\tfrac{3-t}{3}\eta(b,a)\right)\right|dt\right) \\
\leq&\tfrac{\eta^2(b,a)}{54}\left(\left(\int_0^1 t(1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left|f''\left(a+\tfrac{1-t}{3}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&+\left(\int_0^1 t(1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left|f''\left(a+\tfrac{2-t}{3}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\left.+\left(\int_0^1 t(1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left|f''\left(a+\tfrac{3-t}{3}\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right) \\
\leq&\tfrac{\eta^2(b,a)}{54}\left(\int_0^1 t(1-t)\,dt\right)^{1-\frac{1}{q}} \\
&\times\left(\left(\max\left\{\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|^q,\left|f''(a)\right|^q\right\}\int_0^1 t(1-t)\,dt\right)^{\frac{1}{q}}\right. \\
&+\left(\max\left\{\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|^q,\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|^q\right\}\int_0^1 t(1-t)\,dt\right)^{\frac{1}{q}} \\
&\left.+\left(\max\left\{\left|f''(b)\right|^q,\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|^q\right\}\int_0^1 t(1-t)\,dt\right)^{\frac{1}{q}}\right) \\
=&\tfrac{\eta^2(b,a)}{54}\left(\int_0^1 t(1-t)\,dt\right)\left(\left(\max\left\{\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|^q,\left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\
&+\left(\max\left\{\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|^q,\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
&\left.+\left(\max\left\{\left|f''(b)\right|^q,\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}}\right) \\
=&\tfrac{\eta^2(b,a)}{324}\left(\left(\max\left\{\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|^q,\left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\
&+\left(\max\left\{\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|^q,\left|f''\left(a+\tfrac{1}{3}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}} \\
&\left.+\left(\max\left\{\left|f''(b)\right|^q,\left|f''\left(a+\tfrac{2}{3}\eta(b,a)\right)\right|^q\right\}\right)^{\frac{1}{q}}\right),
\end{aligned}
$$

which is the desired result. □

Corollary 7. *In Theorem 3 choosing $\eta(b,a) = b - a$, we obtain*

$$\left|\frac{1}{6}\left(f(a) + 2f\left(\frac{2a+b}{3}\right) + 2f\left(\frac{a+2b}{3}\right) + f(b)\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq \frac{(b-a)^2}{324}\left(\left(\max\left\{\left|f''\left(\frac{2a+b}{3}\right)\right|^q, |f''(a)|^q\right\}\right)^{\frac{1}{q}}\right.$$

$$+\left(\max\left\{\left|f''\left(\frac{a+2b}{3}\right)\right|^q, \left|f''\left(\frac{2a+b}{3}\right)\right|^q\right\}\right)^{\frac{1}{q}}$$

$$\left.+\left(\max\left\{|f''(b)|^q, \left|f''\left(\frac{a+2b}{3}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right).$$

4. Applications to special means

We shall consider the means for arbitrary real numbers a, b

The Arithmetic mean: $A(a,b) = \frac{a+b}{2}$.

The p-Logarithmic mean: $L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a, b > 0, a \neq b$ and $p \in \mathbb{R} \setminus \{0, -1\}$.

Proposition 1. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$\left|A\left(a^5, b^5\right) + 2A\left(\left(\frac{2a+b}{3}\right)^5, \left(\frac{a+2b}{3}\right)^5\right) - 3L_5^5(a,b)\right|$$

$$\leq \frac{5(b-a)^2}{27}\left(\left(\frac{2a+b}{3}\right)^3 + \left(\frac{a+2b}{3}\right)^3 + b^3\right).$$

Proof. The assertion follows from Corollary 3 with $\eta(b,a) = b - a$ applied to the function $f(x) = x^5$. □

Proposition 2. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$\left|A\left(a^3, (a + A(a,b))^3\right) + 2A\left(\left(\frac{3a+A(a,b)}{3}\right)^3, \left(\frac{3a+2A(a,b)}{3}\right)^3\right) - 3L_3^3(a, a + A(a,b))\right|$$

$$\leq \frac{A^2(b,a)}{3}\left(B(p+1, p+1)\right)^{\frac{1}{p}}(a + 3A(a,b)).$$

Proof. The assertion follows from Corollary 4 with $q \geq 1, \eta(b,a) = A(a,b)$, applied to the function $f(x) = x^3$. □

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